Mathematical Formulae

1. Vector Formulae

Bold characters are vector functions and f is a scalar function.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

$$\nabla\cdot(f\mathbf{A})=\nabla f\cdot\mathbf{A}+f\nabla\cdot\mathbf{A}$$

$$\nabla\times(f\mathbf{A})=\nabla f\times\mathbf{A}+f\nabla\times\mathbf{A}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}+(\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}\nabla\cdot\mathbf{B}-\mathbf{B}\nabla\cdot\mathbf{A}+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\nabla\times\nabla f\equiv 0$$

$$\nabla\cdot(\nabla\times\mathbf{A})\equiv 0$$

$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$$

$$\nabla\cdot\mathbf{r}=3,\ \mathbf{r}=\text{ position vector}$$

$$\nabla\times\mathbf{r}=0$$

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}=-4\pi\delta(\mathbf{r}-\mathbf{r}')$$

$$\text{Substantive derivative: }\frac{df}{dt}=\frac{\partial f}{\partial t}+(\mathbf{v}\cdot\nabla)f,\ \mathbf{v}=\text{ velocity}$$

$$\text{Substantive derivative: }\frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{A}$$

$$\text{Substantive derivative: }\frac{d\mathbf{v}}{dt}=\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{\partial\mathbf{v}}{\partial t}+\frac{1}{2}\nabla v^2+(\nabla\times\mathbf{v})\times\mathbf{v}$$

$$\text{Gauss' theorem: }\int_V\nabla\cdot\mathbf{A}dV=\oint_S\mathbf{A}\cdot d\mathbf{S}$$

$$\text{Stokes' theorem }\int_S\nabla\times\mathbf{A}\cdot d\mathbf{S}=\oint_C\mathbf{A}\cdot d\mathbf{l}$$

$$\oint_S(\nabla\times\mathbf{A})\cdot d\mathbf{S}=0\text{ for closed surface}$$

$$\int_V\nabla\times\mathbf{A}dV=\oint_S d\mathbf{S}\times\mathbf{A}=-\oint_S\mathbf{A}\times d\mathbf{S}$$

2. Delta Function

$$\int f(t)\delta(t-t')dt = f(t')$$

$$\delta(at) = \frac{1}{|a|}\delta(t), \quad \int g(t)\delta[f(t)]dt = \frac{g(t')}{\left|\frac{df}{dt}\right|_{t=t'}} \text{ where } f(t')=0$$

$$\begin{aligned}
& \delta(\mathbf{r}-\mathbf{r}') \text{ (3-D delta function)} \\
= \; & \delta(x-x')\delta(y-y')\delta(z-z') \text{ (Cartesian)} \\
= \; & \frac{\delta(r-r')}{rr'}\frac{\delta(\theta-\theta')}{\sin\theta}\delta(\phi-\phi') \text{ (spherical)} \\
= \; & \frac{\delta(r-r')}{rr'}\delta(\cos\theta-\cos\theta')\delta(\phi-\phi') \text{ (spherical)} \\
= \; & \frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')\delta(z-z') \text{ (cylindrical)}
\end{aligned}$$

3. Curvilinear Coordinates

Let $u_i(x,y,z)$ $(i=1,2,3)$ be a system of curvilinear coordinates. The metric coefficients are

$$h_i = \sqrt{\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2},$$

and the length segments in each direction are $h_i du_i \mathbf{e}_i$ ($\mathbf{e}_i$ unit vector).
Area elements

$$d\mathbf{S}_i = h_j h_k du_j du_k \mathbf{e}_j \times \mathbf{e}_k$$

Volume element

$$dV = h_1 h_2 h_3 du_1 du_2 du_3$$

Gradient

$$\nabla f = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial u_i}$$

Divergence

$$\nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right\}$$

Curl

$$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{\mathbf{e}_1}{h_2h_3} & \frac{\mathbf{e}_2}{h_3h_1} & \frac{\mathbf{e}_3}{h_1h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

Scalar Laplacian

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right\}$$

Vector Laplacian (definition)

$$\nabla^2 \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

Spherical Coordinates (r, θ, ϕ)

Transformation

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

Metric coefficients

$$h_r = 1,\ h_\theta = r,\ h_\phi = r \sin\theta.$$

Transformation of the unit vectors

$$\begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{bmatrix}$$

For example,

$$\begin{aligned} \mathbf{e}_x &= \nabla x = \nabla(r \sin\theta \cos\phi) \\ &= \sin\theta\cos\phi \mathbf{e}_r + \cos\theta\cos\phi \mathbf{e}_\theta - \sin\phi \mathbf{e}_\phi \end{aligned}$$

Derivative of the unit vectors

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta,\ \frac{\partial \mathbf{e}_r}{\partial \phi} = \sin\theta \mathbf{e}_\phi,\ \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r,\ \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos\theta \mathbf{e}_\phi,\ \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\sin\theta \mathbf{e}_r - \cos\theta \mathbf{e}_\theta.$$

Gradient

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Curl

$$\nabla \times \mathbf{A} = \begin{vmatrix} \dfrac{\mathbf{e}_r}{r^2\sin\theta} & \dfrac{\mathbf{e}_\theta}{r\sin\theta} & \dfrac{\mathbf{e}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

Scalar Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Vector Laplacian

$$\begin{aligned}
\nabla^2 \mathbf{A} &\equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \\
&= \left(\nabla^2 A_r - \frac{2}{r^2} A_r - \frac{2}{r^2} \frac{\partial A_\theta}{\partial \theta} - \frac{2 \cot \theta}{r^2} A_\theta - \frac{2}{r^2 \sin \theta} \frac{\partial A_\phi}{\partial \phi} \right) \mathbf{e}_r \\
&\quad + \left(\nabla^2 A_\theta - \frac{1}{r^2 \sin^2 \theta} A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\phi}{\partial \phi} \right) \mathbf{e}_\theta \\
&\quad + \left(\nabla^2 A_\phi - \frac{1}{r^2 \sin^2 \theta} A_\phi + \frac{2}{r^2 \sin \theta} \frac{\partial A_r}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\theta}{\partial \phi} \right) \mathbf{e}_\phi
\end{aligned}$$

Note that in non-cartesian coordinates,

$$(\nabla^2 \mathbf{A})_i \neq \nabla^2 A_i$$

Cylindrical Coordinates (ρ, ϕ, z)

Transformation

$$\begin{aligned}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= z
\end{aligned}$$

Metric coefficients

$$h_\rho = 1, \; h_\phi = \rho, \; h_z = 1$$

Derivatives of the unit vectors

$$\frac{\partial \mathbf{e}_\rho}{\partial \phi} = \mathbf{e}_\phi, \; \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_\rho$$

Gradient

$$\nabla f = \frac{\partial f}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{e}_z$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Curl

$$\nabla \times \mathbf{A} = \begin{vmatrix} \dfrac{\mathbf{e}_\rho}{\rho} & \mathbf{e}_\phi & \dfrac{\mathbf{e}_z}{\rho} \\[2ex] \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\[2ex] A_\rho & \rho A_\rho & A_z \end{vmatrix}$$

Scalar Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

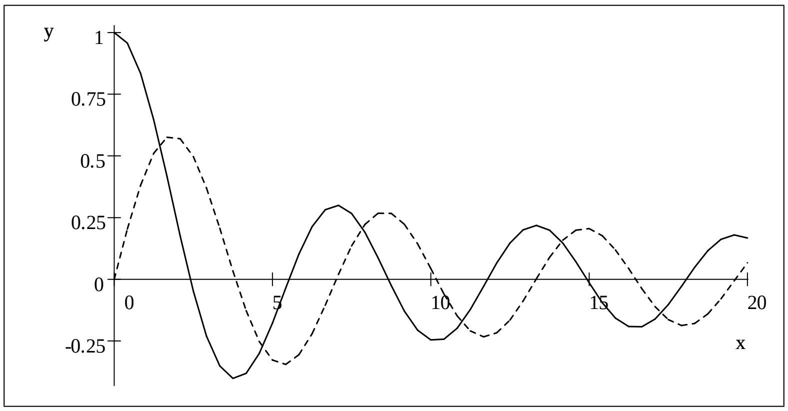
Vector Laplacian

$$\begin{aligned}\nabla^2\mathbf{A} &\equiv \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) \\ &= \left(\nabla^2 A_\rho - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi} - \frac{1}{\rho^2}A_\rho\right)\mathbf{e}_\rho \\ &\quad + \left(\nabla^2 A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi} - \frac{1}{\rho^2}A_\phi\right)\mathbf{e}_\phi + \nabla^2 A_z\mathbf{e}_z\end{aligned}$$

4. Special Functions

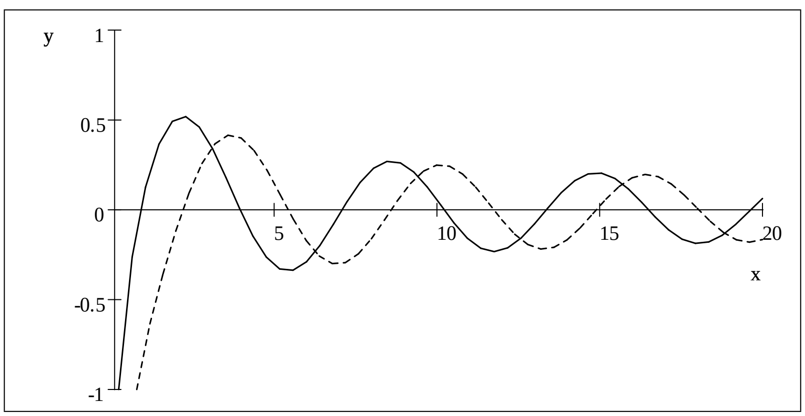
Bessel Functions $Z_m(x) = J_m(x),\ N_m(x)$ **[or** $Y_m(x)$ **in some books]**

$J_0(x), J_1(x)$



$J_0(x)$ (solid line) and $J_1(x)$ (dashed line).

$Y_0(x), Y_1(x)$



$Y_0(x)$ (solid line) and $Y_1(x)$ (dashed line).

Differential equation

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + k^2 - \frac{m^2}{\rho^2}\right)Z_m(k\rho) = 0$$

Wronskian

$$J_m(x)N_m'(x) - J_m'(x)N_m(x) = \frac{2}{\pi x}$$

Series representation of $J_m(x)$

$$\begin{aligned}
J_m(x) &= \left(\frac{x}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{(m+n)!n!}\left(\frac{x}{2}\right)^{2n}, \ J_{-m}(x) = (-1)^m J_m(x) \\
J_m(x) &\simeq \left(\frac{x}{2}\right)^m \text{ for small } x \ll 1 \\
J_0(x) &\simeq 1 - \tfrac{1}{4}x^2 + \tfrac{1}{64}x^4 \text{ for small } x \ll 1 \\
J_1(x) &\simeq \tfrac{1}{2}x - \tfrac{1}{16}x^3 \text{ for small } x \ll 1
\end{aligned}$$

Generating functions

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{m=-\infty}^{\infty} J_m(x) t^m$$

$$e^{ix\sin\theta} = \sum_{m=-\infty}^{\infty} J_m(x) e^{im\theta}$$

n-th root of $J_0(x) = 0$, x_{0n}, and value of $J_1(x_{0n})$

x_{0n}	2.40483	5.52008	8.65373	11.79153	14.93092	18.07106
$J_1(x_{0n})$	0.51915	−0.34026	0.27145	−0.23246	0.20655	−0.18773

n-th root of $J_1(x) = 0$, x_{1n}, and value of $J_0(x_{1n})$

x_{1n}	3.83171	7.01559	10.17347	13.32369	16.47063	19.61586
$J_0(x_{1n})$	−0.40276	0.30012	−0.24970	0.21836	−0.19647	0.18006

For small x ($\ll 1$),

$$J_m(x) \simeq \left(\frac{x}{2}\right)^m, \quad N_m(x) \simeq \frac{2}{\pi}\left(\ln\frac{x}{2} + \gamma_{\rm E}\right), \ \gamma_{\rm E} = 0.5772\cdots \ \text{(Euler's constant)}.$$

Hankel functions of the first and second kinds

$$H_m^{(1,2)}(x) = J_m(x) \pm i N_m(x)$$

Asymptotic forms at large $x \gg 1$

$$J_m(x) \simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{2m+1}{4}\pi\right)$$

$$N_m(x) \simeq \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{2m+1}{4}\pi\right)$$

$$H_m^{(1,2)}(x) \simeq \sqrt{\frac{2}{\pi x}} \exp\left[\pm i\left(x - \frac{2m+1}{4}\pi\right)\right]$$

Recurrence formulae

$$Z_0'(x) = -Z_1(x)$$

$$Z_m'(x) = \frac{1}{2}\left[Z_{m-1}(x) - Z_{m+1}(x)\right]$$

$$Z_m(x) = \frac{x}{2m}\left[Z_{m+1}(x) + Z_{m-1}(x)\right]$$

$$\frac{d}{dx}\left[x^m Z_m(x)\right] = x^m Z_{m-1}(x)$$

Integral representations (There are many. A few are listed.)

$$J_m(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(m\theta - x\sin\theta)d\theta, \quad J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos(xt)}{\sqrt{1-t^2}}dt$$

$$N_0(x) = -\frac{2}{\pi}\int_1^\infty \frac{\cos(xt)}{\sqrt{1-t^2}}dt$$

$$N_m(x) = -\frac{2^{m+1}x^{-m}}{\sqrt{\pi}\Gamma(\frac{1}{2}-m)}\int_1^\infty \frac{\cos xt}{(t^2-1)^{m+(1/2)}}dt$$

Integrals

$$\int_0^\infty J_m(ax)dx = \frac{1}{a}, \quad \int_0^\infty N_m(ax)dx = -\frac{1}{a}\tan\left(\frac{m\pi}{2}\right)$$

$$\int_0^\infty J_0(ax)J_0(bx)dx = \frac{2}{\pi b}K(a/b), \;\; K: \text{ complete elliptic integral of the first kind}$$

$$\int_0^\infty J_0(ax)J_1(bx)dx = \begin{cases} 1/b, & b > a > 0 \\ 1/2b, & a = b > 0 \\ 0, & a > b > 0 \end{cases} \quad \text{(step function)}$$

$$\int_0^\infty xJ_1(ax)J_1(bx)dx = \frac{\delta(a-b)}{a}, \;\; \text{(derivative of the above with respect to } a\text{)}$$

In fact for any integer m,

$$\int_0^\infty xJ_m(ax)J_m(bx)dx = \frac{\delta(a-b)}{a}$$

$$\int_0^\infty J_{\mu+x}(ax)J_{\nu-x}(ax)dx = J_{\mu+\nu}(2a), \;\; \mu+\nu > 1$$

$$\int_0^\infty x^{\mu-1}J_\nu(ax)dx = \frac{2^{\mu-1}\Gamma\left(\frac{\mu+\nu}{2}\right)}{a^\mu\Gamma\left(\frac{\nu-\mu}{2}+1\right)}, \;\; \Gamma: \text{gamma function}$$

$$\int_0^\infty \frac{J_\nu(ax)J_\nu(bx)}{x^2-y^2}dx = \frac{i\pi}{2}J_\nu(by)H_\nu^{(1)}(ay)$$

$$\int_0^\infty e^{-ax}J_0(bx)dx = \frac{1}{\sqrt{a^2+b^2}}$$

$$\int_0^\infty e^{-ax}J_\nu(bx)dx = \frac{(\sqrt{a^2+b^2}-a)^\nu}{b^\nu\sqrt{a^2+b^2}}$$

$$\int_0^\infty e^{-ax}J_\nu(bx)J_\nu(cx)dx = \frac{1}{\pi\sqrt{bc}}Q_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2bc}\right), \;\; Q_\mu: \text{Legendre function of the 2nd kind}$$

$$\int_0^\infty e^{-a^2x^2}J_{2\nu}(px)dx = \frac{\sqrt{\pi}}{2a}\exp\left(-\frac{b^2}{8a^2}\right)I_\nu\left(\frac{b^2}{8a^2}\right)$$

$$\int_0^\infty e^{-a^2x^2}x^2J_0(bx)dx = \frac{1}{2a^2}\exp\left(-\frac{b^2}{4a^2}\right)$$

$$\int_0^\infty e^{-a^2x^2}J_\nu(px)J_\nu(qx)dx = \frac{1}{2a^2}\exp\left(-\frac{p^2+q^2}{4a^2}\right)I_\nu\left(\frac{pq}{2a^2}\right)$$

$$\int_0^\infty \sin(ax)J_\nu(bx)dx = \begin{cases} \frac{1}{\sqrt{b^2-a^2}} \sin\left[\nu \sin^{-1}(a/b)\right], & b > a \\ \frac{b^\nu}{\sqrt{a^2-b^2}(a+\sqrt{a^2-b^2})^\nu} \cos\left(\frac{\nu\pi}{2}\right), & a < b \end{cases}$$

$$\int_0^\infty \sin(ax)J_0(bx)dx = \begin{cases} 0, & b > a \\ \dfrac{1}{\sqrt{a^2-b^2}}, & a > b \end{cases}$$

Sum

$$J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x) = 1$$

$$\sum_{n=-\infty}^{\infty} J_n^2(x) = 1$$

$$\sum_{n=0}^{\infty} n^2 J_n^2(x) = \frac{x^2}{4}$$

$$J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x) = \cos x$$

$$\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x) = \frac{1}{2}\sin x$$

$$\sum_{n=0}^{\infty}(2n+1)J_{2n+1}(x) = \frac{x}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} J_{2n}(2nx) = \frac{x^2}{2}$$

$$\sum_{n=1}^{\infty} J_{2n}(2nx) = \frac{x^2}{2(1-x^2)}$$

$$\sum_{n=1}^{\infty} n^2 J_{2n}(2nx) = \frac{x^2(1+x^2)}{2(1-x^2)^4}$$

$$\sum_{n=1}^{\infty} n^2 \int_0^x J_{2n}(2nt)dt = \frac{x^3}{6(1-x^2)^3}$$

Spherical Bessel Functions $z_l(x) = j_l(x),\ n_l(x)$

Spherical Bessel functions are elementary functions. Some low order forms are:

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x - x\cos x}{x^2}, \quad j_2(x) = \frac{(3-x^2)\sin x - 3x\cos x}{x^3}$$

$$n_0(x) = -\frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x + x\sin x}{x^2}, \quad n_2(x) = -\frac{(3-x^2)\cos x + 3x\sin x}{x^3}$$

Definition

$$j_l(x) \equiv \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x), \quad n_l(x) \equiv \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x)$$

$$h_l^{(1,2)}(x) = j_l(x) \pm i n_l(x)$$

Differential equation

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}\right) z_l(kr) = 0$$

Wronskian

$$j_l(x) n_l'(x) - j_l'(x) n_l(x) = \frac{1}{x^2}$$

For $x \ll 1$,

$$j_l(x) \simeq \frac{x^l}{(2l+1)!!}, \quad n_l(x) \simeq -\frac{(2l-1)!!}{x^{l+1}}$$

For $x \gg 1$,

$$j_l(x) \simeq \frac{1}{x}\cos\left(x - \frac{l+1}{2}\pi\right), \quad n_l(x) \simeq \frac{1}{x}\sin\left(x - \frac{l+1}{2}\pi\right)$$

$$h_l^{(1)}(x) \simeq (-i)^{l+1}\frac{e^{ix}}{x}, \quad h_l^{(2)}(x) \simeq i^{l+1}\frac{e^{-ix}}{x}$$

Recurrence formulae

$$(2l+1) z_l(x) = x[z_{l-1}(x) - z_{l+1}(x)]$$

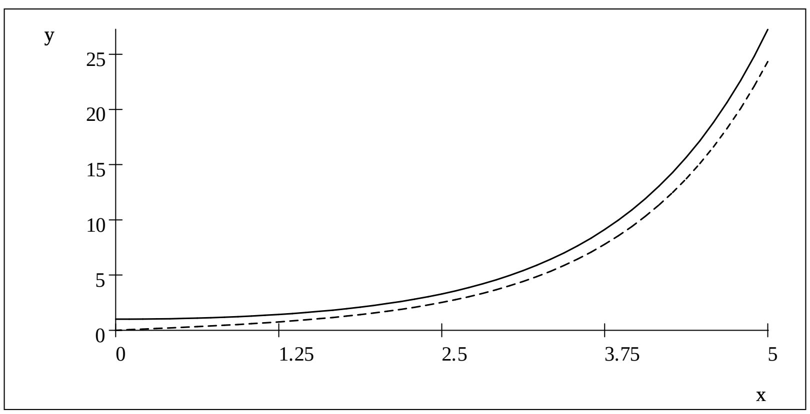
$$(2l+1)\frac{dz_l(x)}{dx} = l z_{l-1}(x) - (l+1) z_{l+1}(x)$$

$$\frac{d}{dx}[x^{l+1} z_l(x)] = x^{l+1} z_{l-1}(x)$$

$$\frac{d}{dx}[x^{-l} z_l(x)] = -x^{-l} z_{l+1}(x)$$

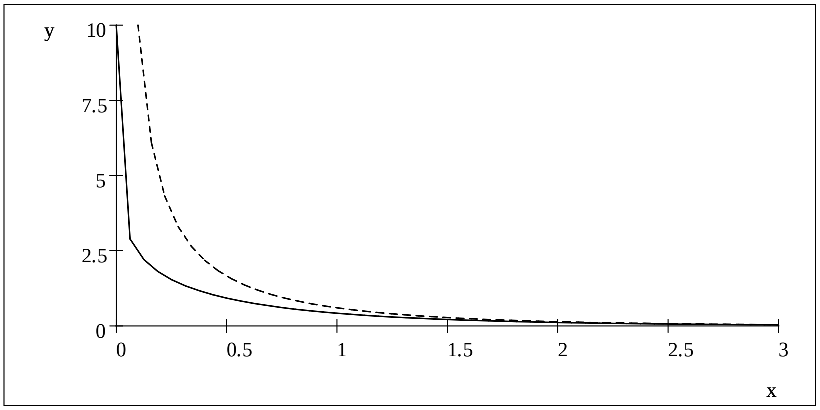
Modified Bessel Functions $I_m(x),\ K_m(x)$

$$I_0(x), I_1(x)$$



$I_0(x)$ (solid line) and $I_1(x)$ (dashed line).

$$K_0(x), K_1(x)$$



$K_0\,(x)$ (solid line) and $K_1\,(x)$ (dashed line).

Definition

$$\begin{aligned} I_m(x) &= e^{-im\pi/2} J_m(ix) \\ &= \left(\frac{x}{2}\right)^m \sum_0^\infty \frac{(x/2)^{2n}}{n!(m+n)!} \end{aligned}$$

$$\begin{aligned} K_m(x) &= (-1)^{m+1} I_m(x) \left(\gamma + \ln \frac{x}{2}\right) + \frac{(-1)^m}{2} \sum_{k-0}^\infty \frac{(x/2)^{m+2k}}{k!(m+k)!} \left[\sum_{n=1}^k \frac{1}{n} + \sum_{n=1}^{k+m} \frac{1}{n}\right] \\ &+ \frac{1}{2} \sum_{r=0}^{m-1} (-1)^r \frac{(m-r-1)!}{r!} \left(\frac{x}{2}\right)^{2r-m} \end{aligned}$$

Differential equation

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + k^2 + \frac{m^2}{\rho^2}\right) \left(\begin{array}{c} I_m(k\rho) \\ K_m(k\rho) \end{array}\right) = 0$$

Wronskian

$$I_m'(x) K_m(x) - I_m(x) K_m'(x) = \frac{1}{x}$$

Series representation of $I_m(x)$

$$I_m(x) = \left(\frac{x}{2}\right)^m \sum_{n=0}^\infty \frac{1}{m!(m+n)!} \left(\frac{x}{2}\right)^{2n}$$

For $x \ll 1$

$$I_0(x) \simeq 1 + \frac{1}{4}x^2 + \cdots$$

$$I_1(x) \simeq \frac{x}{2} + \frac{1}{16}x^3 + \cdots$$

Recurrence formulae

$$I_{m-1}(x) - I_{m+1}(x) = \frac{2m}{x} I_m(x), \quad I_{m-1}(x) - I_{m+1}(x) = 2I_m'(x)$$

$$K_{m-1}(x) - K_{m+1}(x) = -\frac{2m}{x} K_m(x), \quad K_{m-1}(x) + K_{m+1}(x) = -2K_m'(x)$$

$$I_0'(x) = I_1(x), \quad K_0'(x) = -K_1(x)$$

Integral representation

$$K_0(x) = \int_0^\infty \frac{tJ_0(xt)}{t^2+1}dt = \int_0^\infty \frac{\cos xt}{\sqrt{t^2+1}}dt$$

$$K_1(x) = -K_0'(x) = \int_0^\infty \frac{t^2 J_1(xt)}{1+t^2}dt$$

$$K_{1/3}\left(\frac{2x^{3/2}}{3^{3/2}}\right) = \frac{3}{\sqrt{x}}\int_0^\infty \cos(t^3+xt)dt, \text{ (Airy's integral)}$$

Legendre Functions $P_l^m(x)$, $Q_l^m(x)$

Differential equation

$$\left((1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + l(l+1) - \frac{m^2}{1-x^2}\right)\begin{pmatrix} P_l^m(x) \\ Q_l^m(x) \end{pmatrix} = 0$$

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \text{ (Rodrigues' formula)}$$

$$Q_l(x) = \frac{1}{2}P_l(x)\ln\frac{1+x}{1-x} - W_{l-1}(x), \ x \text{ real and } |x| \le 1$$

$$Q_l(z) = \frac{1}{2}P_l(z)\ln\frac{z+1}{z-1} - W_{l-1}(z) \text{ for general complex } z$$

$$W_{-1}(x) = 0,\ W_0(x) = 1,\ W_1(x) = \frac{3}{2}x,\ W_2(x) = \frac{5}{2}x^2 - \frac{2}{3}, \cdots$$

Special values ($m = 0$)

$$P_l(1) = 1,\ P_l(-1) = (-1)^l$$

$$P_l(0) = 0 \text{ for odd } l,\ P_l(0) = (-1)^{l/2}\frac{(l-1)!!}{l!!} \text{ for even } l$$

$$P_l'(1) = \frac{l(l+1)}{2},\ P_l'(0) = -(l+1)P_{l+1}(0)$$

Definition of $P_l^m(x)$, $Q_l^m(x)$ in terms of $P_l(x)$, $Q_l(x)$ (x real, $|x| \le 1$)

$$P_l^m(x) = \left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_l(x), \quad Q_l^m(x) = \left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}Q_l(x)$$

For general complex argument z

$$P_l^m(z) = \left(z^2-1\right)^{m/2}\frac{d^m}{dz^m}P_l(z), \quad Q_l^m(z) = \left(z^2-1\right)^{m/2}\frac{d^m}{dz^m}Q_l(z)$$

Orthogonality of $P_l^m(x)$

$$\int_{-1}^1 \frac{P_l^m(x)P_l^{m'}(x)}{1-x^2}dx = \frac{1}{m}\frac{(l+m)!}{(l-m)!}\delta_{mm'}$$

$$\int_{-1}^1 P_l^m(x)P_{l'}^m(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}$$

Wronskian

$$m=0,\ P_l(x)Q_l'(x)-P_l'(x)Q_l(x)=\frac{1}{1-x^2}$$

$$\begin{aligned}P_l^m(x)\frac{dQ_l^m(x)}{dx}-\frac{dP_l^m(x)}{dx}Q_l^m(x) &= \frac{2^{2m}}{1-x^2}\frac{\Gamma[(l+m+1)/2]\Gamma[(l+m)/2+1]}{\Gamma[(l-m+1)/2]\Gamma[(l-m)/2+1]}\\ &= \frac{1}{1-x^2}\frac{(l+m)!}{(l-m)!},\ 0\le m\le l\end{aligned}$$

Generating function

$$(\cos\theta+i\sin\theta\cos\phi)^l=P_l(\cos\theta)+2\sum_{m=1}^{l} i^m\frac{l!}{(l+m)!}P_l^m(\cos\theta)\cos(m\phi)$$

Low order forms of $P_l^m(x)$, $(x=\cos\theta)$
$m=0$

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{3x^2-1}{2},\ P_3(x)=\frac{5x^3-3x}{2},\cdots$$

$l=1$, $m=1$

$$P_1^1(x)=\sqrt{1-x^2}=\sin\theta$$

$l=2$, $m=1,2$

$$\begin{aligned}P_2^1(x) &= 3\sqrt{1-x^2}x=3\sin\theta\cos\theta=\frac{3}{2}\sin 2\theta\\ P_2^2(x) &= 3(1-x^2)=3\sin^2\theta=\frac{3}{2}(1-\cos 2\theta)\end{aligned}$$

$l=3$, $m=1,2,3$

$$\begin{aligned}P_3^1(x) &= \frac{3}{2}\sqrt{1-x^2}(5x^2-1)=\frac{3}{8}(\sin\theta+5\sin 3\theta)\\ P_3^2(x) &= 15(1-x^2)x=\frac{15}{4}(\cos\theta-\cos 3\theta)\\ P_3^3(x) &= 15(1-x^2)^{3/2}=15\sin^3\theta=\frac{15}{4}(2\sin 2\theta+7\sin 4\theta)\end{aligned}$$

$Q_l^m(x)$ (associated Legendre function of the second kind)

Special case $z=i\sinh\eta$

$$Q_0(i\sinh\eta)=-i\cot^{-1}(\sinh\eta),\ Q_1(i\sinh\eta)=\sinh\eta\cot^{-1}(\sinh\eta)-1$$

$$Q_1^1(i\sinh\eta)=\cosh\eta\cot^{-1}(\sinh\eta)-\tanh\eta$$

Spherical harmonic function $Y_{lm}(\theta,\phi)$

$$\text{Positive } m: Y_{lm}(\theta,\phi)=(-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}\text{ for }0\le m\le l$$

$$\text{Negative } m: Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi) \text{ for } -l \le -m \le 0$$

General form

$$Y_{lm}(\theta,\phi) = (-1)^{\frac{m+|m|}{2}} \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos\theta) e^{im\phi}, \text{ for } -l \le m \le l$$

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$\begin{aligned} Y_{10}(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_{1,\pm1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \end{aligned}$$

$$\begin{aligned} Y_{20}(\theta,\phi) &= \sqrt{\frac{5}{4\pi}}\frac{1}{2}(3\cos^2\theta - 1) \\ Y_{2,\pm1}(\theta,\phi) &= \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi} \\ Y_{2,\pm2}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm i2\phi} \end{aligned}$$

Orthogonality of $Y_{lm}(\theta,\phi)$

$$\oint Y_{lm}(\theta,\phi) Y^*_{l'm'}(\theta,\phi) d\Omega = \delta_{ll'}\delta_{mm'}, \; d\Omega = \sin\theta d\theta d\phi$$

Toroidal functions $P_{l-\frac{1}{2}}(\cosh\eta)$, $Q_{l-\frac{1}{2}}(\cosh\eta)$ satisfy

$$\left(\frac{d^2}{d\eta^2} + \coth\eta\frac{d}{d\eta} - l^2 + \frac{1}{4} - m^2 \text{cosech}^2\eta\right) F(\eta) = 0$$

Integral representations

$$\begin{aligned} P^m_{l-\frac{1}{2}}(\cosh\eta) &= \frac{(-1)^m(2l-1)!!}{\pi 2^{m+1}(2l-2m-1)!!}\int_0^\pi \frac{\cos m\theta}{(\cosh\eta + \cos\theta\sinh\eta)^{l+\frac{1}{2}}} d\theta \\ Q^m_{l-\frac{1}{2}}(\cosh\eta) &= \frac{(-1)^m(2l-1)!!}{2^{m+1}(2l-2m-1)!!}\int_0^\infty \frac{\cosh mt}{(\cosh\eta + \cosh t\sinh\eta)^{l+\frac{1}{2}}} dt \end{aligned}$$

Gamma Function

Definition

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Properties

$$\Gamma(z+1) = z\Gamma(z), \; \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \; \Gamma\left(z+\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)}$$

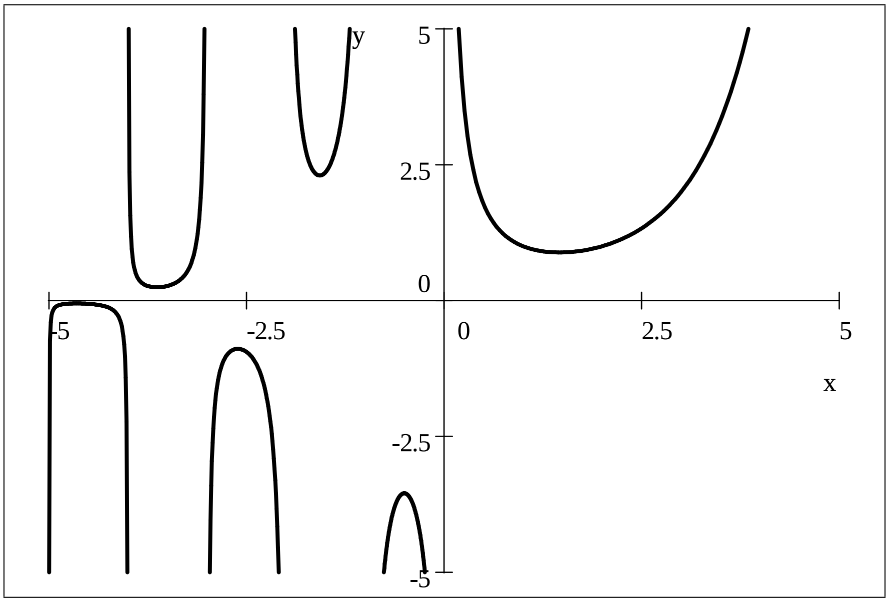
If z is a positive integer, $z = n$,

$$\Gamma(n+1) = n!$$

Special values

$$\Gamma(1) = 1,\ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi},\ \Gamma\left(n + \tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

$\Gamma(x)$



Gamma function $\Gamma(x)$.

Elliptic Integrals $K(k^2)$ and $E(k^2)$

Complete Elliptic Integrals of the First Kind $K(k^2)$ and Second Kind $E(k^2)$

Definitions

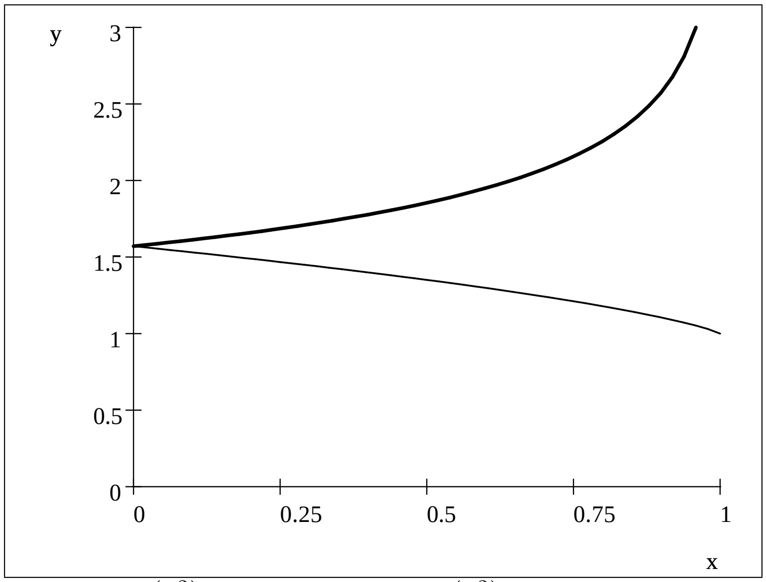
$$K\left(k^2\right) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2\theta}} d\theta,\ E\left(k^2\right) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, d\theta,\ 0 \le k^2 \le 1$$

Special values

$$K(0) = E(0) = \frac{\pi}{2},\ \lim_{\varepsilon \to 0} K(1-\varepsilon) = \ln\left(\frac{4}{\sqrt{\varepsilon}}\right),\ E(1) = 1$$

$$K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - x \sin^2\theta}} d\theta$$

$$E(x) = \int_0^{\pi/2} \sqrt{1 - x \sin^2\theta}\, d\theta$$



$K\left(k^2\right)$ (thick line) and $E\left(k^2\right)$ (thin line).

Relationship between K and E

$$E(k^2) = \left(1 - k^2\right) \frac{d}{dk}\left[kK\left(k^2\right)\right]$$

Integrals that can be reduced to the elliptic integrals

$$\int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{1 - k^2\sin^2\theta}} d\theta = \frac{1}{k^2}\left[K\left(k^2\right) - E\left(k^2\right)\right]$$

$$\int_0^{\pi/2} \frac{\cos^2\theta}{\sqrt{1 - k^2\sin^2\theta}} d\theta = \frac{1}{k^2}\left[E\left(k^2\right) - (1 - k^2)K\left(k^2\right)\right]$$

$$\int_0^{\pi/2} \frac{1}{(1 - k^2\sin^2\theta)^{3/2}} d\theta = \frac{1}{1 - k^2} E\left(k^2\right) = \frac{d}{dk}\left[kK\left(k^2\right)\right]$$

$$\int_0^{\pi/2} \frac{\sin^2\theta}{(1 - k^2\sin^2\theta)^{3/2}} d\theta = \frac{1}{k^2\left(1 - k^2\right)}\left[E\left(k^2\right) - (1 - k^2)K\left(k^2\right)\right]$$

$$\int_0^{\pi/2} \frac{\cos^2\theta}{(1 - k^2\sin^2\theta)^{3/2}} d\theta = \frac{1}{k^2}\left[K\left(k^2\right) - (1 - k^2)E\left(k^2\right)\right]$$

Series Expansion of Elementary Functions

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \quad |x| < 1$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 +$$

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$$

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots, \quad |x| < 1$$

Infinite Products

$$\prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) = \frac{\sinh \pi x}{\pi x}$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) = \frac{\sin \pi x}{\pi x}$$

$$\prod_{n=1}^{\infty} \left(1 + \frac{x^2}{(2n-1)^2}\right) = \cosh \frac{\pi x}{2}$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(2n-1)^2}\right) = \cos \frac{\pi x}{2}$$

$$\prod_{n=-\infty}^{\infty} \left(1 + \frac{x^2}{(a-2n\pi)^2}\right) = \frac{\cosh x - \cos a}{1 - \cos a}$$

$$\prod_{n=-\infty}^{\infty} \left(1 - \frac{x^2}{(a-2n\pi)^2}\right) = \frac{\cos x - \cos a}{1 - \cos a}$$