§3. Electromagnetic Waves

§3.4. EM fields (waves) in conductors

The behaviour of EM waves in a conductor is quite different from that in a source-free medium. The conduction current in a conductor is the cause of the difference. We shall analyze the source terms in the Maxwell’s equations to simplify Maxwell’s equations in a conductor. From this set of equations, we can derive a diffusion equation and investigate the skin effects.

§3.4.1 Skin Effects in Conductors

A. Maxwell’s Equations in a Conductor

Complete Maxwell’s equations:

\[ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{B} = \mu \left( \vec{j} + \frac{\partial \vec{E}}{\partial t} \right) \]

Fact 1: Conducting current dominates over the displacement current

In a conductor, the electric field is the driving source for the conduction current. The collision is the source of impedance. The conduction current is governed by the Ohm’s law:

\[ \vec{j} = \sigma \vec{E} \]

where \( \sigma \) (S/m) is the conductivity.

The AC conductivity differs from the DC conductivity. Let \( \nu \) be the collision frequency of electrons (current carrier in conductors) with ions and \( \omega \) the frequency of the EM waves in the conductor. The equation of motion for electrons is:

\[ m \frac{d\vec{v}}{dt} = -e \vec{E} - m \nu \vec{v} \]

Assume \( \vec{v} = \vec{v}_0 e^{-i\omega t} \) and use \( \partial/\partial t \rightarrow -i\omega \), we obtain

\[ -i\omega m \vec{v} = -e \vec{E} - m \nu \vec{v} \rightarrow \vec{v} = \frac{-e}{m(\nu - i\omega)} \vec{E} \]
Recall the current density is expressed by $\vec{j} = -en\vec{v}$ ($n$ is the electron number density in the conductor),

$$\vec{j}_f = \frac{ne^2}{m(\nu - i\omega)}\vec{E}$$

which gives the AC conductivity in conductor:

$$\sigma(\omega) = \frac{1}{\nu - i\omega} \frac{ne^2}{m}$$

In practical situations, $\omega \ll \nu \sim 10^{14}$ (1/sec) (infrared range). So the DC conductivity

$$\sigma = \frac{ne^2}{m\nu}$$

can be used.

Let’s now compare the magnitude of conduction current with that of the displacement current.

Assume $\vec{E} = \vec{E}_0 e^{-i\omega t}$. Then

$$\left| \frac{\vec{j}_f}{\epsilon \frac{\partial \vec{E}}{\partial t}} \right| = \frac{\sigma E}{\epsilon \omega E} = \frac{\sigma}{\epsilon \omega}$$

In copper, $\sigma = 6 \times 10^7$ (S/m). The condition for $j_f \simeq \epsilon \frac{\partial \vec{E}}{\partial t}$, or $\frac{\sigma}{\epsilon \omega} \simeq 1$ leads to

$$\omega = \frac{\sigma}{\epsilon} = \frac{6 \times 10^7}{8.85 \times 10^{-12}} \sim 7 \times 10^{19} \text{ (rad/sec)}$$

At frequencies $\omega < 10^{12}$ (rad/sec) (communication wave frequency),

$$\frac{\sigma}{\epsilon \omega} \gg 1 \quad \text{or} \quad \left| \vec{j}_f \right| \gg \left| \epsilon \frac{\partial \vec{E}}{\partial t} \right|$$

Ampere’s law in a conductor:

$$\nabla \times \vec{B} = \mu \vec{j}_f = \mu \sigma \vec{E}$$

Fact 2: No significant charge accumulation

Because of the good conductivity, no significant charge accumulation in a conductor is expected ($\rho \simeq 0$).
From the charge conservation and Gauss’s law
\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \]
\[ \nabla \cdot \vec{j}_f = -\frac{\partial \rho_f}{\partial t} \]
we obtain
\[ \frac{\partial \rho_f}{\partial t} = -\sigma \nabla \cdot \vec{E} = -\frac{\sigma}{\varepsilon} \rho_f \]
So
\[ \rho_f = \rho_f(0)e^{-\frac{t}{\tau}} \]
The free charge \( \rho_f(0) \) dissipates in a characteristic time \( \tau = \frac{\varepsilon}{\sigma} \), similar to the static case in which the charge will flow out to the edge of the conductor.
If the transient phase is excluded, \( \rho_f = 0 \) can be assumed in a conductor. For simplicity, we shall consider “good” conductor case in which the displacement current can be ignored.

Maxwell’s equations in a conductor:
\[ \nabla \cdot \vec{E} = 0 \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{B} = \mu \sigma \vec{E} \]

It can be shown that the EM waves in conductor are also TEM (Transverse EM) waves

B. Diffusion Equation
As we did before for waves in source-free media, let’s apply curl operator to the 2nd Maxwell’s equation:
\[ \nabla \times \left( \nabla \times \vec{E} \right) = -\nabla \times \left( \frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \vec{B}) = -\frac{\partial}{\partial t} (\mu \sigma \vec{E}) \]
LHS of the equation
\[ \nabla \times \left( \nabla \times \vec{E} \right) = \nabla \left( \nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\nabla^2 \vec{E} \]
So
\[ \nabla^2 \vec{E} = \mu \sigma \frac{\partial \vec{E}}{\partial t} \quad \text{Diffusion equation} \]
Similarly, the magnetic field also satisfies the same diffusion equation:
\[ \nabla^2 \vec{B} = \mu \sigma \frac{\partial \vec{B}}{\partial t} \quad \text{Diffusion equation} \]
C. Skin Depth

Suppose we have a plane wave. It comes from the $-z$ direction and reaches a large conductor surface at $z = 0$. Outside of a conductor: $\vec{E} = E_0 e^{-i\omega t} e_x$ at $z = 0$.

Assume the wave inside the conductor has the form

$$\vec{E} = \vec{E}_0 e^{ikz - i\omega t}$$

where $k$ is an unknown constant. Recall

$$\nabla \rightarrow ik, \quad \frac{\partial}{\partial t} \rightarrow -i\omega$$

for the waves of the above type, we find from the diffusion equation

$$(ik)^2 \vec{E} = -i\omega \mu \sigma \vec{E} \quad \Rightarrow \quad k^2 = i\omega \mu = \omega \mu e^{\pm i}$$

Choose “+” sign to allow the electric field to damping (to “propagate”) in the $+z$ direction. Separate the real and imaginary parts of $k$:

$$k = k_+ + ik_- \quad k_+ = \pm \sqrt{\frac{\omega \mu \sigma}{2}}$$

If the “good” conductor assumption is not valid, the displacement current should be included in the 4th Maxwell’s equation. The solution for $E$ and $B$ are the same as above with

$$k_{\pm} = \omega \sqrt{\frac{\varepsilon \mu}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\varepsilon \omega} \right)^2} \pm 1 \right]^{1/2}$$

The equations (1) and (2) indicate that the amplitude of $E$ and $B$ fields decays to $1/e$ of their values at $z = 0$ in a distance:

$$\delta = \frac{1}{k_-}$$
where $\delta$ is called skin depth

For a good conductor ($\sigma \gg \varepsilon \omega$):

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}} \text{ (m)}$$

Also, the wavelength is $\lambda = \frac{2\pi}{k} = 2\pi \delta$ in a good conductor. The wave decays significantly within one wavelength. Since $\delta \propto \sqrt{1/\omega \sigma}$, deep penetration occurs for

1. Low frequency
2. poor conductor

Example: skin depth at $f = 60$ Hz for copper.

$$\delta = \sqrt{\frac{2}{2\pi \times 60 \times 4\pi \times 10^{-7} \times 6 \times 10^{7}}} = 8 \times 10^{-3} \text{ m} = 8 \text{ mm}$$
There is no advantage to construct AC transmission lines using wires with a radius much larger than the skin depth because the current flows mainly in the outer part of the conductor.

For a poor conductor \((\sigma \ll \varepsilon \omega)\):

\[
\delta = \frac{2}{\sigma} \sqrt{\frac{\varepsilon}{\mu}} \text{ (m)}
\]

independent of the frequency.

Example:

For sea water, \(\mu = \mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2, \varepsilon \approx 70\varepsilon_0 = 6 \times 10^{-10} \text{ C}^2/\text{N} \cdot \text{m}^2\), and \(\sigma \approx 5 \text{ (}\Omega \cdot \text{m})^{-1}\).

Sea water is a poor conductor for frequency

\[
f = \frac{\omega}{2\pi} \gg \frac{\sigma}{2\pi\varepsilon} = 10^9 \text{ Hz}
\]

or \(\lambda \ll 30 \text{ cm}\). The skin depth is

\[
\delta = \frac{2}{\sigma} \sqrt{\frac{\varepsilon}{\mu}} = \frac{2}{\sigma} \sqrt{\frac{70\varepsilon_0}{\mu_0}}
\]

\[
= \frac{2\sqrt{70}}{\sigma Z} = \frac{2\sqrt{70}}{5 \times 377} \approx 1 \text{ cm}
\]

In the radio frequency range \((f \ll 10^9 \text{ Hz})\) sea water is a good conductor, the skin depth \(\delta = \sqrt{2/(\omega \mu \sigma)}\) is quite short. To reach a depth \(\delta = 10 \text{ m}\), for communication with submarines,

\[
f = \frac{\omega}{2\pi} = \frac{1}{\pi \mu \sigma \delta^2} \approx 500 \text{ Hz}
\]
The wavelength in the air is about
\[ \lambda = \frac{c}{f} = \frac{3 \times 10^8}{500} = 600 \text{ km} \]

The required \( \lambda/4 \) antenna would be gigantic.

§3.4.2 Monochromatic plane waves in a Conductor

A. Transverse waves

The E and B fields in a conductor
\[ \vec{E}(z, t) = \vec{E}_0 e^{-k_- z} e^{i(k_+ z - \omega t)}, \quad \vec{B}(z, t) = \vec{B}_0 e^{-k_- z} e^{i(k_+ z - \omega t)} \]
can be rewritten as
\[ \vec{E}(z, t) = \vec{E}_0 e^{i(kz - \omega t)}, \quad \vec{B}(z, t) = \vec{B}_0 e^{i(kz - \omega t)} \]

They have the same functions as EM wave in vacuum, except that \( k \) is a complex number. Following the same calculation for waves in vacuum, we can derive from
\[ \nabla \cdot \vec{E} = 0, \quad \text{and} \quad \nabla \cdot \vec{B} = 0 \]
the following results:
\[ \vec{k} \cdot \vec{E} = k \vec{e}_z \cdot \vec{E} = 0 \quad \vec{k} \cdot \vec{B} = k \vec{e}_z \cdot \vec{B} = 0 \]

Both \( \vec{E} \) and \( \vec{B} \) are perpendicular to \( \vec{e}_z \), the wave propagation direction.

Let’s assume
\[ \vec{E} = E_0 e^{i(kz - \omega t)} \vec{e}_x = E_0 e^{-k_- z} e^{i(k_+ z - \omega t)} \vec{e}_x \]

From
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
we obtain
\[ \vec{B} = \frac{k}{\omega} E_0 e^{-k_- z} e^{i(k_+ z - \omega t)} \vec{e}_y \]
\[ \quad = \frac{|k|}{\omega} e^{i\phi} E_0 e^{-k_- z} e^{i(k_+ z - \omega t)} \vec{e}_y \]
where
\[ \phi = \tan^{-1} \left( \frac{k_-}{k_+} \right), \quad |k| = \sqrt{k_+^2 + k_-^2} \]

For a good conductor \( \phi = 45^0, \ |k| = \sqrt{\frac{\omega \mu \sigma}} \)

The B fields lags behind the electric fields

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§4.3.3 Reflection of EM Waves on a Conductor Surface

We have seen that the EM waves do not penetrate the conductor deeply. Where do the waves go? Absorbed or reflected?

Back to our example with a plane wave perpendicularly propagating to a conducting surface

\[
\vec{H} = H_y \hat{e}_y = H_0 e^{i(z/\delta - \omega t)} e^{-z/\delta} \hat{e}_y
\]

\[
\nabla \times \vec{H} = J = \sigma E_x \hat{e}_x = \sigma E_0 e^{i(z/\delta - \omega t)} e^{-z/\delta} \hat{e}_x
\]

\[
(\nabla \times \vec{H})_x = [\nabla \times (H_y \hat{e}_y)]_x = \left. \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right|_{y=0} = \frac{1 - i}{\delta} H_y
\]

So

\[
\frac{1 - i}{\delta} H_y = \sigma E_x
\]
Characteristic impedance of the conductor

\[
Z = \frac{E_x}{H_y} = \frac{1 - i}{\sigma \delta} = \frac{1 - i}{\sigma} \sqrt{\frac{\omega \mu \sigma}{2}}
\]

\[
= \frac{1 - i}{\sqrt{2}} \sqrt{\frac{\omega \mu}{\sigma}} = \sqrt{\frac{-i \omega \mu}{\sigma}}
\]

For 4mm microwave, \( f = 75 \) Gzh. If the conductor is aluminum (\( \sigma_{Al} = 2 \times 10^7 \) S/m)

\[
|Z_{Al}| = \sqrt{\frac{\omega \mu}{\sigma}} = \sqrt{\frac{2 \pi \times 75 \times 10^9 \times 4 \pi \times 10^{-7}}{2 \times 10^7}}
\]

\[
= 0.17 \Omega \ll Z_{air} = 377 \Omega
\]

The reflectivity:

\[
\Gamma = \frac{Z_{Al} - Z_{air}}{Z_{Al} + Z_{air}} \approx \frac{-Z_{air}}{Z_{air}} = -1
\]

Almost complete reflection.