P812 Solution No. 1

1. The potential due to a ring charge (charge q, radius a) placed on the $x-y$ plane is

$$\Phi(r,\theta) = \frac{q}{2\pi^2\epsilon_0}\frac{1}{\sqrt{r^2+a^2+2ar\sin\theta}}K(k^2)$$

where $K\left(k^2\right)$ is the complete elliptic integral of the first kind defined by

$$K(k^2) = \int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\alpha}}d\alpha$$

The argument k^2 is

$$k^2 = \frac{4ar\sin\theta}{r^2+a^2+2ar\sin\theta}$$

Using

$$\lim_{k^2\to 1-\delta} K\left(k^2\right) = \ln\left(\frac{4}{\sqrt{\delta}}\right)$$

show that the capacitance of a thin torus of major radius a and minor radius b ($\ll a$) is

$$C \simeq \frac{4\pi^2\varepsilon_0 a}{\ln\left(\dfrac{8a}{b}\right)} \text{ (F)}$$

Solution

The ring surface is an equipotential surface and an arbitrary position can be chosen to find the potential. Let us choose

$$r = a-b,\ \theta = \pi/2$$

which is at inner board of the torus. There,

$$\begin{aligned}
k^2 &= \frac{4ar\sin\theta}{r^2+a^2+2ar\sin\theta}\\
&= \frac{4a(a-b)}{(a-b)^2+a^2+2a(a-b)}\\
&= \frac{4a(a-b)}{4a(a-b)+b^2}\\
&\simeq 1-\left(\frac{b}{2a}\right)^2
\end{aligned}$$

Then $\delta = \left(\frac{b}{2a}\right)^2$ and

$$K\left(k^2\right) \simeq \ln\left(\frac{4}{\sqrt{\delta}}\right) = \ln\left(\frac{8a}{b}\right)$$

The ring potential is

$$\Phi = \frac{q}{2\pi^2\varepsilon_0}\frac{1}{2a}\ln\left(\frac{8a}{b}\right)$$

and the capacitance is

$$C = \frac{q}{\Phi} = \frac{4\pi^2\varepsilon_0 a}{\ln\left(\frac{8a}{b}\right)} \text{ (F)}$$

subject to $a \gg b$ (thin ring).

Alternatively, the potential on the ring surface can be found directly as

$$\Phi = \Phi_1 + \Phi_2$$

where

$$\begin{aligned}
\Phi_1 &= \frac{\lambda}{4\pi\varepsilon_0} \int_{-l}^{l} \frac{1}{\sqrt{z^2+b^2}} dz \\
&= \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\sqrt{z^2+b^2}+z\right)\Big|_0^l \\
&= \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{2l}{b}\right)
\end{aligned}$$

$$\begin{aligned}
\Phi_2 &= \frac{\lambda}{2\pi\varepsilon_0} \int_{\delta}^{\pi} \frac{a d\theta}{2a\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{4a}{l}\right)
\end{aligned}$$

with

$$\delta = \frac{l}{a} \ll 1$$

Then

$$\Phi = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{8a}{b}\right)$$

and the capacitance is

$$C = \frac{q}{\Phi} = \frac{2\pi a\lambda}{\Phi} = \frac{4\pi^2\varepsilon_0 a}{\ln\left(\frac{8a}{b}\right)}$$

2. A plane at $z = 0$ has the following surface charge distribution,

$$\sigma = \sigma_0 \sin(ax)\sin(by), \ \ (\text{C m}^{-2}),$$

and corresponding volume charge distribution of

$$\rho(x,y,z) = \sigma_0 \sin(ax)\sin(by)\,\delta(z), \ \ (\text{C m}^{-3}).$$

Show that the potential

$$\Phi(x,y,z) = \frac{\sigma_0}{2\varepsilon_0 k}\sin(ax)\sin(by)\,e^{-k|z|}, \ \ k = \sqrt{a^2+b^2},$$

satisfies the Poisson's equation,

$$\nabla^2\Phi(x,y,z) = -\frac{\rho(x,y,z)}{\varepsilon_0}.$$

Hint:

$$\frac{d^2}{dz^2}e^{-k|z|} = k^2 e^{-k|z|} - 2k\delta(z).$$

Solution

$\nabla^2\Phi(x,y,z)$ is

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\sigma_0}{2\varepsilon_0 k}\sin(ax)\sin(by)\,e^{-k|z|}\\
=&\ \frac{\sigma_0}{2\varepsilon_0 k}\left(-a^2-b^2+k^2-2k\delta(z)\right)\sin(ax)\sin(by)\,e^{-k|z|}
\end{aligned}$$

Therefore, if

$$k^2=a^2+b^2$$

the potential satisfies the Poisson's equation

$$\nabla^2\Phi(x,y,z)=-\frac{\rho(x,y,z)}{\varepsilon_0}$$

3. Four line charges λ (C m^{-1}) at $(a,0)$, $-\lambda$ at $(0,a)$, λ at $(-a,0)$, and $-\lambda$ at $(0,-a)$ form a two dimensional quadrupole. Show that the potential near the origin $\left(\rho^2=x^2+y^2\ll a^2\right)$ is given by

$$\Phi(x,y)\simeq\frac{\lambda}{\pi\varepsilon_0 a^2}\left(x^2-y^2\right),\ x^2+y^2\ll a^2.$$

Hint: The potential due to a line charge λ along the z axis is

$$\Phi=-\frac{\lambda}{2\pi\varepsilon_0}\ln\rho,$$

where $\rho=\sqrt{x^2+y^2}$. Use

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

Solution

The potential due to the line charge at $(x=a,\ y=0)$ is

$$\begin{aligned}
\Phi_1 &= -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{(x-a)^2+y^2}\\
&= -\frac{\lambda}{4\pi\varepsilon_0}\ln\left(a^2+x^2-2ax+y^2\right)
\end{aligned}$$

In the region $\rho^2=x^2+y^2\ll a^2$,

$$\Phi_1\simeq-\frac{\lambda}{4\pi\varepsilon_0}\left(\ln a-\frac{x^2-y^2}{a^2}-\frac{2x}{a}\right)$$

The potentials due to the line charges at $(x=0,\ y=a)$, $(x=-a,y=0)$, $(x=0,y=-a)$ are

$$\Phi_2\simeq\frac{\lambda}{4\pi\varepsilon_0}\left(\ln a+\frac{x^2-y^2}{a^2}-\frac{2y}{a}\right)$$

$$\Phi_3\simeq-\frac{\lambda}{4\pi\varepsilon_0}\left(\ln a-\frac{x^2-y^2}{a^2}+\frac{2x}{a}\right)$$

$$\Phi_4\simeq\frac{\lambda}{4\pi\varepsilon_0}\left(\ln a+\frac{x^2-y^2}{a^2}+\frac{2y}{a}\right)$$

The total potential is

$$\Phi=\frac{\lambda}{\pi\varepsilon_0}\frac{x^2-y^2}{a^2},\ x^2+y^2\ll a^2$$

4. In the lowest order, each electron cloud in the helium atom may be described by the charge density distribution

$$\rho_c(r) = -\frac{8e}{\pi a^3} \exp\left(-\frac{4r}{a}\right),$$

where $e = 1.6 \times 10^{-19}$ (C) is the electronic charge and $a = 5.3 \times 10^{-11}$ (m) is the Bohr radius.

(a) Show that the interaction potential energy between the two electrons is given by

$$U = \frac{1}{4\pi\varepsilon_0} \frac{5e^2}{4a}.$$

(b) Estimate the ionization energy of He atom. (Experimental value is 79 eV.)

Solution

(a) The interaction potential energy can be found from

$$\int \Phi \rho dV,$$

where $\Phi(r)$ is the potential due to one electron cloud and ρ is the charge density of the other cloud. However, since both clouds are identical, the energy can alternatively be found from

$$\int \varepsilon_0 E^2 dV,$$

where the electric field follows from the Gauss' law,

$$4\pi r^2 E(r) = \frac{1}{\varepsilon_0} \int_0^r \rho_c(r) 4\pi r^2 dr,$$

$$E(r) = -\frac{e}{4\pi\varepsilon_0 a^3} \frac{1}{r^2} \left(a^3 - a^3 e^{-\frac{4}{a}r} - 8ar^2 e^{-\frac{4}{a}r} - 4a^2 r e^{-\frac{4}{a}r}\right).$$

Then

$$\begin{aligned}
&\int \varepsilon_0 E^2 dV \\
&= \frac{e^2}{4\pi\varepsilon_0} \int_0^\infty \frac{1}{r^2} \left(1 - \left(1 + \frac{4r}{a} + \frac{8r^2}{a^2}\right) e^{-\frac{4r}{a}}\right)^2 dr \\
&= \frac{e^2}{4\pi\varepsilon_0} \lim_{r\to\infty} \left(-\frac{1}{4} \frac{4a^3 - 8a^3 e^{-4\frac{r}{a}} - 16 e^{-4\frac{r}{a}} r a^2 + 4a^3 e^{-8\frac{r}{a}} + 21 e^{-8\frac{r}{a}} r a^2 + 40 e^{-8\frac{r}{a}} r^2 a + 32 e^{-8\frac{r}{a}} r^3 - 5}{a^3 r}\right. \\
&= \frac{e^2}{4\pi\varepsilon_0} \times \frac{5}{4a}.
\end{aligned}$$

(b) The potential due to the electron clouds at $r = 0$ (where the protons are) is

$$\begin{aligned}
\Phi(r=0) &= -2 \int_\infty^0 E(r)\, dr \\
&= -\frac{2e}{4\pi\varepsilon_0 a^3} \int_0^\infty \frac{1}{r^2} \left(a^3 - a^3 e^{-\frac{4}{a}r} - 8ar^2 e^{-\frac{4}{a}r} - 4a^2 r e^{-\frac{4}{a}r}\right) dr \\
&= -\frac{e}{\pi\varepsilon_0 a}.
\end{aligned}$$

Then the interaction energy between the protons and electron clouds is

$$U = -\frac{2e^2}{\pi\varepsilon_0 a} = -\frac{1}{4\pi\varepsilon_0}\frac{8e^2}{a}.$$

This is 8 times of that in hydrogen atom as expected. Corresponding kinetic energy of electrons is

$$\frac{1}{4\pi\varepsilon_0}\frac{4e^2}{a}$$

and the the total energy of He atom without electron-electron interaction is

$$-\frac{1}{4\pi\varepsilon_0}\frac{4e^2}{a} = -109 \text{ eV}$$

Then the lowest order ionization energy is 109 eV.

If the electron-electron interaction energy is included, the He energy becomes

$$U = -109 + \frac{5}{4} \times 27.2 = 75 \text{ eV}$$

which is closer to the experimental value of 79 eV. The discrepancy is due to the primitive form of the electron cloud. Quantum mechanical variational method can improve the ionization energy to about 77 eV.