§3. Electromagnetic Waves

§3.3. Reflection and Transmission of EM waves

§3.3.1 Reflection and Transmission at Normal Incidence

A. Fields

Assume an incident light with $\overrightarrow{E}$ polarized in the $x$-direction and $\overrightarrow{k}$ (or $\overrightarrow{v}$) in $z$-direction entering from medium 1 to medium 2. The normal of the boundary surface is in the $z$-direction.

![Diagram of electromagnetic waves](image)

The incident light can be expressed in the form

$$\overrightarrow{E_I} = E_{I0} \exp\left[ i (k_1 z - \omega t) \right] \overrightarrow{e_x}$$

$$\overrightarrow{B_I} = \frac{1}{v_1} \overrightarrow{k}_{I0} \times \overrightarrow{E_I} = \frac{1}{v_1} E_{I0} \exp\left[ i (k_1 z - \omega t) \right] \overrightarrow{e_y}$$

It can be shown that $\overrightarrow{E_R}$ and $\overrightarrow{E_T}$ are all in the $\overrightarrow{e_x}$ direction (assignment). Note also that the reflected wave is in the $-z$ direction ($\overrightarrow{k}_R = -k_1 \overrightarrow{e_z}$), we obtain

$$\overrightarrow{E_R} = E_{R0} \exp\left[ i (-k_1 z - \omega t) \right] \overrightarrow{e_x}$$

$$\overrightarrow{B_R} = \frac{1}{v_1} \overrightarrow{k}_{R0} \times \overrightarrow{E_R} = -\frac{1}{v_1} E_{R0} \exp\left[ i (-k_1 z - \omega t) \right] \overrightarrow{e_y}$$

the transmitted wave is

$$\overrightarrow{E_T} = E_{T0} \exp\left[ i (k_2 z - \omega t) \right] \overrightarrow{e_x}$$

$$\overrightarrow{B_T} = \frac{1}{v_2} \overrightarrow{k}_{T0} \times \overrightarrow{E_T} = \frac{1}{v_2} E_{T0} \exp\left[ i (k_2 z - \omega t) \right] \overrightarrow{e_y}$$
Our job now is to use boundary conditions to find the complex amplitudes of the reflected and transmitted waves in terms of that of incident wave. Note

\[
\begin{align*}
\vec{E}_1 &= \vec{E}_I + \vec{E}_R \\
\vec{B}_1 &= \vec{B}_I + \vec{B}_R \\
\vec{E}_2 &= \vec{E}_T \\
\vec{B}_2 &= \vec{B}_T 
\end{align*}
\]

At \( z = 0 \) on the boundary the time varying terms, \( \exp(-i\omega t) \), are the same for all fields. First B.C.

\[
\vec{E}_1^k = \vec{E}_2^k
\]

leads to

\[
E_{I0} + E_{R0} = E_{T0} \quad (1)
\]

another B.C.

\[
\frac{1}{\mu_1} \vec{B}_1^k = \frac{1}{\mu_2} \vec{B}_2^k
\]

leads to

\[
\frac{1}{\mu_1} \left[ \frac{1}{v_1} E_{I0} - \frac{1}{v_1} E_{R0} \right] = \frac{1}{\mu_2 v_2} E_{T0} \quad (2)
\]

Let

\[
\beta = \frac{\mu_1 v_1}{\mu_2 v_2}
\]

eq. (2) becomes

\[
E_{I0} - E_{R0} = \beta E_{T0} \quad (3)
\]

Solve for \( E_{R0} \) and \( E_{T0} \) from eqs. (1) and (3)

\[
\begin{align*}
E_{R0} &= \frac{1 - \beta}{1 + \beta} E_{I0} \\
E_{T0} &= \frac{2}{1 + \beta} E_{I0}
\end{align*}
\]

For \( \mu_1 \sim \mu_0 \), and \( \mu_2 \sim \mu_0 \)

\[
\beta \sim \frac{\mu_0 v_1}{\mu_0 v_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}
\]

So that

\[
\begin{align*}
E_{R0} &= \frac{n_1 - n_2}{n_1 + n_2} E_{I0} \\
E_{T0} &= \frac{2n_1}{n_1 + n_2} E_{I0}
\end{align*}
\]

• Phases
— $\vec{E}_{T}$ and $\vec{E}_{I}$ are always in phase.
— if $n_1 > n_2$ (glass to air), $\vec{E}_{R}$ and $\vec{E}_{I}$ are in phase.
— if $n_1 < n_2$ (air to glass), $\vec{E}_{R}$ and $\vec{E}_{I}$ are out of phase by 180° (note that $-1 = e^{i\pi}$).

B. Intensities

We have derived the formula for EM wave intensity

$$I = \frac{1}{2} v \varepsilon E_0^2$$

Therefore

$$I_R = \frac{1}{2} v_1 \varepsilon_1 E_{R0}^2 = \frac{1}{2} v_1 \varepsilon_1 \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 E_{f0}^2 = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 I_I$$

$$I_T = \frac{1}{2} v_2 \varepsilon_2 E_{R0}^2 = \frac{1}{2} v_2 \varepsilon_2 \left( \frac{2n_1}{n_1 + n_2} \right)^2 E_{f0}^2 = \frac{v_2 \varepsilon_2}{v_1 \varepsilon_1} \left( \frac{2n_1}{n_1 + n_2} \right)^2 I_I$$

since

$$\frac{v_2}{v_1} = \frac{n_1}{n_2}$$

$$\frac{n_1}{n_2} = \frac{v_2}{v_1} \sim \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} \quad \varepsilon_2 \varepsilon_1 = \left( \frac{n_2}{n_1} \right)^2$$

$$I_T = \frac{n_1}{n_2} \left( \frac{n_2}{n_1} \right)^2 \left( \frac{2n_1}{n_1 + n_2} \right)^2 I_I = \frac{4n_1 n_2}{(n_1 + n_2)^2} I_I$$

- Reflection coefficient

$$R = \frac{I_R}{I_I} = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

- Transmission coefficient

$$T = \frac{I_T}{I_I} = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

It is easy to show that

$$R + T = 1$$

satisfying the energy conservation law. This is true even if we do not assume $\mu_1 \sim \mu_2 \sim \mu_0$.

C. One example
Light entering from air \((n_1 \sim 1)\) to glass \((n_2 \sim 1.5)\)

\[
R = \left( \frac{1 - 1.5}{1 + 1.5} \right)^2 = 0.04
\]
\[
T = 1 - R = 0.96
\]

The same coefficient is valid for Light exiting from glass \((n_2 \sim 1.5)\) to air \((n_1 \sim 1)\).

The effective transmission coefficient for lights going through one optical component is

\[
T_1 = T^2
\]

After going through 5 components

\[
T_5 = T^{10} = (0.96)^{10} \sim 52\%
\]

Half of the intensity gets lost! Anti-reflection coating on the optical components can reduce the reflection coefficient.

### §3.3.2 Reflection and Transmission at Oblique Incidence

Select the \(z\)-axis normal to the boundary and the incident wave vector \(\mathbf{k}_I\) on the \(xz\) plane. We do not assume any particular directions of wave vectors for the reflected \(\mathbf{k}_R\) and transmitted \(\mathbf{k}_T\). The wave frequencies for all waves are the same and are determined by the source. The numbers are related through

\[
\omega = k_I v_1 = k_R v_1 = k_T v_2
\]

or

\[
k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T,
\]

Now at \(z = 0\)

\[
\begin{align*}
\vec{E}_I &= \vec{E}_{I0} \exp \left[ i \left( \mathbf{k}_I \cdot \mathbf{r} - \omega t \right) \right] = \vec{E}_{I0} \exp \left[ i \left( k_{Ix} x + k_{Iy} y + k_{Iz} z - \omega t \right) \right] \\
\vec{B}_I &= \frac{1}{v_1} k_{I0} \times \vec{E}_I = \frac{1}{v_1} k_{I0} \times \vec{E}_{I0} \exp \left[ i \left( k_{Ix} x - \omega t \right) \right]
\end{align*}
\]

The reflected waves

\[
\begin{align*}
\vec{E}_R &= \vec{E}_{R0} \exp \left[ i \left( k_{Rx} x + k_{Ry} y - \omega t \right) \right] \\
\vec{B}_R &= \frac{1}{v_1} k_{R0} \times \vec{E}_R = \frac{1}{v_1} k_{R0} \times \vec{E}_{R0} \exp \left[ i \left( k_{Rx} x + k_{Ry} y - \omega t \right) \right]
\end{align*}
\]
Note that the signs of $k_{Rx}$ and $k_{Ry}$ may be either positive or negative.

The transmitted waves

$$\vec{E}_T = \vec{E}_{T0} \exp [i (k_{Tx}x + k_{Ty}y - \omega t)]$$
$$\vec{B}_T = \frac{1}{v_2} \vec{k}_{T0} \times \vec{E}_T = \frac{1}{v_2} \vec{k}_{T0} \times \vec{E}_{T0} \exp [i (k_{Tx}x + k_{Ty}y - \omega t)]$$

Examine one of the boundary conditions

$$\vec{E}_1^k = \vec{E}_2^k$$

or

$$\vec{E}_{I0}^k \exp [i (k_{Ix}x - \omega t)] + \vec{E}_{R0}^k \exp [i (k_{Rx}x + k_{Ry}y - \omega t)] = \vec{E}_{T0}^k \exp [i (k_{Tx}x + k_{Ty}y - \omega t)]$$

or

$$\vec{E}_{I0}^k \exp [i (k_{Ix}x - \omega t)] = \vec{E}_{T0}^k \exp [i (k_{Tx}x + k_{Ty}y - \omega t)] - \vec{E}_{R0}^k \exp [i (k_{Rx}x + k_{Ry}y - \omega t)]$$

Since the exponent on the left hand side is independent of $y$, we require

$$k_{Ty} = k_{Ry} = 0$$

or all wave vectors, $\vec{k}_I$, $\vec{k}_R$ and $\vec{k}_T$ are on the $xz$ plane.

**First Law:** The incident, reflected and transmitted wave vectors form a plane (plane of incidence), which includes the normal to the surface.

Furthermore, we also require

$$k_{Rx} = k_{Tx} = k_{Ix}$$

or

$$k_1 \sin \vartheta_R = k_2 \sin \vartheta_T = k_1 \sin \vartheta_I$$

or

$$\sin \vartheta_R = \sin \vartheta_I, \quad \vartheta_R = \vartheta_I$$

**Second Law:** The angle of incidence is equal to the angle of reflection.

We also obtain

$$\frac{\sin \vartheta_T}{\sin \vartheta_I} = \frac{k_1}{k_2} = \frac{\omega/v_1}{\omega/v_2} = \frac{c/v_1}{c/v_2} = \frac{n_1}{n_2}$$

**Third Law (Snell’s law):**

$$\frac{\sin \vartheta_T}{\sin \vartheta_I} = \frac{n_1}{n_2}$$