

## Mathematical Formulae

### 1. Vector Formulae

Bold characters are vector functions and  $f$  is a scalar function.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

$$\nabla \cdot (f\mathbf{A}) = \nabla f \cdot \mathbf{A} + f\nabla \cdot \mathbf{A}$$

$$\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f\nabla \times \mathbf{A}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$\nabla \times \nabla f \equiv 0$$

$$\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla \cdot \mathbf{r} = 3, \quad \mathbf{r} = \text{position vector}$$

$$\nabla \times \mathbf{r} = 0$$

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

Substantive derivative:  $\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla)f$ ,  $\mathbf{v}$  = velocity

Substantive derivative:  $\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}$

Substantive derivative:  $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla v^2 + (\nabla \times \mathbf{v}) \times \mathbf{v}$

Gauss' theorem:  $\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{S}$

Stokes' theorem  $\int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l}$

$$\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = 0 \text{ for closed surface}$$

$$\int_V \nabla \times \mathbf{A} dV = \oint_S d\mathbf{S} \times \mathbf{A} = - \oint_S \mathbf{A} \times d\mathbf{S}$$

## 2. Delta Function

$$\int f(t)\delta(t-t')dt = f(t')$$

$$\delta(at) = \frac{1}{|a|}\delta(t), \quad \int g(t)\delta[f(t)]dt = \frac{g(t')}{\left|\frac{df}{dt}\right|_{t=t'}} \quad \text{where } f(t') = 0$$

$$\begin{aligned} & \delta(\mathbf{r} - \mathbf{r}') \text{ (3-D delta function)} \\ &= \delta(x - x')\delta(y - y')\delta(z - z') \text{ (Cartesian)} \\ &= \frac{\delta(r - r')}{rr'} \frac{\delta(\theta - \theta')}{\sin\theta} \delta(\phi - \phi') \text{ (spherical)} \\ &= \frac{\delta(r - r')}{rr'} \delta(\cos\theta - \cos\theta')\delta(\phi - \phi') \text{ (spherical)} \\ &= \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi')\delta(z - z') \text{ (cylindrical)} \end{aligned}$$

## 3. Curvilinear Coordinates

Let  $u_i(x, y, z)$  ( $i = 1, 2, 3$ ) be a system of curvilinear coordinates. The metric coefficients are

$$h_i = \sqrt{\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2},$$

and the length segments in each direction are  $h_i du_i \mathbf{e}_i$  ( $\mathbf{e}_i$  unit vector).

Area elements

$$d\mathbf{S}_i = h_j h_k du_j du_k \mathbf{e}_j \times \mathbf{e}_k$$

Volume element

$$dV = h_1 h_2 h_3 du_1 du_2 du_3$$

Gradient

$$\nabla f = \sum_{i=1}^3 \frac{1}{h_i} \frac{\partial f}{\partial u_i} \mathbf{e}_i$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_3 h_1 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right\}$$

Curl

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\mathbf{e}_1}{h_2 h_3} & \frac{\mathbf{e}_2}{h_3 h_1} & \frac{\mathbf{e}_3}{h_1 h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Scalar Laplacian

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right\}$$

Vector Laplacian (definition)

$$\nabla^2 \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

### Spherical Coordinates $(r, \theta, \phi)$

Transformation

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

Metric coefficients

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin \theta.$$

Transformation of the unit vectors

$$\begin{aligned} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{bmatrix} &= \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix} \\ \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix} &= \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{bmatrix} \end{aligned}$$

For example,

$$\begin{aligned} \mathbf{e}_x &= \nabla x = \nabla(r \sin \theta \cos \phi) \\ &= \sin \theta \cos \phi \mathbf{e}_r + \cos \theta \cos \phi \mathbf{e}_\theta - \sin \phi \mathbf{e}_\phi \end{aligned}$$

Derivative of the unit vectors

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_r}{\partial \phi} = \sin \theta \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \quad \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos \theta \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\sin \theta \mathbf{e}_r - \cos \theta \mathbf{e}_\theta.$$

Gradient

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

Curl

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\mathbf{e}_r}{r^2 \sin \theta} & \frac{\mathbf{e}_\theta}{r \sin \theta} & \frac{\mathbf{e}_\phi}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin \theta A_\phi \end{vmatrix}$$

Scalar Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

Vector Laplacian

$$\begin{aligned}
 \nabla^2 \mathbf{A} &\equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \\
 &= \left( \nabla^2 A_r - \frac{2}{r^2} A_r - \frac{2}{r^2} \frac{\partial A_\theta}{\partial \theta} - \frac{2 \cot \theta}{r^2} A_\theta - \frac{2}{r^2 \sin \theta} \frac{\partial A_\phi}{\partial \phi} \right) \mathbf{e}_r \\
 &\quad + \left( \nabla^2 A_\theta - \frac{1}{r^2 \sin^2 \theta} A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\phi}{\partial \phi} \right) \mathbf{e}_\theta \\
 &\quad + \left( \nabla^2 A_\phi - \frac{1}{r^2 \sin^2 \theta} A_\phi + \frac{2}{r^2 \sin \theta} \frac{\partial A_r}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\theta}{\partial \phi} \right) \mathbf{e}_\phi
 \end{aligned}$$

Note that in non-cartesian coordinates,

$$(\nabla^2 \mathbf{A})_i \neq \nabla^2 A_i$$

## Cylindrical Coordinates $(\rho, \phi, z)$

Transformation

$$\begin{aligned}
 x &= \rho \cos \phi \\
 y &= \rho \sin \phi \\
 z &= z
 \end{aligned}$$

Metric coefficients

$$h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1$$

Derivatives of the unit vectors

$$\frac{\partial \mathbf{e}_\rho}{\partial \phi} = \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_\rho$$

Gradient

$$\nabla f = \frac{\partial f}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{e}_z$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Curl

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

Scalar Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

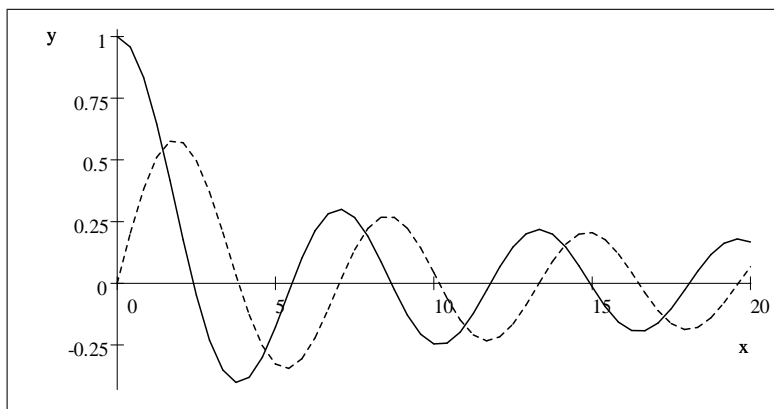
## Vector Laplacian

$$\begin{aligned}\nabla^2 \mathbf{A} &\equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \\ &= \left( \nabla^2 A_\rho - \frac{2}{\rho^2} \frac{\partial A_\phi}{\partial \phi} - \frac{1}{\rho^2} A_\rho \right) \mathbf{e}_\rho \\ &\quad + \left( \nabla^2 A_\phi + \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial \phi} - \frac{1}{\rho^2} A_\phi \right) \mathbf{e}_\phi + \nabla^2 A_z \mathbf{e}_z\end{aligned}$$

## 4. Special Functions

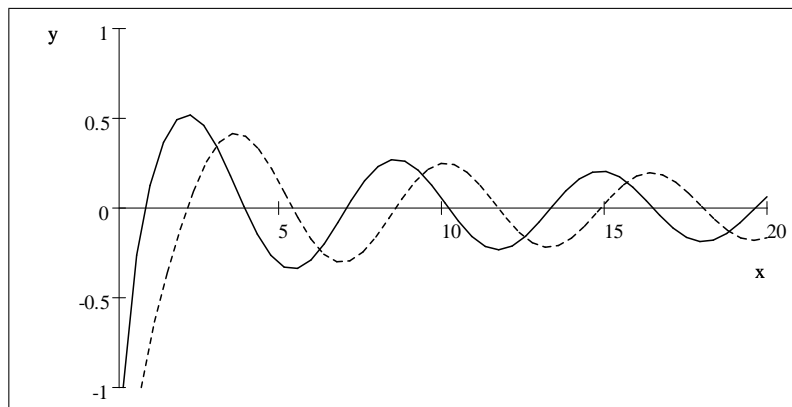
**Bessel Functions**  $Z_m(x) = J_m(x), N_m(x)$  [or  $Y_m(x)$  in some books]

$J_0(x), J_1(x)$



$J_0(x)$  (solid line) and  $J_1(x)$  (dashed line).

$Y_0(x), Y_1(x)$



$Y_0(x)$  (solid line) and  $Y_1(x)$  (dashed line).

Differential equation

$$\left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + k^2 - \frac{m^2}{\rho^2} \right) Z_m(k\rho) = 0$$

Wronskian

$$J_m(x)N'_m(x) - J'_m(x)N_m(x) = \frac{2}{\pi x}$$

Series representation of  $J_m(x)$

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{(m+n)!n!} \left(\frac{x}{2}\right)^{2n}, \quad J_{-m}(x) = (-1)^m J_m(x)$$

$$J_m(x) \simeq \left(\frac{x}{2}\right)^m \text{ for small } x \ll 1$$

$$J_0(x) \simeq 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 \text{ for small } x \ll 1$$

$$J_1(x) \simeq \frac{1}{2}x - \frac{1}{16}x^3 \text{ for small } x \ll 1$$

Generating functions

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{m=-\infty}^{\infty} J_m(x)t^m$$

$$e^{ix \sin \theta} = \sum_{m=-\infty}^{\infty} J_m(x)e^{im\theta}$$

$n$ -th root of  $J_0(x) = 0$ ,  $x_{0n}$ , and value of  $J_1(x_{0n})$

$x_{0n}$	2.40483	5.52008	8.65373	11.79153	14.93092	18.07106
$J_1(x_{0n})$	0.51915	-0.34026	0.27145	-0.23246	0.20655	-0.18773

$n$ -th root of  $J_1(x) = 0$ ,  $x_{1n}$ , and value of  $J_0(x_{1n})$

$x_{1n}$	3.83171	7.01559	10.17347	13.32369	16.47063	19.61586
$J_0(x_{1n})$	-0.40276	0.30012	-0.24970	0.21836	-0.19647	0.18006

For small  $x$  ( $\ll 1$ ),

$$J_m(x) \simeq \left(\frac{x}{2}\right)^m, \quad N_m(x) \simeq \frac{2}{\pi} \left(\ln \frac{x}{2} + \gamma_E\right), \quad \gamma_E = 0.5772 \dots \text{ (Euler's constant).}$$

Hankel functions of the first and second kinds

$$H_m^{(1,2)}(x) = J_m(x) \pm iN_m(x)$$

Asymptotic forms at large  $x \gg 1$

$$J_m(x) \simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{2m+1}{4}\pi\right)$$

$$N_m(x) \simeq \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{2m+1}{4}\pi\right)$$

$$H_m^{(1,2)}(x) \simeq \sqrt{\frac{2}{\pi x}} \exp\left[\pm i\left(x - \frac{2m+1}{4}\pi\right)\right]$$

Recurrence formulae

$$Z'_0(x) = -Z_1(x)$$

$$Z'_m(x) = \frac{1}{2} [Z_{m-1}(x) - Z_{m+1}(x)]$$

$$Z_m(x) = \frac{x}{2m} [Z_{m+1}(x) + Z_{m-1}(x)]$$

$$\frac{d}{dx} [x^m Z_m(x)] = x^m Z_{m-1}(x)$$

Integral representations (There are many. A few are listed.)

$$J_m(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(m\theta - x \sin \theta) d\theta, \quad J_0(x) = \frac{2}{\pi} \int_0^1 \frac{\cos(xt)}{\sqrt{1-t^2}} dt$$

$$N_0(x) = -\frac{2}{\pi} \int_1^\infty \frac{\cos(xt)}{\sqrt{1-t^2}} dt$$

$$N_m(x) = -\frac{2^{m+1} x^{-m}}{\sqrt{\pi} \Gamma(\frac{1}{2} - m)} \int_1^\infty \frac{\cos xt}{(t^2 - 1)^{m+(1/2)}} dt$$

Integrals

$$\int_0^\infty J_m(ax) dx = \frac{1}{a}, \quad \int_0^\infty N_m(ax) dx = -\frac{1}{a} \tan\left(\frac{m\pi}{2}\right)$$

$$\int_0^\infty J_0(ax) J_0(bx) dx = \frac{2}{\pi b} K(a/b), \quad K : \text{complete elliptic integral of the first kind}$$

$$\int_0^\infty J_0(ax) J_1(bx) dx = \begin{cases} 1/b, & b > a > 0 \\ 1/2b, & a = b > 0 \\ 0, & a > b > 0 \end{cases} \quad (\text{step function})$$

$$\int_0^\infty x J_1(ax) J_1(bx) dx = \frac{\delta(a-b)}{a}, \quad (\text{derivative of the above with respect to } a)$$

In fact for any integer  $m$ ,

$$\int_0^\infty x J_m(ax) J_m(bx) dx = \frac{\delta(a-b)}{a}$$

$$\int_0^\infty J_{\mu+x}(ax) J_{\nu-x}(ax) dx = J_{\mu+\nu}(2a), \quad \mu + \nu > 1$$

$$\int_0^\infty x^{\mu-1} J_\nu(ax) dx = \frac{2^{\mu-1} \Gamma(\frac{\mu+\nu}{2})}{a^\mu \Gamma(\frac{\nu-\mu}{2} + 1)}, \quad \Gamma : \text{gamma function}$$

$$\int_0^\infty \frac{J_\nu(ax) J_\nu(bx)}{x^2 - y^2} dx = \frac{i\pi}{2} J_\nu(by) H_\nu^{(1)}(ay)$$

$$\int_0^\infty e^{-ax} J_0(bx) dx = \frac{1}{\sqrt{a^2 + b^2}}$$

$$\int_0^\infty e^{-ax} J_\nu(bx) dx = \frac{(\sqrt{a^2 + b^2} - a)^\nu}{b^\nu \sqrt{a^2 + b^2}}$$

$$\int_0^\infty e^{-ax} J_\nu(bx) J_\nu(cx) dx = \frac{1}{\pi \sqrt{bc}} Q_{\nu-\frac{1}{2}}\left(\frac{a^2 + b^2 + c^2}{2bc}\right), \quad Q_\mu : \text{Legendre function of the 2nd kind}$$

$$\int_0^\infty e^{-a^2 x^2} J_{2\nu}(px) dx = \frac{\sqrt{\pi}}{2a} \exp\left(-\frac{b^2}{8a^2}\right) I_\nu\left(\frac{b^2}{8a^2}\right)$$

$$\int_0^\infty e^{-a^2 x^2} x^2 J_0(bx) dx = \frac{1}{2a^2} \exp\left(-\frac{b^2}{4a^2}\right)$$

$$\int_0^\infty e^{-a^2 x^2} J_\nu(px) J_\nu(qx) dx = \frac{1}{2a^2} \exp\left(-\frac{p^2 + q^2}{4a^2}\right) I_\nu\left(\frac{pq}{2a^2}\right)$$

$$\int_0^\infty \sin(ax) J_\nu(bx) dx = \begin{cases} \frac{1}{\sqrt{b^2-a^2}} \sin[\nu \sin^{-1}(a/b)], & b > a \\ \frac{b^\nu}{\sqrt{a^2-b^2}(a+\sqrt{a^2-b^2})^\nu} \cos\left(\frac{\nu\pi}{2}\right), & a < b \end{cases}$$

$$\int_0^\infty \sin(ax) J_0(bx) dx = \begin{cases} 0, & b > a \\ \frac{1}{\sqrt{a^2-b^2}}, & a > b \end{cases}$$

Sum

$$J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) = 1$$

$$\sum_{n=-\infty}^{\infty} J_n^2(x) = 1$$

$$\sum_{n=0}^{\infty} n^2 J_n^2(x) = \frac{x^2}{4}$$

$$J_0(x) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(x) = \cos x$$

$$\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x) = \frac{1}{2} \sin x$$

$$\sum_{n=0}^{\infty} (2n+1) J_{2n+1}(x) = \frac{x}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} J_{2n}(2nx) = \frac{x^2}{2}$$

$$\sum_{n=1}^{\infty} J_{2n}(2nx) = \frac{x^2}{2(1-x^2)}$$

$$\sum_{n=1}^{\infty} n^2 J_{2n}(2nx) = \frac{x^2(1+x^2)}{2(1-x^2)^4}$$

$$\sum_{n=1}^{\infty} n^2 \int_0^x J_{2n}(2nt) dt = \frac{x^3}{6(1-x^2)^3}$$

### Spherical Bessel Functions $z_l(x) = j_l(x), n_l(x)$

Spherical Bessel functions are elementary functions. Some low order forms are:

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x - x \cos x}{x^2}, \quad j_2(x) = \frac{(3-x^2) \sin x - 3x \cos x}{x^3}$$

$$n_0(x) = -\frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x + x \sin x}{x^2}, \quad n_2(x) = -\frac{(3-x^2) \cos x + 3x \sin x}{x^3}$$

Definition

$$j_l(x) \equiv \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x), \quad n_l(x) \equiv \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x)$$

$$h_l^{(1,2)}(x) = j_l(x) \pm in_l(x)$$

Differential equation

$$\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2} \right) z_l(kr) = 0$$

Wronskian

$$j_l(x)n_l'(x) - j_l'(x)n_l(x) = \frac{1}{x^2}$$

For  $x \ll 1$ ,

$$j_l(x) \simeq \frac{x^l}{(2l+1)!!}, \quad n_l(x) \simeq -\frac{(2l-1)!!}{x^{l+1}}$$

For  $x \gg 1$ ,

$$j_l(x) \simeq \frac{1}{x} \cos\left(x - \frac{l+1}{2}\pi\right), \quad n_l(x) \simeq \frac{1}{x} \sin\left(x - \frac{l+1}{2}\pi\right)$$

$$h_l^{(1)}(x) \simeq (-i)^{l+1} \frac{e^{ix}}{x}, \quad h_l^{(2)}(x) \simeq i^{l+1} \frac{e^{-ix}}{x}$$

Recurrence formulae

$$(2l+1)z_l(x) = x[z_{l-1}(x) - z_{l+1}(x)]$$

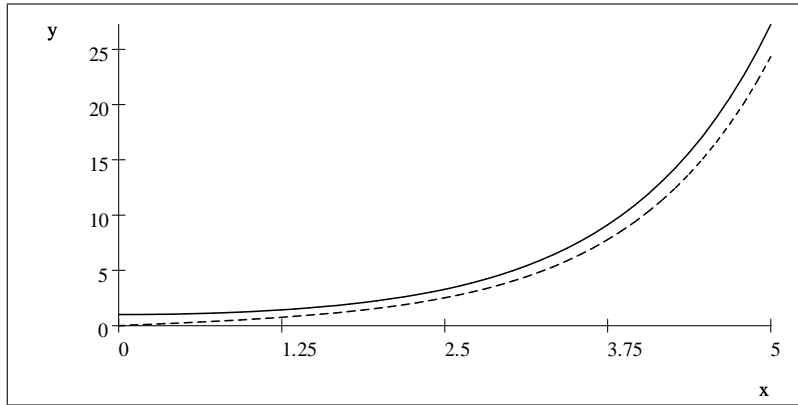
$$(2l+1) \frac{dz_l(x)}{dx} = lz_{l-1}(x) - (l+1)z_{l+1}(x)$$

$$\frac{d}{dx}[x^{l+1}z_l(x)] = x^{l+1}z_{l-1}(x)$$

$$\frac{d}{dx}[x^{-l}z_l(x)] = -x^{-l}z_{l+1}(x)$$

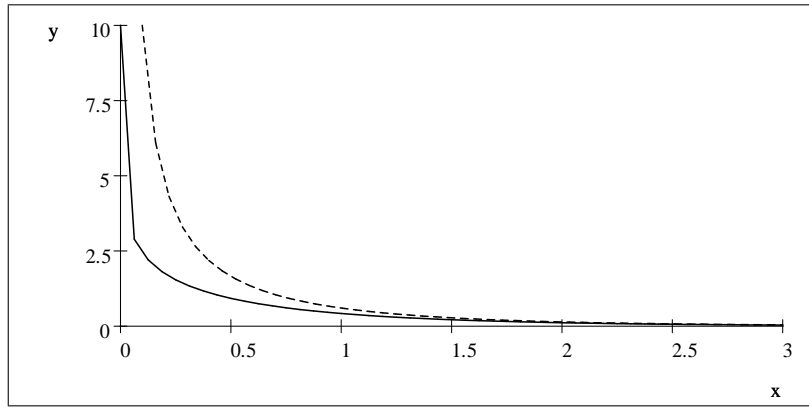
**Modified Bessel Functions**  $I_m(x)$ ,  $K_m(x)$

$$I_0(x), I_1(x)$$



$I_0(x)$  (solid line) and  $I_1(x)$  (dashed line).

$$K_0(x), K_1(x)$$



$K_0(x)$  (solid line) and  $K_1(x)$  (dashed line).

Definition

$$\begin{aligned} I_m(x) &= e^{-im\pi/2} J_m(ix) \\ &= \left(\frac{x}{2}\right)^m \sum_0^{\infty} \frac{(x/2)^{2n}}{n!(m+n)!} \end{aligned}$$

$$\begin{aligned} K_m(x) &= (-1)^{m+1} I_m(x) \left(\gamma + \ln \frac{x}{2}\right) + \frac{(-1)^m}{2} \sum_{k=0}^{\infty} \frac{(x/2)^{m+2k}}{k!(m+k)!} \left[ \sum_{n=1}^k \frac{1}{n} + \sum_{n=1}^{k+m} \frac{1}{n} \right] \\ &\quad + \frac{1}{2} \sum_{r=0}^{m-1} (-1)^r \frac{(m-r-1)!}{r!} \left(\frac{x}{2}\right)^{2r-m} \end{aligned}$$

Differential equation

$$\left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + k^2 + \frac{m^2}{\rho^2} \right) \begin{pmatrix} I_m(k\rho) \\ K_m(k\rho) \end{pmatrix} = 0$$

Wronskian

$$I'_m(x)K_m(x) - I_m(x)K'_m(x) = \frac{1}{x}$$

Series representation of  $I_m(x)$

$$I_m(x) = \left(\frac{x}{2}\right)^m \sum_{n=0}^{\infty} \frac{1}{m!(m+n)!} \left(\frac{x}{2}\right)^{2n}$$

For  $x \ll 1$

$$I_0(x) \simeq 1 + \frac{1}{4}x^2 + \dots$$

$$I_1(x) \simeq \frac{x}{2} + \frac{1}{16}x^3 + \dots$$

Recurrence formulae

$$I_{m-1}(x) - I_{m+1}(x) = \frac{2m}{x} I_m(x), \quad I_{m-1}(x) - I_{m+1}(x) = 2I'_m(x)$$

$$K_{m-1}(x) - K_{m+1}(x) = -\frac{2m}{x} K_m(x), \quad K_{m-1}(x) + K_{m+1}(x) = -2K'_m(x)$$

$$I'_0(x) = I_1(x), \quad K'_0(x) = -K_1(x)$$

Integral representation

$$K_0(x) = \int_0^\infty \frac{tJ_0(xt)}{t^2 + 1} dt = \int_0^\infty \frac{\cos xt}{\sqrt{t^2 + 1}} dt$$

$$K_1(x) = -K'_0(x) = \int_0^\infty \frac{t^2 J_1(xt)}{1 + t^2} dt$$

$$K_{1/3} \left( \frac{2x^{3/2}}{3^{3/2}} \right) = \frac{3}{\sqrt{x}} \int_0^\infty \cos(t^3 + xt) dt, \quad (\text{Airy's integral})$$

**Legendre Functions**  $P_l^m(x)$ ,  $Q_l^m(x)$

Differential equation

$$\left( (1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} + l(l+1) - \frac{m^2}{1-x^2} \right) \begin{pmatrix} P_l^m(x) \\ Q_l^m(x) \end{pmatrix} = 0$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \quad (\text{Rodrigues' formula})$$

$$Q_l(x) = \frac{1}{2} P_l(x) \ln \frac{1+x}{1-x} - W_{l-1}(x), \quad x \text{ real and } |x| \leq 1$$

$$Q_l(z) = \frac{1}{2} P_l(z) \ln \frac{z+1}{z-1} - W_{l-1}(z) \quad \text{for general complex } z$$

$$W_{-1}(x) = 0, \quad W_0(x) = 1, \quad W_1(x) = \frac{3}{2}x, \quad W_2(x) = \frac{5}{2}x^2 - \frac{2}{3}, \dots$$

Special values ( $m = 0$ )

$$P_l(1) = 1, \quad P_l(-1) = (-1)^l$$

$$P_l(0) = 0 \text{ for odd } l, \quad P_l(0) = (-1)^{l/2} \frac{(l-1)!!}{l!!} \text{ for even } l$$

$$P'_l(1) = \frac{l(l+1)}{2}, \quad P'_l(0) = -(l+1)P_{l+1}(0)$$

Definition of  $P_l^m(x)$ ,  $Q_l^m(x)$  in terms of  $P_l(x)$ ,  $Q_l(x)$  ( $x$  real,  $|x| \leq 1$ )

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad Q_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} Q_l(x)$$

For general complex argument  $z$

$$P_l^m(z) = (z^2 - 1)^{m/2} \frac{d^m}{dz^m} P_l(z), \quad Q_l^m(z) = (z^2 - 1)^{m/2} \frac{d^m}{dz^m} Q_l(z)$$

Orthogonality of  $P_l^m(x)$

$$\int_{-1}^1 \frac{P_l^m(x) P_{l'}^{m'}(x)}{1-x^2} dx = \frac{1}{m} \frac{(l+m)!}{(l-m)!} \delta_{mm'}$$

$$\int_{-1}^1 P_l^m(x) P_{l'}^m(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}$$

Wronskian

$$m = 0, P_l(x)Q_l'(x) - P_l'(x)Q_l(x) = \frac{1}{1-x^2}$$

$$\begin{aligned} P_l^m(x) \frac{dQ_l^m(x)}{dx} - \frac{dP_l^m(x)}{dx} Q_l^m(x) &= \frac{2^{2m}}{1-x^2} \frac{\Gamma[(l+m+1)/2]\Gamma[(l+m)/2+1]}{\Gamma[(l-m+1)/2]\Gamma[(l-m)/2+1]} \\ &= \frac{1}{1-x^2} \frac{(l+m)!}{(l-m)!}, \quad 0 \leq m \leq l \end{aligned}$$

Generating function

$$(\cos \theta + i \sin \theta \cos \phi)^l = P_l(\cos \theta) + 2 \sum_{m=1}^l i^m \frac{l!}{(l+m)!} P_l^m(\cos \theta) \cos(m\phi)$$

Low order forms of  $P_l^m(x)$ , ( $x = \cos \theta$ )

$m = 0$

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3x^2 - 1}{2}, P_3(x) = \frac{5x^3 - 3x}{2}, \dots$$

$l = 1, m = 1$

$$P_1^1(x) = \sqrt{1-x^2} = \sin \theta$$

$l = 2, m = 1, 2$

$$\begin{aligned} P_2^1(x) &= 3\sqrt{1-x^2}x = 3 \sin \theta \cos \theta = \frac{3}{2} \sin 2\theta \\ P_2^2(x) &= 3(1-x^2) = 3 \sin^2 \theta = \frac{3}{2}(1 - \cos 2\theta) \end{aligned}$$

$l = 3, m = 1, 2, 3$

$$\begin{aligned} P_3^1(x) &= \frac{3}{2} \sqrt{1-x^2}(5x^2 - 1) = \frac{3}{8}(\sin \theta + 5 \sin 3\theta) \\ P_3^2(x) &= 15(1-x^2)x = \frac{15}{4}(\cos \theta - \cos 3\theta) \\ P_3^3(x) &= 15(1-x^2)^{3/2} = 15 \sin^3 \theta = \frac{15}{4}(2 \sin 2\theta + 7 \sin 4\theta) \end{aligned}$$

$Q_l^m(x)$  (associated Legendre function of the second kind)

Special case  $z = i \sinh \eta$

$$Q_0(i \sinh \eta) = -i \cot^{-1}(\sinh \eta), \quad Q_1(i \sinh \eta) = \sinh \eta \cot^{-1}(\sinh \eta) - 1$$

$$Q_1^1(i \sinh \eta) = \cosh \eta \cot^{-1}(\sinh \eta) - \tanh \eta$$

Spherical harmonic function  $Y_{lm}(\theta, \phi)$

$$\text{Positive } m : Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi} \text{ for } 0 \leq m \leq l$$

Negative  $m$  :  $Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi)$  for  $-l \leq -m \leq 0$

General form

$$Y_{lm}(\theta, \phi) = (-1)^{\frac{m+|m|}{2}} \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos \theta) e^{im\phi}, \text{ for } -l \leq m \leq l$$

$$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{1,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$$Y_{20}(\theta, \phi) = \sqrt{\frac{5}{4\pi}} \frac{1}{2} (3 \cos^2 \theta - 1)$$

$$Y_{2,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$$

$$Y_{2,\pm 2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{\pm i2\phi}$$

Orthogonality of  $Y_{lm}(\theta, \phi)$

$$\oint Y_{lm}(\theta, \phi) Y_{l'm'}^*(\theta, \phi) d\Omega = \delta_{ll'} \delta_{mm'}, \quad d\Omega = \sin \theta d\theta d\phi$$

Toroidal functions  $P_{l-\frac{1}{2}}(\cosh \eta)$ ,  $Q_{l-\frac{1}{2}}(\cosh \eta)$  satisfy

$$\left( \frac{d^2}{d\eta^2} + \coth \eta \frac{d}{d\eta} - l^2 + \frac{1}{4} - m^2 \operatorname{cosech}^2 \eta \right) F(\eta) = 0$$

Integral representations

$$P_{l-\frac{1}{2}}^m(\cosh \eta) = \frac{(-1)^m (2l-1)!!}{\pi 2^{m+1} (2l-2m-1)!!} \int_0^\pi \frac{\cos m\theta}{(\cosh \eta + \cos \theta \sinh \eta)^{l+\frac{1}{2}}} d\theta$$

$$Q_{l-\frac{1}{2}}^m(\cosh \eta) = \frac{(-1)^m (2l-1)!!}{2^{m+1} (2l-2m-1)!!} \int_0^\infty \frac{\cosh mt}{(\cosh \eta + \cosh t \sinh \eta)^{l+\frac{1}{2}}} dt$$

## Gamma Function

Definition

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Properties

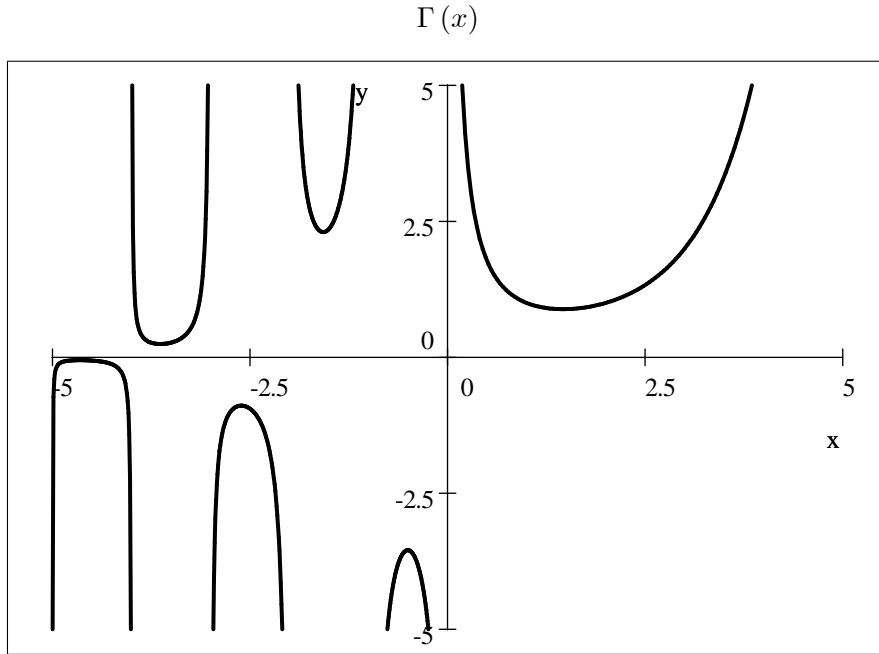
$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad \Gamma\left(z+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)}$$

If  $z$  is a positive integer,  $z = n$ ,

$$\Gamma(n+1) = n!$$

Special values

$$\Gamma(1) = 1, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \sqrt{\pi}$$



Gamma function  $\Gamma(x)$ .

### Elliptic Integrals $K(k^2)$ and $E(k^2)$

Complete Elliptic Integrals of the First Kind  $K(k^2)$  and Second Kind  $E(k^2)$

Definitions

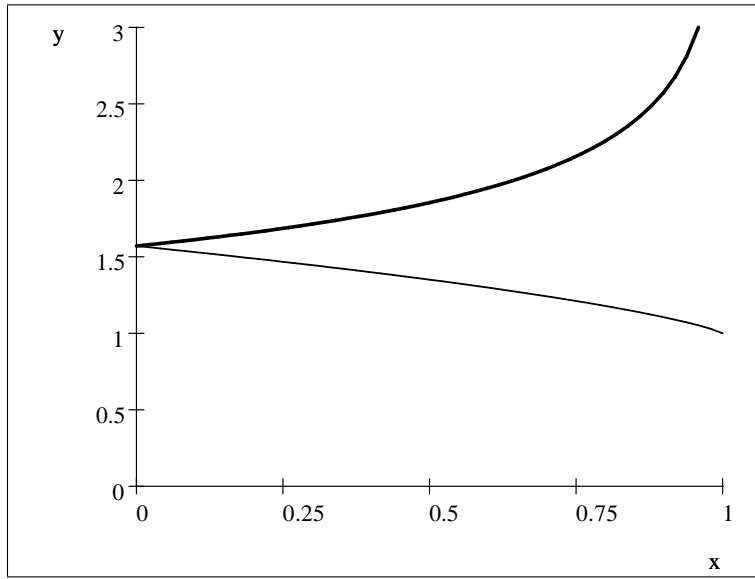
$$K(k^2) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta, \quad E(k^2) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta, \quad 0 \leq k^2 \leq 1$$

Special values

$$K(0) = E(0) = \frac{\pi}{2}, \quad \lim_{\varepsilon \rightarrow 0} K(1 - \varepsilon) = \ln\left(\frac{4}{\sqrt{\varepsilon}}\right), \quad E(1) = 1$$

$$K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - x \sin^2 \theta}} d\theta$$

$$E(x) = \int_0^{\pi/2} \sqrt{1 - x \sin^2 \theta} d\theta$$



$K(k^2)$  (thick line) and  $E(k^2)$  (thin line).

Relationship between  $K$  and  $E$

$$E(k^2) = (1 - k^2) \frac{d}{dk} [kK(k^2)]$$

Integrals that can be reduced to the elliptic integrals

$$\int_0^{\pi/2} \frac{\sin^2 \theta}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta = \frac{1}{k^2} [K(k^2) - E(k^2)]$$

$$\int_0^{\pi/2} \frac{\cos^2 \theta}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta = \frac{1}{k^2} [E(k^2) - (1 - k^2)K(k^2)]$$

$$\int_0^{\pi/2} \frac{1}{(1 - k^2 \sin^2 \theta)^{3/2}} d\theta = \frac{1}{1 - k^2} E(k^2) = \frac{d}{dk} [kK(k^2)]$$

$$\int_0^{\pi/2} \frac{\sin^2 \theta}{(1 - k^2 \sin^2 \theta)^{3/2}} d\theta = \frac{1}{k^2(1 - k^2)} [E(k^2) - (1 - k^2)K(k^2)]$$

$$\int_0^{\pi/2} \frac{\cos^2 \theta}{(1 - k^2 \sin^2 \theta)^{3/2}} d\theta = \frac{1}{k^2} [K(k^2) - (1 - k^2)E(k^2)]$$

## Series Expansion of Elementary Functions

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots, \quad |x| < 1$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots$$

$$\begin{aligned} \cosh x &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots \\ \sinh x &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots \\ \tanh x &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots \\ \ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots, \quad |x| < 1 \end{aligned}$$

## Infinite Products

$$\prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) = \frac{\sinh \pi x}{\pi x}$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) = \frac{\sin \pi x}{\pi x}$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{x^2}{(2n-1)^2}\right) = \cosh \frac{\pi x}{2}$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(2n-1)^2}\right) = \cos \frac{\pi x}{2}$$

$$\prod_{n=-\infty}^{\infty} \left(1 + \frac{x^2}{(a-2n\pi)^2}\right) = \frac{\cosh x - \cos a}{1 - \cos a}$$

$$\prod_{n=-\infty}^{\infty} \left(1 - \frac{x^2}{(a-2n\pi)^2}\right) = \frac{\cos x - \cos a}{1 - \cos a}$$