§3. Electromagnetic Waves

§3.5. Plasma waves

A plasma is an ionized gas consisting of charged particles (e.g., electrons and ions). Various waves can be excited easily in a plasma. Wave phenomena have been an important subject in the plasma research community.

The plasma is nearly charge neutral. So the $\nabla \cdot \vec{E} = 0$ still holds. However, neither the conduction current nor the displacement current can be ignored. Waves in plasma is different from the waves in vacuum and in conductors.

§3.5.1 Effective Permittivity in a Plasma

In vacuum, the phase velocity of an EM wave is:

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

This can be generalized for phase velocity of waves in matter:

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon \mu}}$$

In plasma $\mu = \mu_0$, but $\epsilon \neq \epsilon_0$.

Electric field, magnetic field, and generally any quantity in plasma can be divided into two parts: DC part that does not depend on time and parts that associated with waves:

$$Q_{\text{total}} = Q_{\text{DC}} + Q(\vec{r}, t)$$

We study now only the part associated with waves, $Q$, and assume

$$Q = Q_0 e^{i(k \cdot \vec{r} - \omega t)}$$

In plasma, the current is predominately carried by electrons, as in a conductor, because the mass of a electron is small compared with that of ions. The electrons in plasma experience the electric force and suffer from collision with ions. The velocity of electrons has been derived before (when we study waves in conductors):

$$\vec{v} = -\frac{e}{m(\nu - i\omega)} \vec{E}$$

$$\vec{J} = \frac{ne^2}{m(\nu - i\omega)} \vec{E}$$

if there is no DC magnetic field (So we can ignore $\vec{v} \times \vec{B}$ term in the equation of motion).
In most cases, the electron density, and thus the collision frequency $\nu$, are much smaller than in conductors. So:

$$ j \simeq -\frac{ne^2}{mi\omega} \vec{E} = i \frac{ne^2}{m\omega} \vec{E} $$

The 4th Maxwell’s equation becomes:

$$ \nabla \times \vec{B} = \mu_0 \left( i \frac{ne^2}{m\omega} \vec{E} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) $$

$$ = \mu_0 \left( i \frac{ne^2}{m\omega} \vec{E} - i \omega \epsilon_0 \vec{E} \right) $$

$$ = -i \omega \epsilon_0 \mu_0 \left( 1 - \frac{ne^2}{m\epsilon_0 \omega^2} \right) \vec{E} $$

Let

$$ \omega_p = \sqrt{\frac{ne^2}{m\epsilon_0}} $$

Plasma frequency

and recall $\vec{H} = \vec{B}/\mu_0$, we find

$$ \nabla \times \vec{H} = -i \omega \epsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2} \right) \vec{E} $$

For low frequency wave, $\omega_p \gg \omega$, conduction current dominates (like waves in a conductor).

For high frequency wave, $\omega_p \ll \omega$, displacement current dominates (like waves in vacuum).

If we define an effective permittivity in plasma,

$$ \epsilon(\omega) = \epsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2} \right), $$

The 4th Maxwell’s equation for a monochromatic plane wave in plasma becomes

$$ \nabla \times \vec{H} = -i \omega \epsilon \vec{E} = \epsilon \frac{\partial \vec{E}}{\partial t}, $$

similar to the equation for a monochromatic plane wave in the vacuum, except $\epsilon_0 \rightarrow \epsilon$. Therefore, all the results for waves in vacuum can be used in plasma wave with the modification $\epsilon_0 \rightarrow \epsilon$. Note that the effective permittivity in plasma depends on the frequency $\omega$. Note also $\epsilon < \epsilon_0$.

§3.5.2 Dispersion Relation
The phase velocity of plasma waves:

\[ v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \]

\[ = \frac{1}{\sqrt{\mu_0\epsilon_0}} \frac{1}{\sqrt{1 - \frac{\omega_0^2}{\omega^2}}} \]

\[ = \frac{c}{\sqrt{1 - \frac{\omega_0^2}{\omega^2}}} > c \]  \hspace{1cm} (1)

In plasma, \( \epsilon \) depends on \( \omega \) and thus the phase velocity depends also on \( \omega \). The wave in this case is dispersive.

A square wave contains the fundamental frequency and its higher harmonics. If the phase velocity depends on frequency, it will spread out while it propagates.

The dependence of \( \omega \) on \( k \) is called dispersion relation.

We can solve eq. (1) for \( \omega \)

\[ \frac{\omega^2}{k^2} = \frac{c^2}{1 - \frac{\omega_0^2}{\omega^2}} \]

\[ \omega^2 \left( 1 - \frac{\omega_0^2}{\omega^2} \right) = c^2 k^2 \]

\[ \omega^2 = \omega_p^2 + (ck)^2 \]  \hspace{1cm} dispersion relation in plasma
The dispersion relation of EM waves in vacuum is a straight line (non-dispersive) with a slope \( \tan \alpha = c \).

The dispersion relation of EM waves in plasma is above the line \( \omega = ck \) because

\[
\omega = \sqrt{\omega_p^2 + (ck)^2} \geq ck,
\]

but approaches the line \( \omega = ck \) when \( k \) becomes large because

\[
\omega = \sqrt{\omega_p^2 + (ck)^2} \simeq ck \quad \text{when} \quad k \gg \frac{\omega_p}{c}.
\]

**Propagation and Reflection of EM Waves in Plasma**

Assume a plane wave propagating in +z direction.

\[
\vec{E} = \vec{E}_0e^{i(kz-\omega t)}
\]

From the dispersion relation, we obtain

\[
k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}
\]

- If \( \omega > \omega_p \) (like in vacuum)
  
  \( k \) is real, \( \vec{E} = \vec{E}_0e^{i(kz-\omega t)} \), wave propagates without decay.

- If \( \omega < \omega_p \) (like in conductor)

  \( k = i\frac{1}{c}\sqrt{\omega_p^2 - \omega^2} = |k| \) is imaginary.

\[
\vec{E} = \vec{E}_0e^{-i\omega t}e^{-|k|z}
\]

Waves decays in z direction and will be reflected.

If \( \omega \ll \omega_p \), \( k \simeq \frac{\omega_p}{c} = i\frac{1}{\delta} \)

\[
\delta = \frac{c}{\omega_p} \quad \text{skin depth in plasma}
\]

**Short Wave Communication**

Ionospheric plasma (height: 50 km to 100 km) has a typical density of \( 10^{13}/m^3 \).

\[
f_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{ne^2}{m\epsilon_0}} = 28 \text{ MHz}
\]

Short wave radio (\( f \sim 10 \text{ MHz} \)) relies on the multiple reflection between the ionospheric plasma layer and the earth to reach a distant receiver.
Earth is “conductor” \((\sigma \sim 10^{-2} \text{ S/m})\) as long as the impedance
\[
|Z| = \left| \frac{-i\omega\mu_0}{\sigma} \right| \ll Z_{\text{air}} = \sqrt{\frac{\mu_0}{\epsilon_0}}
\]
which requires
\[
\omega \ll \frac{\sigma}{\epsilon_0}
\]
or
\[
f \ll \frac{\sigma}{2\pi\epsilon_0} = \frac{10^{-2}}{2\pi \times 8.85 \times 10^{-12}} = 180 \text{ MHz}
\]
For \(f = 10 \text{ MHz}\), the earth is good conductor.

§3.5.3 Group Velocity

According to Einstein’s relativity, nothing should propagate faster than the light speed \(c\).

In plasma, phase velocity
\[
v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega^2}{c^2}}} > c
\]
But the phase velocity does not correspond to the information (energy) propagation velocity. The information is propagating at the group velocity
\[
v_g = \frac{d\omega}{dk}
\]
For non-dispersive waves, e.g., EM waves in vacuum,
\[
\omega = ck \quad v_p = v_g = c.
\]
In plasma, wave is dispersive. The group velocity is
\[
v_g = \frac{d\omega}{dk} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} < c.
\]