Impact of electron thermal effects on Farley-Buneman waves at arbitrary aspect angles

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[1] Using a standard 8-moment expansion of the fluid equations, we have extended fluid calculations of the effect of electron temperature feedback on the linear Farley-Buneman instability theory to include the contribution of field-aligned heat flow and field-aligned wave electric fields through nonzero-aspect angles. Our treatment includes arbitrary ion magnetization and neutral winds but is limited, for simplicity, to zero flow angles. Our results indicate that, for 3-m waves, electron thermal corrections are significant (threshold phase velocities greater than predicted by isothermal theory by 20% or more) at aspect angles less than 0.35° for altitudes less than 103 km. Similar but smaller numbers are obtained for 10-m waves. The impact of the thermal corrections normally also decreases rapidly with increasing altitude. These results provide a straightforward explanation for equatorial observations of 3-m phase velocities that are at least 30% greater than the isothermal ion-acoustic speed below 105 km in the equatorial regions. The altitude dependence of the observed phase speeds also mimics very well the behavior seen in our calculations. The implication is that the aspect angles of the structures must be smaller than 0.3°. This is consistent with recent findings from radar interferometry.

1. Introduction

[2] Owing to its key importance in research on ionospheric irregularities, the theory of Farley-Buneman (or modified two-stream) instability has continued to evolve and has gained in complexity over the years. This has been true of both linear and nonlinear approaches. Here we concentrate on the linear treatment. The increased degree of sophistication in the linear fluid theories was at first limited to isothermal treatments, culminating in the generalization presented by Fejer et al. [1984]. Gradually, thermal processes were also added to the calculations both in kinetic and, mostly, in fluid treatments of the problem. The addition of thermal effects led to the discovery of additional feedback mechanisms which had either a positive or a negative effect on threshold properties, depending on conditions [e.g., Eruckhimov et al., 1982, 1983; Gurevich and Karashtin, 1984; Kissack et al., 1995, 1997; Dimant and Sudan, 1995a, 1995b, 1995c, 1997; Shalimov and Haldoupis, 1995; Robinson, 1998; St.-Maurice and Kissack, 2000; Kagan and Kelley, 2000].

[3] In the more recent past, two distinct approaches have emerged in order to tackle the nonisothermal electron corrections in Farley-Buneman linear instability calculations. In the series of papers written by Dimant and Sudan, the focus has been on an improved kinetic starting point for the instability calculations. Thus, in order to take better care of the electron collisions, the simple BGK collision operator that had been used in previous kinetic studies was replaced by a more sophisticated collision operator. This operator, which is Fokker-Plank-like, has the virtue to correctly describe the fast isotropization of the electrons in velocity space while changing their energy over much longer timescales. The new theoretical approach developed by Dimant and Sudan had the virtue that it made specific and explicit predictions for the phase velocity and the threshold conditions so that its results could readily be checked either against other theories or against observations. We will do this comparison of theories and observations below.

[4] In the other camp, to which the present paper belongs, there is a series of papers based on generalized fluid equations and which has culminated in complexity with the work of Shalimov and Haldoupis [1995] and the subsequent work of Kissack et al. [1995, 1997]. This tried-and-true set of transport equations is based on Grad’s set of fluid equations and is closed, in the more comprehensive theories, at the heat flow level (see, for example,
Shunk and Nagy [2000] for more on the validity of this scheme. The theory that we specifically use also describes the effects of collisions self-consistently, using Burgers's [1969] expressions for collision integrals. Ad hoc, but well-documented, terms must also be added in order to describe the very important inelastic electron cooling process. While, in practice, we only use here a calibrated proportionality constant with the elastic collision frequency to describe this otherwise complex process, more comprehensive expressions listed in Shunk and Nagy [2000] and updated in a recent series of papers by Pavlov and coworkers [Pavlov, 1998a, 1998b; Pavlov and Berrington, 1999] could be used if deemed necessary. The important point here is that the handling of systematic sets of fluid equations has the advantage that processes like inelastic collisions can be dealt-with self-consistently with expressions that describe integrals of the collision operator as accurately as feasible.

There is no question that the Dimant and Sudan theory and the generalized fluid theory make different predictions regarding phase velocities and threshold conditions (our work pertains to the so-called “short wavelength limit” $\left(\omega - kv_o \gg \delta_{w}w_{o}\right)$ studied by Dimant and Sudan [1995a]). The differences between theoretical predictions should not be surprising, given their very different starting points. At least, the results are often qualitatively similar (for example, similar deviations from the isothermal expressions for the phase velocity at zero-aspect angles). However, there are also some differences of substance. For example, Figure 1 of Dimant and Sudan [1995a], the authors’ preferred electron collision-frequency model produces what we describe below as “super-adiabatic” threshold speeds, just like the generalized fluid theory does. However, with our systematic fluid results, the super-adiabatic speeds gradually disappear as the altitude climbs above approximately 105 km. This behavior in turn is similar to equatorial type I observations [St.-Maurice et al., 2003]. This indicates to us that our fluid theory is “on the right track,” while the new theory based on a Fokker-Plank collision operator is running into difficulties. This comparison of our systematic fluid theoretical approach with the observations encourage us to pursue our fluid line of calculations further in the present paper in order to answer some key questions regarding the aspect angle corrections to the St.-Maurice and Kissack [2000] calculations in view of the observations described in the St.-Maurice et al. [2003] and Swartz [1997] papers.

There can be no question that, with the exception of the Farley and Providakes [1989] “fast track” approach, an analysis of the physical processes has often been difficult to track for the nonisothermal problem. This is because, not too surprisingly, the equations become bulky with intricate formulae that make it difficult to understand and extract the physics behind the math. For example, Dimant and Sudan [1997] had to write a separate publication devoted almost entirely to a physical explanation of the theory they had developed [Dimant and Sudan, 1995a, 1995b]. Likewise, St.-Maurice and Kissack [2000] considered thermal corrections to Farley-Buneman waves at zero flow angles using a subset of generalized treatments presented by Kissack et al. [1995, 1997]. To make the analysis as easy and clear as possible, St.-Maurice and Kissack [2000] also had to limit themselves not just to zero flow angle conditions but also to the lower E region (nonmagnetized ions) and zero-aspect angles. This, nevertheless, turned out to be more useful than expected, when St.-Maurice et al. [2003] were able to use these results to explain the otherwise puzzling behavior of the phase velocity of what they labeled as “two-step type-I waves” in the lower equatorial electrojet [see, e.g., Pfaff et al., 1988]. The waves were moving at speeds up to 50% greater than the isothermal ion-acoustic speed along the flow, and at the altitudes over which electron energy corrections were deemed to be important. There was even a solid quantitative agreement between the threshold speed predictions made by the zero-aspect angle predictions of St.-Maurice and Kissack [2000] and the upper limit observed in the magnitude of the two-step type-I phase velocities.

The success of the zero-aspect angle theory in explaining the phase velocities observed by St.-Maurice et al. [2003] immediately raises an important question, namely: for what range of angles is the zero-aspect angle theory applicable, and how large should the corrections be for small, but nonzero, aspect angles? Should the corrections matter at, say, 0.2°, or at 1°? This question becomes relevant in view of observations by Kudeki and Farley [1989], which indicated that the aspect angles of two-step type-I waves were less than 0.25°, even though Kudeki and Farley [1989] could not tell how much less than 0.25° they were. The work of St.-Maurice et al. [2003] raises the interesting prospect that the phase velocity observations might be useful for tracking down the aspect angle at saturation, which in turn has implications for the wave saturation mechanism. Clearly, the first step in answering this question requires that we reintroduce a finite aspect angle effect in the theory, still under the zero flow angles conditions of the observations so as to minimize the complexity of the system and maximize its understanding. To further maximize the understanding, the final expressions should be formulated in such a way that the traditional expressions should immediately be recoverable from the generalized treatment, just as presented by Dimant and Sudan [1995a] and St.-Maurice and Kissack [2000]. In this way, additional effects due to frequency broadening and/or frequency shifts produced by turbulence could be, to first order, integrated reasonably easily, following for example a heuristic prescription proposed by St.-Maurice et al. [2003].

In the present work, in addition to providing a description based on finite aspect angles, we have generalized the zero flow angle expressions obtained by St.-Maurice and Kissack [2000] to include an arbitrary ion magnetization and a neutral wind. The later addition has been motivated by a recent compilation of various observations by Larsen [2002], who has demonstrated that the neutral winds in the lower E region are much larger, and therefore far more important, than earlier suspected.

With the above limited generalization in mind, we now present our mathematical model and our derivation of the linear dispersion relation for nonisothermal electrons and isothermal ions in section 2. In section 3 we derive the solution to this dispersion relation for the complex frequency. In section 4 we discuss the threshold conditions of the instability under conditions appropriate to the equatorial ionosphere. We end with a brief summary of our results in section 5, where, for the reader’s convenience, we also
2. Mathematical Model and Derivation of the Generalized Dispersion Relation

[10] For our mathematical treatment we basically follow the lines of the work done by St.-Maurice and Kissack [2000]. We use a generalized two-fluid treatment for nonisothermal electrons and isothermal ions. We assume that there is only one type of ions and that they possess a single elementary charge. We also assume quasi-neutrality conditions. The electron equations include the equations for continuity, momentum, energy and heat flow balance ((1)–(4)). The ions are described by standard continuity and momentum equations ((13)–(15)). As stated in section 1, one of our objectives is also to derive the dispersion relation in a form in which the thermal corrections can be easily seen as a generalization of the classical expression found elsewhere in the earlier literature [e.g., Farley, 1963; Buneman, 1963; Sudan et al., 1973; Sudan, 1983].

[11] Prior to embarking on our derivation, it seems important to stress that in what follows we neglect zeroth-order density and pressure gradients as well as zeroth order heat flows. However, perturbed density gradients, perturbed pressure gradients and perturbed heat flows are all kept and found to play a very important role in the results. The retention of perturbed heat flow terms in particular is a fairly new feature that was introduced in recent years. It has been systematically explored by Kissack et al. [1995, 1997]. Suffice it to state that the perturbed heat flow terms tend, on the one hand, to diminish the influence of adiabatic heating by smoothing out temperature fluctuations driven by adiabatic effects. The smoothing is done through ordinary heat conduction inside the fluctuations. On the other hand, heat flows can also enhance adiabatic tendencies through thermal diffusion, which acts as a source of momentum for the electron gas. See St.-Maurice and Kissack [2000] for a thorough discussion of the origin of this last physical process and for a description of the competition between the various mechanisms.

2.1. Perturbed Electron Equations

[12] In our derivation of the dispersion relation we proceed from the linearized system of the following equations [e.g., St.-Maurice and Kissack, 2000]:

Electron continuity equation

$$\frac{\partial}{\partial t} \cdot (\mathbf{u}_e + \nabla) n_1 + \nabla \cdot \mathbf{u}_e = 0$$

Electron momentum equation

$$\nu_e \mathbf{u}_e + \omega_e \mathbf{u}_e \times \mathbf{b} = -V_f^2 \nabla (n_1 + \tau_e \varphi_e) - g \nu_e (\mathbf{u}_e - \mathbf{u}_n) \tau_e$$

Electron energy balance

$$\frac{\partial}{\partial t} \cdot (\mathbf{u}_e + \nabla) \left[ \frac{3}{2} \tau_e - n_1 \right] + \nabla \cdot \mathbf{q}_e = g \frac{m \nu_e}{T_e} (\mathbf{u}_e - \mathbf{u}_n) \tau_e$$

Electron heat flow equation

$$\alpha \nu_e \mathbf{q}_e + \omega_e \mathbf{q}_e \times \mathbf{b} = -g \nu_e \mathbf{u}_e - \frac{5}{2} \frac{V_f^2}{T_e^2} \nabla \tau_e - g \nu_e \mathbf{u}_e \tau_e$$

In the above we have used

$$\alpha = 1 + 2g(1 + g)/5$$

and

$$g = \frac{T_e \partial \nu_e}{\nu_e \partial T_e} \tau_e$$

In addition, zeroth-order and first-order perturbations have been labeled with the use of subscripts 0 and 1 respectively; \( \mathbf{u}_e, \mathbf{u}_i \) and \( \mathbf{u}_n \) are electron, ion and neutral gas velocities; \( n_1 = N_1/N_0 \) is the normalized dimensionless plasma density fluctuation; \( T_{ce} = T_e/T_0 \) is the normalized dimensionless electron temperature fluctuation; the electric field is assumed to be electrostatic so that \( E_1 = -\nabla \Phi_1 \), where \( \Phi_1 \) is the perturbed electric potential and \( \varphi_1 = e \Phi_1/T_0 \) is its dimensionless version, with \( e \) as the elementary charge of an electron; \( \mathbf{q}_e \) is the electron heat flow; \( V_{fe} \) is the electron thermal velocity; \( \nu_e \) and \( \omega_e \) are the electron collision and cyclotron frequencies, respectively; \( \beta_e \) is the ratio of the rate at which energy is lost through (inelastic) collisions with neutrals over the rate at which momentum is lost through elastic collisions with neutrals; finally, \( \mathbf{b} \) is a unit vector along the geomagnetic field \( \mathbf{B}_0 \). We use energy units for temperature, so that \( T \equiv k_T \) where \( k_T \) is the Boltzmann constant.

[13] Since we neglect zeroth order pressure gradients and heat flows in the momentum equation, and since the electrons are strongly magnetized, the steady state zeroth-order electron velocity is simply given by

$$\mathbf{u}_e \equiv \mathbf{E}_0 \times \mathbf{B}_0 / B_0^2.$$
Likewise, the components of the electron perturbed heat flow derived from (2) and (4) are

\[ q_{el1} = \frac{V_{Te}^2}{e \nu_e (1 + 2g/5)} \nabla_s [gn_1 + (g - 5/2) \tau_{el} - g \varphi_1] \]  
(9x)

\[ q_{el1} = -\frac{\nu_e V_{Te}^2}{\omega_e} \nabla_s [gn_1 + (g + 5\alpha/2) \tau_{el} - g \varphi_1] + \frac{5V_{Te}^2}{2\omega_e} \nabla \tau_{el} + g \frac{\nu_e}{\omega_e} u_{e0} \tau_{el} \]  
(9y)

Using equations (8) and (9), and neglecting terms of order \( (v_e/\omega_e)^2 \ll 1 \) (\( s > 1 \)), the divergence of the electron current and the divergence of the electron heat flow respectively take the following form, to leading order:

\[ \nabla \cdot u_{el1} = -\frac{V_{Te}^2}{\nu_e (1 + 2g/5)} \nabla_s^2 [\alpha (n_1 - \varphi_1) + (\alpha - g) \tau_{el}] \]

\[ -\frac{\nu_e V_{Te}^2}{\omega_e} \nabla_s^2 [n_1 - \varphi_1 + (1 + g) \tau_{el}] \]  
(10)

\[ \nabla \cdot q_{el1} = \frac{V_{Te}^2}{\nu_e (1 + 2g/5)} \nabla_s^2 [g(n_1 - \varphi_1) + (g - 5/2) \tau_{el}] \]

\[ -\frac{\nu_e V_{Te}^2}{\omega_e} \nabla_s^2 [g(n_1 - \varphi_1) + (g + 5\alpha/2) \tau_{el}] \]  
(11)

We also need the zeroth order ion velocity, which is easily found from a standard analysis of the steady state zeroth order ion momentum equation and yields the result

\[ u_{i0} = u_n + \xi_i u_n \times b + \xi_i^* (b \cdot u_{i0}) b \]

\[ + \xi_i E_0 + \xi_i^* E_0 \times b + \xi_i^2 (E_0 \cdot b) \]

\[ = (1 + \xi_i^2) \]

\[ \frac{u_n + \xi_i u_n \times b + \xi_i^* (b \cdot u_{i0}) b}{(1 + \xi_i^2) B_0} \]  
(15)

where \( \xi_i = \omega_i/v_{i0} \). For the altitudes of interest we may neglect \( \xi_i^2 \) terms compared to 1. This gives \( u_{i0} \equiv u_n + \xi_i u_n \times b + \xi_i^* E_0/B_0 \) which for altitudes near 100 km and below reduces further to \( u_{i0} \equiv u_n \).

[18] From (14) one may find the first-order ion velocity, namely,

\[ u_{i1} = -\frac{T_{ei}}{MV_{im}} (\nabla_\parallel + \nabla_\perp - \xi_i (b \times \nabla_\parallel))(n_1 + \varphi_1). \]  
(16)

where

\[ \xi_i = \omega_i/v_{i0} \]

\[ v_{i0} = v_{i0} + \omega_i - \omega - \mathbf{k} \cdot u_{i0}. \]  
(17)

In (16) and below we again have neglected all terms of order \( \xi_i^2 \) compared to 1. The last term in (17) comes from \( v_{i0}^* = v_{i0} + \partial i/\partial t + u_{i0} \cdot \nabla \) and anticipates the Fourier transform that we are about to use.

[18] Below, we will not use the expression for the polarization electric field directly as such. Rather, we use the combination \( (1 - \frac{\omega_1}{\omega}) \). However, since the polarization electric field plays an important role in the instability generation [see, e.g., Kagan, 2002] we present an expression for \( \varphi_1 \) in the Appendix B.

[20] The perturbed variable \((1 - \frac{\omega_1}{\omega})\) can be found from the momentum and continuity equations of either the electrons or the ions. Using the ion equations (13) and (14) for now, we get the intermediate result

\[ 1 - \frac{\varphi_1}{\omega} = \frac{M}{T_{e0} k^2 c^2_s} \left( c^2_{s1} k^2 s^2 - (\omega - k \cdot u_{i0})^2 + iv_{i0} (\omega - k \cdot u_{i0}) \right), \]  
(18)

where \( c^2_{s1} = (T_{e0} + T_{i0})/M \) is the square of the isothermal (rather than actual) ion-acoustic speed (we specifically add a
subscript “i” to the classical ion-acoustic speed to remind the reader that this speed is only correct for the isothermal case. In (18) we have introduced

\[ k^*^2 = k_i^2 + k_\perp^2 / \left( \omega_i^2 / \nu_{in}^2 + 1 \right). \]  \hfill (19)

### 2.3. Generalized Form of the Dispersion Relation

[21] We can now substitute (18) in the preliminary form of the dispersion relation, equation 12. This gives us the dispersion relation in a form where the thermal corrections to the classical case are grouped in the last term through their proportionality to \( \frac{2\pi}{n_i} \), namely,

\[ \omega - k \cdot u_{in} + \frac{\psi_T}{\nu_{in}} \left\{ -e^2_c k^*^2 + (\omega - k \cdot u_{in})^2 - i \nu_{in}(\omega - k \cdot u_{in}) \right\} \]

\[ - i(1 + g)A \left( \frac{k_i^2}{k_\perp^2} \right) D_{ci} k_i^2 \frac{\tau_{el}}{n_i} = 0, \] \hfill (20)

where \( D_{ci} = T_{ci}/\nu_{ci} m_{ci}^2 \) and

\[ \psi_T = \psi_0 k_i^2 [1 + \alpha \beta] / k^*^2, \quad \psi_0 = \frac{\omega_c \nu_{in}}{\omega_i \omega_{ci}} \] \hfill (21)

\[ A \left( \frac{k_i^2}{k_\perp^2} \right) = \left[ 1 + \frac{(\alpha - g)}{(1 + g)} \beta \right] / [1 + \alpha \beta], \] \hfill (22a)

and where

\[ \beta = \frac{1}{(1 + 2g/5) k_i^2 \omega_{ci}^2} \] \hfill (22b)

is a function of the aspect angle \( k_i/k_\perp \) and altitude only and depends on thermal processes only through \( g \). It should be noted that in (20) \( \psi_T \) and \( A \left( \frac{k_i^2}{k_\perp^2} \right) \) are also functions of the aspect angle and altitude. In (21) the corrections involving \( g \) in the coefficients \( \alpha \) and \( \beta \) are due to thermal diffusion, namely, to heat flow effects showing up, through collisions, in the electron energy balance (\( \alpha \)) and the electron momentum balance (\( \beta \)). Furthermore the following points should be stressed:

[22] 1. \( k^*^2 = k_i^2 \) for \( \omega_i < \nu_{in} \) (easily satisfied for the altitudes of interest in the present work).

[23] 2. For \( g = 0 \) (no electron thermal diffusion), we recover the traditional expression \( \psi_T \equiv \psi \), where \( \psi = \psi_0[k_i^2 + k_\perp^2 \omega_{ci}^2/v_{ci}^2] / k^2 \).

[24] 3. \( A \left( \frac{k_i^2}{k_\perp^2} \right) = 1 \) for either \( g = 0 \), or \( k_i = 0 \) (the latter being the case treated by St.-Maurice and Kissack [2000]).

[25] 4. Our value of \( \alpha \beta \) is greater than the \((5/7)k_i^2 \omega_{ci}^2/k_i^2 \omega_i^2 \) value obtained by Pecseli et al. [1989], but less than what Dimant and Sudan [1995a] recovered with their semidiffusive model: for instance, with their choice of \( g = 1 \) we end up with \((9/7)(k_i^2 \omega_{ci}^2/k_i^2 \omega_i^2) \) while Dimant and Sudan get \((5/3)(k_i^2 \omega_{ci}^2/k_i^2 \omega_i^2) \). The two theories do agree exactly, however, in the absence of thermal diffusion, namely, for the \( g = 0 \) case (which is also the traditional isothermal result). In practice, furthermore, the disagreements for \( g \) of order \( 1 \) have little impact on observations, since they amount to changes less than 10\% in aspect angle magnitudes.

[26] It is clear from (20) that we still need \( \tau_{el}/n_i \), which we can obtain by using (11) and our plane wave decomposition, followed by a substitution into the perturbed electron energy balance (3). At this point, we assume the wave vector to be along the flow direction, thereby dropping the second term on the right-hand-side of (3). This assumption means that we are, for the purpose at hand, removing from our analysis interesting frictional heating feedback terms discussed by Dimant and Sudan [1997] at nonzero flow angles. The resulting temperature fluctuation balance then takes the form

\[ \left\{ \frac{3}{2} (\omega - k \cdot u_{in}) + \eta_i + \frac{(g - 5/2)}{1 + 2g/5} D_{ci} k_i^2 \right\} \]

\[ + \left[ 5/2 + 2g + g^2 \right] D_{ci} k_i^2 \frac{\tau_{el}}{n_i} + \left\{ - \frac{1}{(1 + 2g/5)} \frac{\omega_{ci} k_i^2}{k_i} + 1 \right\} \]

\[ \cdot gD_{ci} k_i^2 \left( \frac{1 - \frac{\eta_i}{\nu_{in}}}{} - i(\omega - k \cdot u_{in}) = 0 \right. \] \hfill (23)

Here we have introduced \( \eta_i = \psi_0 \delta_i - \eta \) and \( g(\nu_{in} - \nu_{in})^2 / \omega_i^2 \).

Taking the zeroth-order electron energy balance into account, this expression becomes

\[ \eta_i = \delta_i \nu_{ci} [1 - g + gT_{ci}/T_{ci}] \] \hfill (24)

For the sake of simplicity and in order to align ourselves with the presentation of St.-Maurice and Kissack [2000], we furthermore substitute our expressions for the electron continuity and momentum equations into the perturbed electron energy balance. For this purpose, in (23) we substitute an expression for \((1 - \frac{\eta_i}{\nu_{in}}) \) that can be derived from the electron continuity and momentum equations instead of the ion continuity and momentum equations. In this way we obtain:

\[ \left( 1 - \frac{\eta_i}{\nu_{in}} \right) D_{ci} k_i^2 \frac{\tau_{el}}{n_i} = - \left\{ i(\omega - k \cdot u_{in}) \right\} \]

\[ - \left[ 1 + g + \frac{(\alpha - g)}{(1 + 2g/5)} \frac{\omega_{ci}^2 k_i^2}{k_i^2 \omega_i^2} \right] D_{ci} k_i^2 \frac{\tau_{el}}{n_i} \]

\[ \times \left\{ 1 + \frac{\alpha}{(1 + 2g/5)} \frac{\omega_{ci}^2}{v_{ci}^2} \right\} \] \hfill (25)

This allows us to express \( \tau_{el}/n_i \) in a form that makes it relatively easy to compare with the case \( k_i = 0 \) and \( u_{in} = 0 \), considered by St.-Maurice and Kissack [2000]. That is to say, we can write

\[ \frac{\tau_{el}}{n_i} = (1 + g)B \left( \frac{k_i^2}{k_\perp^2} \right)(\omega - k \cdot u_{in}) \]

\[ \cdot \left\{ -i \left[ \eta_i + C \left( \frac{k_i^2}{k_\perp^2} \right) D_{ci} k_i^2 \right] + \frac{3}{2} (\omega - k \cdot u_{in}) \right\} \]

\[ \times \left\{ \eta_i + C \left( \frac{k_i^2}{k_\perp^2} \right) D_{ci} k_i^2 \right\}^{-1} \] \hfill (26)
where, in terms of the parameter $\beta$ defined in (22a), we have

$$B\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) = \frac{1 + (\alpha - g)\beta}{1 + \alpha \beta}$$  

(27)

$$C\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) = 5/2 + 2g + g^2 + (g - 5/2)\beta - g(1 + g)A\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right)$$  

(28)

Note that for $k_{||} = 0$, $B\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) = 1$ and $C\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) = 5/2 + g$.

In that case we immediately recover from (26) result (44) of St.-Maurice and Kissack [2000].

3. Solution to the Dispersion Relation: Growth Rate and Frequency

[27] It is easy to see from (20) that thermal corrections to the growth rate and frequency of the Farley-Buneman instability respectively depend on $\text{Re}\left(\frac{\tau_{el}}{n_{1}}\right)$ and $\text{Im}\left(\frac{\tau_{el}}{n_{1}}\right)$. More precisely, under the usual assumption that the growth rate is much smaller than the oscillation frequency, it is easy to show that the growth rate $\gamma$ and oscillation frequency $\omega_{r}$ ($\omega = \omega_{r} - i\gamma$) obtained from (20) are given by

$$\gamma = \frac{\psi_{r}}{1 + \psi_{f}}\left\{(\omega - k \cdot u_{0})^2 - c_{e}^2 k^{*2}\right\}$$

$$\cdot \left[1 + \frac{T_{e0}}{T_{e0} + T_{i0}}(1 + g)A\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) \text{Re}\left(\frac{\tau_{el}}{n_{1}}\right)\right]$$  

(29)

$$\omega_{r} = \frac{1}{1 + \psi_{f}}\left\{k \cdot u_{0} + \psi_{f} k \cdot u_{0} + \psi_{r} T_{e0}\right\}$$

$$\times \frac{k^{*2}}{v_{m}A}\left(1 + g\right)A\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) \text{Im}\left(\frac{\tau_{el}}{n_{1}}\right).$$  

(30)

We might note, in passing, that the correction term

$$\left(1 + g\right)A\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) \text{Re}\left(\frac{\tau_{el}}{n_{1}}\right)$$

in (29) is playing the role of $P_{e,\parallel}$ and/or the series expansion $R$ of Dimant and Sudan [1995a, 1995b].

3.1. Final Growth Rate Expression

[28] The trouble with (29) and (30) is that $\text{Re}\left(\frac{\tau_{el}}{n_{1}}\right)$ in (29) and $\text{Im}\left(\frac{\tau_{el}}{n_{1}}\right)$ in (30) are themselves functions of the growth rate and frequency. However, with the assumption of small growth rates, the complication remains tractable. Thus, starting with the growth rate expression, from equation (26) we find

$$\text{Re}\left(\frac{\tau_{el}}{n_{1}}\right) = \left(1 + g\right)B\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right)$$

$$\cdot \left\{\frac{3}{2}(\omega - k \cdot u_{0})^2 - \gamma \left[C\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) D_{cl} k_{\perp}^2 + \bar{n}\right]\right\}$$

$$\times \left\{\frac{9}{4}(\omega - k \cdot u_{0})^2 + \left[C\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) D_{cl} k_{\perp}^2 + \bar{n}\right]^2\right\}^{-1}.$$  

(31)

Substituting (31) into (29) we finally arrive at

$$\gamma = \frac{\psi_{r}}{1 + \psi_{f}}\left(1 + \psi_{r}\right)\nu_{m}\left(\omega - k \cdot u_{0}\right)^2 - c_{e}^2 k^{*2}$$

$$\cdot \left[1 + \frac{T_{e0}}{T_{e0} + T_{i0}}(1 + g)A\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) \text{Re}\left(\frac{\tau_{el}}{n_{1}}\right)\right],$$  

(32)

where

$$F\left(\frac{\tau_{el}}{n_{1}}\right) = (1 + g)B\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right)\frac{3}{2}(\omega - k \cdot u_{0})^2 + \frac{9}{4}(\omega - k \cdot u_{0})^2$$

$$\times \left[C\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) D_{cl} k_{\perp}^2 + \bar{n}\right]^2 \right\}^{-1}.$$  

(33)

and

$$0 = \frac{\psi_{r} c_{e}^{2} k^{*2}}{1 + \psi_{f}}\frac{T_{e0}}{\nu_{m} (T_{e0} + T_{i0})}$$

$$\cdot \left(1 + g\right)A\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right)B\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) C\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) D_{cl} k_{\perp}^2 + \bar{n}\right]\right\}^{-1}.$$  

(34)

3.2. Final Expression for the Oscillation Frequency

[29] The expression for the frequency of oscillations likewise depends on $\text{Im}\left(\frac{\tau_{el}}{n_{1}}\right)$, which again from (26), can be seen, for small growth rates, to be given by

$$\text{Im}\left(\frac{\tau_{el}}{n_{1}}\right) = -\left(1 + g\right)B\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) (\omega - k \cdot u_{0})$$

$$\times \left[C\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) D_{cl} k_{\perp}^2 + \bar{n}\right]\right\}^{-1}.$$  

(35)

[30] As a result, our final expression for the frequency becomes

$$\omega = \frac{1}{1 + \psi_{f}}\left\{k \cdot u_{0} + \psi_{f} k \cdot u_{0} - \psi_{r} T_{e0}\right\}$$

$$\times \left(1 + g\right)A\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) B\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) (\omega - k \cdot u_{0})$$

$$\times \left[C\left(\frac{k_{\perp}^{2}}{k_{||}^{2}}\right) D_{cl} k_{\perp}^2 + \bar{n}\right]\right\}^{-1}.$$  

(36)

[31] This expression can be solved for the frequency and then inserted into (32) to find the growth rate, just as is done in the standard linear treatment of the instability theory. This means that we are finally in a position to study the effect of electron thermal corrections on the waves using our systematic perturbation of the transport equations.

4. Threshold Conditions for the Farley-Buneman Instability in the Presence of Electron Thermal Corrections

[32] The principal goal of this work is to assess the impact of nonzero-aspect angles on the Farley-Buneman waves in
the presence of nonisothermal electrons using our systematic set of fluid equations. In a standard treatment of the problem we would at this point study the growth rate and frequency as functions of altitude, aspect angle and electric field strength. However, we will not do this here because it has become rather clear in research on ionospheric irregularities that the waves saturate at their threshold speed [e.g., Sahr and Fejer, 1996]. For this reason we focus our study here not just on the drift threshold conditions but also on the zero growth rate conditions found both at threshold as well as at stronger electric field cases.

4.1. Relation Between Eigenfrequency and the Particle Drifts

Threshold conditions are defined by the zero-growth rate, which is easily derived by setting the growth rate to zero in (32). However, we also need to relate the resulting expression for the frequency to an electric field. For this reason we start with an analysis of (36), which can be rewritten in the form

$$\omega_0 - k \cdot u_{\text{th}} = \frac{1}{(1 + \psi_F)} \left[ \psi_F k \cdot (u_0 - u_{\text{th}}) \right] + \frac{\psi_F T_0}{v_{\text{th}} M} k^2 (1 + g)^2$$

$$\times A \left[ \frac{k^2}{k^2_f} \right] B \left[ \frac{k^2}{k^2_f} \right] (\omega_0 - k \cdot u_{\text{th}})$$

$$\times \left[ \eta + C \left( \frac{k^2}{k^2_f} \right) D_{\chi,1} k^2 \right] \times \left[ \eta + C \left( \frac{k^2}{k^2_f} \right) D_{\chi,2} k^2 \right]$$

$$+ \frac{9}{4} \left( \omega_0 - k \cdot u_{\text{th}} \right)^2 \right]^{1/2} \right). \tag{37}$$

Equation (37) has to be solved as a function of both \(u_0 = (u_0 - u_{\text{th}})\) and the aspect angle. If it were not for the second term on the right-hand-side of (37) we would recover the well-known isothermal limit and be able to immediately relate the frequency to the electron and ion drift as well as to the aspect angle. However, we now have an additional term introducing a correction to the well-known result. This correction is indeed caused by the fact that the electrons are no longer treated as isothermal in our work. In order to assess the importance of this correction we first introduce the dimensionless frequency

$$\varpi = - \frac{(\omega_0 - k \cdot u_{\text{th}}) (1 + \psi_F)}{k \cdot u_0}. \tag{38}$$

This dimensionless frequency is such that \(\varpi = 1\) when the eigenfrequency is described by its classical expression (the modifications to the isothermal value of \(\psi_F\) notwithstanding). With this definition, (37) can be written as

$$\varpi = 1 + \frac{2}{3} \frac{\psi_F T_0}{v_{\text{th}} M k \cdot (u_0 - u_{\text{th}})} k^2 (1 + g)^2 A \left( \frac{k^2}{k^2_f} \right) B \left( \frac{k^2}{k^2_f} \right)$$

$$\sqrt{\frac{2}{3} X + \frac{1}{2} X^2}, \tag{39}$$

where

$$X = \left( \eta + C \left( \frac{k^2}{k^2_f} \right) D_{\chi,1} k^2 \right) \left( \omega_0 - k \cdot u_{\text{th}} \right). \tag{40}$$

Finally, we note that for larger wavelengths, like 10 m, the electron collision frequency is greater, since \(\psi_F v_{\text{th}} = \nu_{\text{th}}(1 + \alpha \beta) \omega_0 \omega_\text{th}\). By contrast, for larger relative drifts between ions and electrons, the correction becomes smaller. In Figure 1 we show the results (equation (41)) for 3-m waves in the form of a contour plot as function of altitude and magnitude of the relative drift between ions and electrons based on \(g = 5/6\). This value comes out from the semi-empirical expressions from Shunk and Nagy [2000] and is somewhat temperature-dependent. For the range of temperatures of interest here, it has been used as a reasonable approximation by a number of authors, like Shalimov and Haldoupis [1995] and Shunk and Nagy [2000], and references therein. More details on \(g = 5/6\) are also given by Kissack et al. [1997].

The correction is more noticeable at lower altitudes where the electron collision frequency is greater, since \(\psi_F v_{\text{th}} = \nu_{\text{th}}(1 + \alpha \beta) \omega_0 \omega_\text{th}\). By contrast, for larger relative drifts between ions and electrons, the correction becomes smaller. In Figure 1 we show the results (equation (41)) for 3-m waves in the form of a contour plot as function of altitude and magnitude of the relative drift between ions and electrons based on \(g = 5/6\). This value comes out from the semi-empirical expressions from Shunk and Nagy [2000] and is somewhat temperature-dependent. For the range of temperatures of interest here, it has been used as a reasonable approximation by a number of authors, like Shalimov and Haldoupis [1995] and Shunk and Nagy [2000], and references therein. More details on \(g = 5/6\) are also given by Kissack et al. [1997].

The percentage difference for the maximum correction to the dimensionless oscillation frequency in Figure 1 is always less than 5% and in practice will not exceed 3% even at zero-aspect angles. Finally, we note that for larger wavelengths, like 10 m, the results are even smaller, since they scale with \(k\). We conclude from this that thermal corrections do not introduce significant differences in the oscillation frequency. Namely, just as Pecseli et al. [1989], Shalimov and Haldoupis...
[1995], and Dimant and Sudan [1995a] found before us, it is quite appropriate to use
\[ \omega_r = (k \cdot u_e - \psi_r k \cdot u_i)/(1 + \psi_T) \]  
(42)
for any aspect angle, any altitude and any electric field strength. Nevertheless notice that, in agreement with the nonisothermal work of Pecseli et al. [1989] and the work by Dimant and Sudan [1995a], \( \psi_T \) differs from parameter \( \psi \) used in the isothermal case (see equation (21)). As described in section 2.3, however, our numerical correction to \( \psi \) is not quite the same as that which was obtained by these other authors but it is at least the same as that of Dimant and Sudan’s in the \( g = 0 \) case.

4.2. Threshold Conditions in Terms of the E × B Drift

[36] Having obtained a relation between the frequency and the electron and ion drifts, we can now seek the values that such drifts must have at threshold. To this end we simply substitute (33) into (32) for the zero growth rate using expression (42) for the eigenfrequency. For growth rate calculations the frequency must also be expressed in the ion frame of reference. Using \( \omega_r - k \cdot u_{0b} = -k \cdot u_0(1 + \psi_T) \), we introduce next the dimensionless variable
\[ Y = \left( \frac{V_{ph}}{c_{i,1}} \right)^2, \]
(43)
where \( V_{ph} \) is the phase velocity in the ion reference frame. Under the zero growth rate requirement we arrive at the following quadratic equation for \( Y \),
\[ Y^2 + a_1 Y - a_0 = 0, \]
(44)
giving us
\[ Y = \left( -a_1 + \sqrt{a_1^2 + 4a_0} \right)/2, \]
(45)
where we have retained only the positive root since, from (43), we must have \( Y \geq 0 \). The coefficients in (44) and (45) are
\[ a_0 = 4 \left( \frac{k_i^2}{k_T^2} \right) D_{c,i} k_i^2 + \frac{\Pi}{2} \psi_T^2 c_{i,1}^2 k_i^2, \]
(46)
\[ a_1 = a_0 - 1 - \frac{2}{3} \frac{T_{i,0}}{T_{e,0} + T_{i,0}} (1 + g)^2 A \frac{k_i^2}{k_T^2} B \frac{k_i^2}{k_T^2}. \]
(47)
Equation (45) gives us the threshold magnitudes for the dimensionless phase velocity \( V_{ph}/c_{i,1} \) in the ion frame of reference.

[37] We display \( V_{ph}/c_{i,1} \) as a function of altitude for various aspect angles (in degrees) in Figure 2a and Figure 3a for 3-m and 10-m Farley-Buneman waves respectively. Our calculations were made for a coordinate system moving with the neutrals \( u_{0b} = 0 \). In Figures 2b and 3b we show how \( V_{ph}/c_{i,1} \) depends on the aspect angle at different altitudes. We have included in Figures 2a and 3a a zero-aspect angle case where we also set \( g = 0 \) (gray thick solid line) to illustrate the importance of thermal diffusion in amplifying the results, as discussed earlier by St.-Maurice and Kissack [2000]. In order to facilitate comparison with the case studied by St.-Maurice et al. [2003] we also used a zero flow angle (recall that we have already assumed in the derivation that the wave vector \( k \) is parallel to the relative ion-electron drift velocity) and used \( g = 5/6 \) and \( \eta = 0.003 \).\( \psi_r \) (thus neglecting the temperature dependence of \( \psi_r \)). The atmospheric parameters we have used come from MSIS [see St.-Maurice et al., 2003, Table 1].

[38] Figures 2 and 3 reveal that when the electric field is just large enough to excite the instability (1) the threshold phase velocity comes closer to the isothermal ion acoustic speed with increasing altitude without ever quite reaching it (6–10% enhancements remain even at 110 km) and (2) the effect of thermal corrections is larger at 3 m than at 10 m. In particular, the calculations show that enhancements greater than 20% (which are easily detectable) can be expected below 102 km at 3 m as long as the aspect angle of the waves is less than approximately 0.35°. Indeed, the 0.1° and 0.2° aspect angle curves of Figure 2a are very reminiscent of the kind of behavior seen by St.-Maurice et al. [2003] in two-step type I waves at the equator.

[39] It is important not to confuse the minimum threshold speed of Farley-Buneman waves with the minimum electron drift conditions required to trigger the instability. At higher altitudes these speeds are comparable because \( \psi_T \) is small. Lower down the ratio of the drift divided by the threshold speed increases rapidly. In Figure 4a we have therefore produced contours as a function of altitude and aspect angle of the threshold electron drift \( u_{0b} \) required to excite 3-m Farley-Buneman waves at the threshold phase speed. The threshold drift is \( 1 + \psi_T \) greater than the phase speeds (equation (42)). In addition \( \psi_T \) rapidly becomes much less than 1 at zero-aspect angle when the altitude becomes greater than 100 km: as shown by the contour plot in Figure 4b, \( \psi_T \approx 1 \) near 97 km at zero-aspect angle, or near 99 km at 0.5° aspect angle, and near 110 km at 0.75° aspect angle. Therefore the phase speed likewise comes increasingly close to the threshold speed at higher altitudes at small aspect angles. However, as the aspect angle increases, and/or as the altitude decreases, there has to be an increase in the constant of proportionality between the threshold speed and the phase speed, through \( 1 + \psi_T \).

[40] To explicitly show the effect of thermal diffusion on the threshold speeds, we have also plotted in Figure 4c altitude profiles of the threshold electron drift for different aspect angles together with the profile for \( g = 0 \) at the zero-aspect angle. As stated above, the thermal corrections, which are amplified by thermal diffusion, are most important at lower altitudes: the difference, which is 500 m/s near 94 km, becomes negligible near 110 km.

4.3. Threshold Phase Velocities for Electric Fields Greater Than the Required Threshold

[41] The threshold speed can be obtained more directly by finding the zero of (32) without having to solve first for the eigenfrequency. This raises an intriguing issue in view of the fact that observations indicate that waves often move at the threshold speed even when the electric field is strong enough to excite waves that could be moving much faster than that. Several nonlinear theories have proposed that the frequency could match the threshold speed by having the
Figure 2. Threshold magnitudes of $V_{ph}/c_{s,i}$ (a) as a function of altitude for different aspect angles and (b) as a function of aspect angle for different altitudes for 3-m Farley-Buneman waves. Isothermal ion acoustic speed of the medium shown by black solid lines. Gray solid line in Figure 2a: $g = 0$ case at zero-aspect angle, to illustrate that without thermal diffusion the magnitudes of $V_{ph}^{th}$ would be much smaller. Calculations made for a coordinate system moving with the neutral wind ($u_{0} = 0$), at zero flow angle (wave vector $k$ along the current velocity), for $g = 5/6$ and $\Pi = 0.003$ $\nu_e$. The atmospheric parameters are from the MSIS model [St.-Maurice et al., 2003, Table 1].
Figure 3. Same as Figure 2, but for 10-m Farley-Buneman waves.
eigenfrequency changed nonlinearly through, for example, anomalous diffusion [Robinson, 1986; St.-Maurice, 1987] or mode-coupling [Hamza and St.-Maurice, 1995a, 1995b]. The question that then arises, particularly at high latitudes, is this: if the electric field exceeds the threshold value, should we be using the results discussed in section 4.2 or is this: if the electric field exceeds the threshold value, given that (32) has some explicit electron drift term into it, should we find the solution to zero growth independently, should we be using the results discussed in section 4.2 or is this: if the electric field exceeds the threshold value, given that (32) has some explicit electron drift term into it, by contrast to the isothermal case? [42] To address this point we have repeated the calculations presented in section 4.2 but, instead of using an electron drift that produced just the right threshold speed, we actually used a speed three times that value and introduced it in (32) to determine the zero growth condition (assuming therefore that the eigenfrequency was shifted to that value through some unspecified nonlinear processes). The results of our calculations are presented for 3-m waves in Figure 5. The figure in part offers an equivalent look at the results presented in Figure 2, but this time in the form of a contour plot with altitude on one axis and aspect angle on the other. In other words, we have produced contours of $V_{ph}/c_{s,i}$ associated with the threshold electron drifts described in Figures 4a–4c (black lines) together with the $V_{ph}/c_{s,i}$ contours for electron drifts three times as large as the computed threshold value (gray lines). In some ways the results are rather similar. Overall, the threshold phase speeds are of course greater in the strongly driven case. However, the numbers are not changing that much at lower altitudes or for aspect angles greater than 0.3°. The largest differences are at the higher altitudes, where for small enough aspect angles (<0.1°) a strong enhancement in the phase speed can now be registered even at 110 km (45% at the zero-aspect angle and 25% at the 0.1° aspect angle). Mathematically we recall that this is happening because the frequency is no longer very close to $k \cdot u_0$ when the electron drift is much larger than the threshold drift velocity, which makes the last term in the denominator of (33) less competitive and increases $F$ considerably. The mismatch between the phase velocity and the electron drift therefore allows the high altitudes to look more like the low altitudes at small aspect angles. This higher altitude super-adiabatic result was noted before by St.-Maurice and Kissack [2000], which led them to suggest that so-called “type IV” waves at high latitudes could be zero-aspect angle irregularities observed during electron heating events.

5. Summary and Conclusions

[43] We have used a comprehensive set of well-established transport equations to determine just how thermal feedback from the electrons affects two-fluid models of the Farley-Buneman instability and we have computed the modifications brought to the threshold phase velocity of the waves that should be observed along the electron drift $E \times B$ drift direction. We have determined precisely how this phase velocity varies with aspect angle. We have studied the dependence of our solution on the wavelength and altitude as well. For aspect angles greater than 0.3° and/or altitudes above 108 km, the threshold phase speed should normally only exceed the isothermal ion acoustic speed by 8 to 10%. However, the ratio of the threshold phase speed to the isothermal ion-acoustic speed increases rapidly with decreasing altitude. By the time we go down to 102 km altitude or lower, noticeable corrections (20% or higher) are found for aspect angles less than 0.3° at 3 m. For 10-m waves the corrections become noticeable (20% or higher) near 100 km (and below) and at aspect angles less than 0.25°.

[44] Since thermal corrections are important and may have implications not just at low latitudes but also at higher latitudes, we are now presenting a summary of the equations that ought to be numerically solved in order to determine the nonisothermal threshold speed. We recall that in linear or weakly turbulent situations, the eigenfrequency is still given by equation (42), that is, is of the form $\omega_y = (k \cdot u_e - \psi_T k \cdot u_e)/(1 + \psi_T)$. We must stress however, that $\psi_T$ differs from $\psi$ and this correction factor should not be forgotten, as shown in equation (21).
The zero growth rate condition can be found by solving directly an equation easily derived from (32) and (33), namely,

\[
\frac{V_{ph}^2}{c_{ij}^2} = 1 + \frac{(1 + g)^2 A \left( \frac{k_{ij}^2}{k_{T1}^2} \right) B \left( \frac{k_{ij}^2}{k_{T1}^2} \right) \left( \frac{V_{ph} - u_{ij}}{c_{ij}} \right)^2}{3 \left\{ \left( \frac{V_{ph}}{c_{ij}} \right)^2 + \left( \frac{u_{ij}}{c_{ij}} \right)^2 + K^2 \right\}},
\]

where

\[
K = \frac{2}{3} \left[ C \left( \frac{k_{ij}^2}{k_{T1}^2} \right) D_{e\perp} k_{e\perp}^2 + \frac{1}{k_{cij}} \right].
\]

The constants A, B, and C are given by

\[
A \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \left[ 1 + \frac{(\alpha - g)}{(1 + g)} \right] / (1 + \alpha \beta),
\]

\[
B \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \frac{1 + (\alpha - g) \beta}{1 + \alpha \beta}
\]

\[
C \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \frac{5/2 + 2g + g^2 + (g - 5/2) \beta - g(1 + g) A \left( \frac{k_{ij}^2}{k_{T1}^2} \right)}{1 + \alpha \beta},
\]

where

\[
\beta = \frac{1}{1 + 2g/5} \frac{k_{ij}^2 c_{ij}^2}{k_{T1}^2 c_{T1}^2}.
\]

The constants A, B, and C are given by

\[
A \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \left[ 1 + \frac{(\alpha - g)}{(1 + g)} \right] / (1 + \alpha \beta),
\]

\[
B \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \frac{1 + (\alpha - g) \beta}{1 + \alpha \beta}
\]

\[
C \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \frac{5/2 + 2g + g^2 + (g - 5/2) \beta - g(1 + g) A \left( \frac{k_{ij}^2}{k_{T1}^2} \right)}{1 + \alpha \beta},
\]

where

\[
\beta = \frac{1}{1 + 2g/5} \frac{k_{ij}^2 c_{ij}^2}{k_{T1}^2 c_{T1}^2}.
\]

The constants A, B, and C are given by

\[
A \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \left[ 1 + \frac{(\alpha - g)}{(1 + g)} \right] / (1 + \alpha \beta),
\]

\[
B \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \frac{1 + (\alpha - g) \beta}{1 + \alpha \beta}
\]

\[
C \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \frac{5/2 + 2g + g^2 + (g - 5/2) \beta - g(1 + g) A \left( \frac{k_{ij}^2}{k_{T1}^2} \right)}{1 + \alpha \beta},
\]

where

\[
\beta = \frac{1}{1 + 2g/5} \frac{k_{ij}^2 c_{ij}^2}{k_{T1}^2 c_{T1}^2}.
\]

The constants A, B, and C are given by

\[
A \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \left[ 1 + \frac{(\alpha - g)}{(1 + g)} \right] / (1 + \alpha \beta),
\]

\[
B \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \frac{1 + (\alpha - g) \beta}{1 + \alpha \beta}
\]

\[
C \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \frac{5/2 + 2g + g^2 + (g - 5/2) \beta - g(1 + g) A \left( \frac{k_{ij}^2}{k_{T1}^2} \right)}{1 + \alpha \beta},
\]

where

\[
\beta = \frac{1}{1 + 2g/5} \frac{k_{ij}^2 c_{ij}^2}{k_{T1}^2 c_{T1}^2}.
\]

The constants A, B, and C are given by

\[
A \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \left[ 1 + \frac{(\alpha - g)}{(1 + g)} \right] / (1 + \alpha \beta),
\]

\[
B \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \frac{1 + (\alpha - g) \beta}{1 + \alpha \beta}
\]

\[
C \left( \frac{k_{ij}^2}{k_{T1}^2} \right) = \frac{5/2 + 2g + g^2 + (g - 5/2) \beta - g(1 + g) A \left( \frac{k_{ij}^2}{k_{T1}^2} \right)}{1 + \alpha \beta},
\]

where

\[
\beta = \frac{1}{1 + 2g/5} \frac{k_{ij}^2 c_{ij}^2}{k_{T1}^2 c_{T1}^2}.
\]
Note that \( g = \left[ \frac{\partial T_e}{\partial r} \right] \approx 5/6 \) for the temperatures of interest, so that we have \( \alpha = 29/18 \), and get for aspect angles large enough to make \( \beta \gg 1 \), \( \frac{d^2 A_{\text{in}}}{\partial \theta^2} \approx \frac{\alpha - \beta}{\alpha + 1 - g} \approx 0.208, \)
\[ B \left( \frac{d^2 A_{\text{in}}}{\partial \theta^2} \right) \approx \frac{\alpha - \beta}{\alpha + 1 - g} \approx 0.483 \]
and
\[ C \left( \frac{d^2 A_{\text{in}}}{\partial \theta^2} \right) = \frac{\alpha - \beta}{3}. \]
In our calculations we also have used \( \eta = 0.003 \) \( v_{\text{cm}} \) and have assumed \( T_{e0} = T_{i0} = T_n. \)

Two final comments are in order. First, it is important to note that, without thermal diffusion, the effects discussed here would be much smaller, and probably very difficult to observe. This can be seen from the case \( g = 0 \) presented in Figures 2a, 3a and 4c. As discussed by \textit{St.-Maurice et al.} [2003], this \( g = 0 \) case is nothing other than the adiabatic limit at zero-aspect angle at the lower altitudes. This limit was first explored in a fluid context by \textit{Farley and Providakes} [1989] and by \textit{Pecseli et al.} [1989], both of whom neglected heat flows associated with temperature fluctuations. That we recover the adiabatic limit can be seen from the fact that for \( g = 0 \) and for zero-aspect angle conditions we get \( A = B = 1 \). Then, at the lower altitudes, the phase velocity deviates enough from \( u_{\text{cm}} \) that we can neglect \( K \) in the denominator of (49), that is to say, heat conduction driven by temperature gradients and inelastic collisions are no longer affecting the solution. At this point the phase velocity becomes \( 2/\sqrt{3} \) greater than the isothermal ion-acoustic speed because adiabatic heating produces temperature fluctuations with a “gas constant,” \( T_{\text{p}} = 5/3. \) We note that \textit{Dimant and Sudan} [1995a] equally recovered this adiabatic limit for the same \( g = 0 \) situation. In physical terms, when \( g = 0 \), the electrons behave like “Maxwell molecules,” that is, particles with constant collision frequencies. When this happens, the heat flow is no longer coupled to the momentum and thermal diffusion effects disappear (see \textit{St.-Maurice and Kissack} [2000] for a thorough discussion). This is why at zero-aspect angles near 100 km the adiabatic limit seriously underestimates the phase speed at threshold. Thermal diffusion greatly enhances the adiabatic effects, again as discussed in detail by \textit{St.-Maurice and Kissack} [2000]. In that sense we should call the zero-aspect angles results at low altitudes “super-adiabatic.”

A second comment concerns a comparison of our results against the specific results of \textit{Dimant and Sudan} [1995a]. There are two major differences that can immediately be inferred even just at zero-aspect angles. First, our super-adiabatic corrections are not as large as theirs. Judging from their Figure 1, the above authors get a correction of order 1.7 times the isothermal ion acoustic speed where we ourselves get a factor 1.47. Certainly, the observations described by \textit{St.-Maurice et al.} [2003] are in very good agreement with a value of 1.47. A second more worrisome problem in the comparison is that \textit{Dimant and Sudan}’s [1995a] super-adiabatic speeds appear at all altitudes above a nominal 105 km altitude (the exact altitude is hard to tell because they produced plots of the threshold drift required to excite the waves, rather than a plot of the threshold phase speed) whereas ours start to decrease above 102 km. The reason for the decrease in our work is clear: as the phase velocity approaches the relative ion-electron drift (\( \psi_T \) becoming substantially smaller than 1), inelastic collisions and heat flows induced by temperature gradients become important and wipe out the temperature fluctuations. It is as though the Dimant and Sudan work, like \textit{Farley and Providakes} [1989] and \textit{Pecseli et al.} [1989] before it, does not properly take into account the effects of cooling by inelastic collisions and of heat flows due to temperature gradients.

### Appendix A: Conditions Under Which Ions Can Be Considered Isothermal

A species can be considered isothermal as long as its relative temperature fluctuations can be considered small when compared to the relative density fluctuations. This can be seen clearly from equations (19) or (37) of \textit{St.-Maurice and Kissack} [2000]. In the ion frame of reference the relative ion temperature perturbations at the altitudes of interest may be found from the balance between ion frictional heating and cooling through heat exchange with the neutrals. Using the representation given by (15) in the text both for the zeroth order and the perturbed flow, and taking the leading order result at the small parameter \( \xi \) we obtain for this energy balance:

\[
\frac{\delta T_i}{T_i} \approx \frac{4}{3} \frac{\delta n_i}{c_{ij}^2} \approx \frac{8}{3} \frac{\xi}{E_0} \frac{E_0}{B_0} \frac{B_0}{c_{ij}^2} = \frac{8}{3} \frac{\xi}{E_0} \frac{E_0}{B_0} \frac{B_0}{c_{ij}^2},
\]

where we have assumed that the zeroth order electron and ion temperatures are basically equal. Using the well-known Farley-Buneman relation between perturbed electric field and perturbed density (applicable as long as (42) is applicable),

\[
\frac{\delta E}{E} \approx \frac{\delta n}{\omega_i (1 + \xi_0)} \frac{1}{n}
\]

will then give us

\[
\frac{\delta T_i}{T_i} \approx \frac{4}{3} \frac{\delta n_i}{c_{ij}^2} \approx \frac{8}{3} \frac{\xi}{E_0} \frac{E_0}{B_0} \frac{B_0}{c_{ij}^2} \frac{\delta n}{n}.
\]

For \( v_{\text{in}} \) greater than 3 \( \omega_i \) (easily applicable for the present work), this gives temperature fluctuations less than 10% as large as the density fluctuations, which would lead to undetectable effects in practice for ambient \( E \times B \) drifts of the order of the ion acoustic speed. Likewise, the ratio remains less than 10% lower down, even for \( E \times B \) drifts 2 or 3 times the isothermal ion-acoustic speed.

Another way to look at result (A3) is to think about it in terms of an ion to electron temperature fluctuation ratio: as long as the electron temperature fluctuations are more important than the ion ones, we could be justified to drop the ion term. However, since in the main text we are finding the relative electron temperature correction to be of the order of the relative density fluctuations below 103 km, it is easy to infer from (A3) that the ion temperature fluctuations are at least 10 times if not 100 times less the electron temperature perturbations in the region of interest. In other words,

\[
\frac{\delta T_e}{\delta T_i} \approx \frac{2 \omega_i^2}{v_{\text{in}}^2} \frac{1}{(1 + \xi_0)} \frac{(E_0^2 / B_0^2)}{c_{ij}^2}. \]
As an aside, note that if we were to use for electrons the balance between inelastic cooling and frictional heating as we did for ions, we would get quite a different result instead, namely, $\frac{\beta_T}{\alpha_e} \approx \frac{v_{te}}{V_{ke}}$. This would make the ion temperature fluctuations comparable to the electron ones, but with the catch that both species would contribute to a less than 10% change in the threshold speed determination. That is to say, both species would be basically isothermal.

Appendix B: Polarization Electric Field Potential

\[ \varphi_1 = \left\{ \begin{array}{ll} \frac{\alpha V_{ke}^2}{\nu_e (1 + 2g/5) k_w^2} - \frac{\nu_{in} V_{ke}^2}{(\omega_i + \omega_m)} k_w^2 + \frac{\nu_{in} V_{ke}^2}{(\omega_i + \omega_m)} k_i^2 & \\
+ i \frac{\nu_e (1 + 2g/5)}{\nu_e (1 + g/5)} k_w^2 + \frac{1 - g}{\omega_i} \frac{\nu_e V_{ke}^2}{\omega_i} k_w^2 & \\
+ \frac{(\alpha - g) V_{ke}^2}{v_e (1 + 2g/5)} k_w^2 + \frac{1 - g}{\omega_i} \frac{\nu_e V_{ke}^2}{\omega_i} k_i^2 & \\
+ i \frac{\nu_e (1 + g/5)}{\nu_e (1 + 2g/5)} k_i^2 + \frac{1 - g}{\omega_i} \frac{\nu_e V_{ke}^2}{\omega_i} k_i^2 \end{array} \right\} \psi_1. \]  

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