Chapter 1

Electrostatics I. Potential due to Prescribed Charge Distribution, Dielectric Properties, Electric Energy and Force

1.1 Introduction

In electrostatics, charges are assumed to be stationary. Electric charges exert force on other charges through Coulomb’s law which is the generic law in electrostatics. For a given charge distribution, the electric field and scalar potential can be calculated by applying the principle of superposition.

Dielectric properties of matter can be analyzed as a collection of electric dipoles. Atoms having no permanent dipole moment can be polarized if placed in an electric field. Most molecules have permanent dipole moments. In the absence of electric field, dipole moments are oriented randomly through thermal agitation. In an electric field, permanent dipoles tend to align themselves in the direction of the applied field and weaken the field. Some crystals exhibit anisotropic polarization and the permittivity becomes a tensor. The well known double diffraction phenomenon occurs through deviation of the group velocity from the phase velocity in direction as well as in magnitude.

1.2 Coulomb’s Law

Normally matter is charge-neutral macroscopically. However, charge neutrality can be broken relatively easily by such means as mechanical friction, bombardment of cosmic rays, heat (e.g., candle flame is weakly ionized), etc. The first systematic study of electric force among charged bodies was made by Cavendish and Coulomb in the 18th century. (Cavendish’s work preceded Coulomb’s. However, Coulomb’s work was published earlier. Lesson: it is important to publish your work as early as possible.) The force to act between two charges, \( q_1 \) at \( r_1 \) and \( q_2 \) at \( r_2 \), follows
the well known Coulomb’s law,

\[ F = \text{const.} \frac{q_1 q_2}{r^2} = \text{const.} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}, \quad (\text{N}) \quad (1.1) \]

where \( r = |\mathbf{r}_1 - \mathbf{r}_2| \) is the relative distance between the charges. In CGS-ESU (cm-gram-second electrostatic unit) system, the constant is chosen to be unity. Namely, if two equal charges \( q_1 = q_2 = q \) separated by \( r = 1 \text{ cm} \) exert a force of 1 dyne \( = 10^{-5} \text{ N} \) on each other, the charge is defined as 1 ESU \( \approx \frac{1}{3} \times 10^{-9} \text{ C} \). The electronic charge in ESU is \( e = 4.8 \times 10^{-10} \text{ ESU} \). In MKS-Ampere (or SI) unit system, 1 Coulomb of electric charge is defined from

\[ 1 \text{ Coulomb} = 1 \text{ Ampere} \cdot 1 \text{ sec}, \]

where 1 Ampere of electric current is defined as follows. If two infinitely long parallel currents of equal amount separated by 1 meter exert a force per unit length of \( 2 \times 10^{-7} \text{ N/m} \) on each other (attracting if the currents are parallel and repelling if antiparallel), the current is defined to be 1 Ampere. Since the force per unit length to act on two parallel currents \( I_1 \) and \( I_2 \) separated by a distance \( d \) is given by

\[ \frac{F}{l} = B_1 I_2 = \frac{\mu_0 I_1 I_2}{2\pi d}, \quad (\text{N/m}) \quad (1.2) \]

the magnetic permeability \( \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \) in MKS-Ampere unit system is an assigned constant introduced to define 1 Ampere current. (The permittivity \( \varepsilon_0 \) is a measured constant that should be determined experimentally.)

In MKS-Ampere unit system, the measured proportional constant in Coulomb’s law is approximately \( 9.0 \times 10^9 \text{ N m}^2 \text{ C}^{-2} \). It is customary to write the constant in the form

\[ \text{const.} = \frac{1}{4\pi \varepsilon_0}, \]
Figure 1-2: In MKS unit system, \( I = 1 \) Ampere current is defined if the force per unit length between infinite parallel currents 1 m apart is \( 2 \times 10^{-7} \) N/m. The magnetic permeability \( \mu_0 = 4\pi \times 10^{-7} \) H/m is an assigned constant to define 1 Ampere current.

where

\[
\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2} \left( = \frac{\text{Farad}}{m} \right),
\]

is the vacuum permittivity. Since

\[
c^2 = \frac{1}{\varepsilon_0 \mu_0}, \quad m^2 s^{-2},
\]

the permittivity \( \varepsilon_0 \) can be deduced from the speed of light in vacuum that can be measured experimentally as well.

### 1.3 Coulomb Electric Field and Scalar Potential

The Coulomb’s law may be interpreted as a force to act on a charge placed in an electric field produced by other charges, since the Coulomb force can be rewritten as

\[
F = \frac{1}{4\pi \varepsilon_0} \frac{q_1(r_2 - r_1)}{|r_1 - r_2|^3} \times q_2
= q_2E_{q_1}, \quad (N)
\]

where

\[
E_{q_1} = \frac{1}{4\pi \varepsilon_0} \frac{q_1(r_2 - r_1)}{|r_1 - r_2|^3}, \quad \left( \frac{N}{C} = \frac{V}{m} \right) \tag{1.4}
\]

is the electric field produced by the charge \( q_1 \) at a distance \( r_1 - r_2 \). The field is associated with the charge \( q_1 \) regardless of the presence or absence of the second charge \( q_2 \). (The factor \( 4\pi \) is the solid
angle pertinent to spherical coordinates. If it is ignored as in the CGS-ESU system, it pops up in planar coordinates. Corresponding Maxwell’s equation is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \text{ in MKS-Ampere,}$$
or

$$\nabla \cdot \mathbf{E} = 4\pi \rho, \text{ in CGS-ESU.}$$

Which unit system to choose is a matter of convenience and it is nonsense to argue one is superior to other.)

Since two charges $q_1$ and $q_2$ exert the Coulomb force on each other, assembling a two-charge system requires work. A differential work needed to move the charge $q_2$ against the Coulomb force is

$$dW = -\mathbf{F} \cdot d\mathbf{r}_2 = -\frac{1}{4\pi\varepsilon_0} \frac{q_1q_2(r_2 - r_1)}{|r_1 - r_2|^3} \cdot d\mathbf{r}_2.$$  

Noting

$$\nabla_{r_2} \frac{1}{|r_1 - r_2|} = \frac{r_1 - r_2}{|r_1 - r_2|^3},$$

integration from $r_2 = \infty$ to $r_2$ can be readily carried out,

$$W = \frac{1}{4\pi\varepsilon_0} \frac{q_1q_2}{|r_1 - r_2|}, \quad (J)$$

This is the potential energy of two-charge system. Note that it can be either positive or negative. If $W$ is positive, the charge system has stored a potential energy that can be released if the system is disassembled. Energy released through nuclear fission process is essentially of electrostatic nature associated with a system of protons closely packed in a small volume. On the other hand, if $W$ is negative, an energy $|W|$ must be given to disassemble the system. For example, to ionize a hydrogen atom at the ground state, an energy of 13.6 eV is required. In a hydrogen atom, the total system energy is given by

$$W = \frac{1}{2} m v^2 - \frac{\mathbf{e}^2}{4\pi\varepsilon_0 r_B} = -\frac{\mathbf{e}^2}{8\pi\varepsilon_0 r_B} < 0,$$

where $r_B = 5.3 \times 10^{-11}$ m is the Bohr radius and

$$\frac{1}{2} m v^2 = \frac{\mathbf{e}^2}{8\pi\varepsilon_0 r_B} = 13.6 \text{ eV},$$
is the electron kinetic energy. The electric potential energy is

$$-\frac{\mathbf{e}^2}{4\pi\varepsilon_0 r_B} = -27.2 \text{ eV}.$$
The potential energy of two-charge system may be written as

\[ W = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \quad (J) \]

which defines the potential associated with a point charge \( q_1 \) at a distance \( r \),

\[ \Phi = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}, \quad (J \text{ C}^{-1} = \text{ V}), \quad (1.6) \]

and the electric field is related to the potential through

\[ \mathbf{E} = -\nabla \Phi, \quad (\text{V m}^{-1}). \quad (1.7) \]

Both potential \( \Phi \) and electric field \( \mathbf{E} \) are created by a charge \( q \) regardless of the presence or absence of a second charge.

### 1.4 Maxwell’s Equations in Electrostatics

In electrostatics, electric charges are stationary being held by forces other than of electric origin such as molecular binding force. Since charges are stationary, no electric currents and thus no magnetic fields are present, \( \mathbf{B} = 0 \).

For a distributed charge density \( \rho(\mathbf{r}) \), the electric field can be calculated from

\[ \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\mathbf{r}') (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' = -\frac{1}{4\pi\varepsilon_0} \nabla \int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV', \quad (1.8) \]

since the differential electric field due to a point charge \( dq = \rho(\mathbf{r}') dV' \) located at \( \mathbf{r}' \) is

\[ d\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}'). \]

Note that

\[ \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}, \]

where \( \nabla' \) is the gradient with respect to \( \mathbf{r}' \). Eq. (1.8) allows one to calculate the electric field for a given charge density distribution \( \rho(\mathbf{r}) \).

In order to find a differential equation to be satisfied by the electric filed, we first note that the surface integral of the electric field

\[ \oint \mathbf{E} \cdot d\mathbf{S}, \]

can be converted to a volume integral of the divergence of the field (Gauss’ mathematical theorem,
not to be confused with Gauss’ law),

\[ \oint_S \mathbf{E} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{E} dV \]

\[ = -\frac{1}{4\pi \varepsilon_0} \int_V dV \nabla^2 \int_{V'} \frac{\rho(r')}{|r - r'|} dV'. \]

However, the function \(1/|r - r'|\) satisfies a singular Poisson’s equation,

\[ \nabla^2 \frac{1}{|r - r'|} = -4\pi \delta(r - r'). \] (1.9)

Therefore,

\[ \oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V dV \int_{V'} \delta(r - r') \rho(r') dV' = \frac{1}{\varepsilon_0} \int \rho(r) dV. \] (1.10)

This is known as Gauss’ law for the electric field. Note that Gauss’ law is a consequence of the Coulomb’s inverse square law. From

\[ \int \nabla \cdot \mathbf{E} dV = \frac{1}{\varepsilon_0} \int \rho(r) dV, \] (1.11)

it also follows that

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}. \] (1.12)

This is the differential form of the Gauss’ law for the longitudinal electric field and constitutes one of Maxwell’s equations.

For the electric field due to a static charge distribution,

\[ \mathbf{E}(r) = \frac{1}{4\pi \varepsilon_0} \int_V \frac{\rho(r')(r - r')}{|r - r'|^3} dV' = -\frac{1}{4\pi \varepsilon_0} \nabla \int_V \frac{\rho(r')}{|r - r'|} dV', \]

its curl identically vanishes because

\[ \nabla \times \mathbf{E} = -\frac{1}{4\pi \varepsilon_0} \nabla \times \nabla \int \frac{\rho(r')}{|r - r'|} dV' \equiv 0. \]

(Note that for any scalar function \(F), \nabla \times \nabla F \equiv 0.\) Therefore, the second Maxwell’s equation in electrostatics is

\[ \nabla \times \mathbf{E} = 0. \] (1.13)

This is a special case of the more general Maxwell’s equation

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \] (1.14)

which determines the transverse component of the electric field. Evidently, if all charges and fields are stationary, there can be no magnetic field. It should be remarked that a vector field can be uniquely determined only if both its divergence and curl are specified. (This is known as Helmholtz’s
In electrodynamics, two vector fields, the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, are to be found for given charge and current distributions $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$. Therefore it is not accidental that four Maxwell’s equations emerge specifying the four functions $\nabla \cdot \mathbf{E}$, $\nabla \times \mathbf{E}$, $\nabla \cdot \mathbf{B}$, and $\nabla \times \mathbf{B}$ which determine the longitudinal and transverse components of $\mathbf{E}$ and $\mathbf{B}$.

**Digression:** A vector $\mathbf{A}$ can be decomposed into the longitudinal and transverse components $\mathbf{A}_l$ and $\mathbf{A}_t$ which are, by definition, characterized by

$$\nabla \times \mathbf{A}_l = 0, \quad \nabla \cdot \mathbf{A}_t = 0.$$

The longitudinal component can be calculated from

$$\mathbf{A}_l(\mathbf{r}) = -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot \mathbf{A}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV',$$

since

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi \delta(\mathbf{r} - \mathbf{r}').$$

The transverse component is given by

$$\mathbf{A}_t(\mathbf{r}) = \mathbf{A}(\mathbf{r}) - \mathbf{A}_l(\mathbf{r})$$

$$= \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{\mathbf{A}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV',$$

where the identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A},$$

is exploited. It is known that for a vector to be uniquely defined, its divergence and curl have to be specified. Since $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_l$, divergence of a vector specifies the divergence of the longitudinal component. Likewise, $\nabla \times \mathbf{A} = \nabla \times \mathbf{A}_t$ specifies the curl of the transverse component. As a concrete example, let us consider the electric field. Its divergence is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0},$$

and the solution to this differential equation is

$$\mathbf{E}_t(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0} \nabla \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'.$$

The transverse component is specified by the magnetic field as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

The vanishing curl of the static electric field allows us to write the electric field in terms of a
gradient of a scalar potential \( \Phi \),

\[
\mathbf{E} = -\nabla \Phi,
\]

(1.15)

for the curl of a gradient of a scalar function identically vanishes,

\[
\nabla \times \nabla \Phi \equiv 0.
\]

For a given charge distribution, the potential has already been formulated in Eq. (1.8),

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'.
\]

(1.16)

In terms of the scalar function \( \Phi(\mathbf{r}) \), Eq. (1.12) can be rewritten as

\[
\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0},
\]

(1.17)

which is known as Poisson’s equation. In general, solving the scalar differential equation for \( \Phi \) is easier than solving vector differential equations for \( \mathbf{E} \).

The physical meaning of the scalar potential is the amount of work required to move adiabatically a unit charge from one position to another. Consider a charge \( q \) placed in an electric field \( \mathbf{E} \). The force to act on the charge is \( \mathbf{F} = q\mathbf{E} \) and if the charge moves over a distance \( dl \), the amount of energy gained (or lost, depending on the sign of \( q\mathbf{E} \cdot dl \)) by the charge is

\[
q\mathbf{E} \cdot dl = -q\nabla \Phi \cdot dl.
\]

Therefore, the work to be done by an external agent against the electric force to move the charge from position \( \mathbf{r}_1 \) to \( \mathbf{r}_2 \) is

\[
W = -q \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot dl = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \nabla \Phi \cdot dl = q [\Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1)],
\]

(1.18)

where \( \Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1) \) is the potential difference between the positions \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \). The work is independent of the choice of the path of integration. The potential is a relative scalar quantity and a constant potential can be added or subtracted without affecting the electric field.

In a conductor, an electric field drives a current flow according to the Ohm’s law,

\[
\mathbf{J} = \sigma \mathbf{E}, \quad (\text{A m}^{-2})
\]

(1.19)

where \( \sigma \) (S m\(^{-1}\)) is the electrical conductivity. In static conditions (no time variation and no flow of charges), the electric field in a conductor must therefore vanish. A steady current flow and constant electric field can exist in a conductor if the conductor is a part of closed electric circuit. However, such a circuit is not static because of the presence flow of charge. For the same reason, a charge given to an isolated conductor must entirely reside on the outer surface of the conductor. Since \( \mathbf{E} \) (static) = 0 in a conductor, the volume charge density \( \rho \) must also vanish according to
\( \rho = \varepsilon_0 \nabla \cdot \mathbf{E} = 0 \). A charge given to a conductor can only appear as a surface charge \( \sigma \) (C m\(^{-2}\)) on the outer surface. The time scale for a conductor to establish such electrostatic state may be estimated from the charge conservation equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{1.20}
\]

The current density can be estimated from the equation of motion for electrons

\[
m \frac{\partial \mathbf{v}}{\partial t} = -e \mathbf{E} - m \nu_e \mathbf{v}, \tag{1.21}
\]

where \( \nu_e \) (s\(^{-1}\)) is the electron collision frequency with the lattice ions. Noting \( \mathbf{J} = -ne \mathbf{v} \) with \( n \) the density of conduction electrons, we obtain

\[
\left( \frac{\partial}{\partial t} + \nu_e \right) \mathbf{J} = \frac{ne^2}{m} \mathbf{E}. \tag{1.22}
\]

Therefore, the equation for the excess charge density in a conductor becomes

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} + \nu_e \right) \rho + \omega_p^2 \rho = 0, \quad \omega_p^2 = \frac{ne^2}{\varepsilon_0 m}, \tag{1.23}
\]

which describes a damped plasma oscillation \( \rho_0 e^{-\gamma t - i \omega_p t} \) with an exponential damping factor \( \gamma = \nu_e / 2 \). The typical electron collision frequency in metals is of order \( \nu_e \simeq 10^{13} \) sec and the time constant to establish electrostatic state is indeed very short. (If the low frequency Ohm’s law is used, an unrealistically short transient time emerges,

\[
\frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon_0} \rho = 0, \quad \rho = \rho_0 e^{-t/\tau},
\]

where \( \tau = \varepsilon_0 / \sigma \simeq 10^{-19} \) sec.)

The principle that a charge given to a conductor must reside entirely on its outer surface is also a consequence of Coulomb’s \( 1/r^2 \) law. For a conductor of an arbitrary shape, the surface charge distribution is so arranged that the electric field inside the conductor vanishes everywhere. In Fig. 1-3, one conducting spherical shell surrounds a smaller one. The spheres are connected through a thin wire. A charge originally given to the inner sphere will end up at the outer surface of the larger sphere and the electric field inside should vanish if Coulomb’s law is correct. Experiments have been conducted to measure residual electric fields inside and it has been established that the power \( \alpha \) in the Coulomb’s law \( F \propto 1/r^\alpha \) is indeed very close to 2 within one part in \( 10^{16} \). \( \alpha \) is probably exactly equal to 2.0. If not, there would be a grave consequence that a photon should have a finite mass. This is because if a photon has a mass \( m_p \), corresponding Compton wavenumber \( k_C = m_p c / h \) will modify the wave equation for all potentials, including the Coulomb potential, in the form

\[
\left( \nabla^2 - k_C^2 - \frac{1}{\varepsilon_0^2} \frac{\partial^2}{\partial t^2} \right) \Phi = -\frac{\rho}{\varepsilon_0}. \tag{1.24}
\]
Figure 1-3: Charge initially given to the inner conductor will all end up at the outer surface of the outer conductor. The absence of the electric field inside the sphere is a consequence of Coulomb’s law.

In static case \( \frac{\partial}{\partial t} = 0 \), this reduces to

\[
(\nabla^2 - k_C^2) \Phi = -\frac{\rho}{\varepsilon_0},
\]

and for a point charge \( \rho(r) = q\delta(r) \), yields a Debye or Yukawa type potential,

\[
\Phi(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} \exp(-k_C r),
\]

and electric field

\[
E_r = \frac{q}{4\pi\varepsilon_0} \left( \frac{1}{r^2} + \frac{k_C}{r} \right) e^{-k_C r}.
\]

The power \( \alpha \) in the Coulomb’s law \( E_r \propto q/r^\alpha \) experimentally established is \( \alpha - 2 = \mathcal{O}(10^{-16}) \), and this corresponds to an upper limit of photon mass of \( m_p < 10^{-51} \) kg. The lower limit of photon Compton wavelength is \( \lambda_C > 4 \times 10^5 \) km at which distance a significant deviation from the Coulomb’s law, if any, is expected. (Incidentally, in a plasma, the static scalar potential does have a Debye form,

\[
\Phi(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} \exp(-k_D r),
\]

where

\[
k_D = \sqrt{\frac{ne^2}{\varepsilon_0 T}},
\]

is the inverse Debye shielding distance, \( n \) is the plasma density and \( T \) (in Joules) is the plasma
temperature. Likewise, in a superconductor, static magnetic field obeys

$$ (\nabla^2 - k_L^2) \mathbf{B} = 0, \quad (1.30) $$

where $k_L = \omega_p/c$ is the London skin depth with $\omega_p = \sqrt{ne^2/\varepsilon_0 m_e}$ the electron plasma frequency.)

The potential energy of a two-charge system is

$$ W = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = 2 \times \frac{1}{2} \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \quad (1.31) $$

where

$$ \frac{1}{2} \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \quad (1.32) $$

is the potential energy associated with each charge. For many charge system, this can be generalized in the form

$$ W = \frac{1}{2} \sum_j \phi_j q_j = \frac{1}{2} \frac{1}{4\pi \varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (1.33) $$

where

$$ \phi_j = \frac{1}{4\pi \varepsilon_0} \sum_{i \neq j} \frac{q_i}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (1.34) $$

is the potential at the location of charge $q_j$ due to all other charges. For distributed charge with a local charge density $\rho(\mathbf{r})$ (C/m$^3$), the potential energy of a differential charge $dq = \rho dV$, which can be regarded as a point charge if $dV$ is sufficiently small, is

$$ dW = \frac{1}{2} \phi dq = \frac{1}{2} \phi \rho dV. $$

Since $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$,

$$ dW = \frac{1}{2} \varepsilon_0 \phi \nabla \cdot \mathbf{E} dV. \quad (1.35) $$

Integrating over the entire volume, we find

$$ W = \frac{1}{2} \varepsilon_0 \int_V \phi \nabla \cdot \mathbf{E} dV = \frac{1}{2} \varepsilon_0 \int_V [\nabla \cdot (\phi \mathbf{E}) - (\nabla \cdot \phi) \mathbf{E}] dV = \frac{1}{2} \varepsilon_0 \int_S \phi \mathbf{E} \cdot d\mathbf{S} + \int_V \frac{1}{2} \varepsilon_0 E^2 dV. \quad (1.36) $$

where use is made of Gauss’ theorem,

$$ \int_V \nabla \cdot \mathbf{F} dV = \oint_S \mathbf{F} \cdot d\mathbf{S}, \quad (1.37) $$

for an arbitrary well defined vector field $\mathbf{F}$. The surface integral vanishes because at infinity, both potential and electric field vanish. Therefore, the potential energy associated with a distributed
charge system is
\[ W = \int \frac{1}{2} \varepsilon_0 E^2 dV, \]  
which is positive definite. In the expression for the potential energy of discrete charge system, the potential energy due to a point charge itself is excluded while in the integral form, spatial distribution of charge is assumed even for point charges and the so-called self energy is included. The quantity
\[ u_e = \frac{1}{2} \varepsilon_0 E^2, \quad (J/m^3) \]  
is the electric energy density associated with an electric field. This expression for the energy density holds regardless of the origin of the electric field, and can be used for fields induced by time varying magnetic field as well.

The self-energy of an ideal point charge evidently diverges because the electric field proportional to \( 1/r^2 \) does in the limit of \( r \to 0 \). However, if the electron is assumed to have a finite radius \( r_e \), the self-energy turns out to be of the order of
\[ W_e \simeq \frac{1}{4\pi \varepsilon_0} \frac{e^2}{r_e}. \]  
This should not exceed the rest energy of the electron \( m_e c^2 \). Equating these two, the following estimate for the electron radius emerges,
\[ r_e \simeq \frac{1}{4\pi \varepsilon_0} \frac{e^2}{mc^2} = 2.85 \times 10^{-15} \text{ m.} \]  
Although this result should not be taken seriously, the scattering cross section of free electron placed in an electromagnetic wave (the process known as Thomson scattering) does turn out to be
\[ \sigma = \frac{8}{3} \pi r_e^2, \]  
and the concept of electron radius is not totally absurd. For proton whose radius is also of order \( 10^{-15} \text{ m} \), the analogy obviously breaks down.

The Poisson’s equation in Eq. (1.17) is the Euler’s equation to make the energy-like integral stationary
\[ U = \int \left( \frac{1}{2} \varepsilon_0 E^2 - \rho \Phi \right) dV = \int \left( \frac{1}{2} \varepsilon_0 (\nabla \Phi)^2 - \rho \Phi \right) dV. \]  
Indeed, the variation of this integral
\[ \delta U = \int (\varepsilon_0 \nabla \Phi \cdot \nabla \delta \Phi - \rho \delta \Phi) dV \]
\[ = - \int (\varepsilon_0 \nabla^2 \Phi + \rho) \delta \Phi dV, \]
becomes stationary ($\delta U = 0$) when the Poisson’s equation holds,

$$\nabla^2 \Phi + \frac{\rho}{\varepsilon_0} = 0.$$  

In other words, electrostatic fields are realized in such a way that the total electric energy becomes minimum. In general, electric force act so as to reduce the energy in a closed (isolated) system. Macroscopically, electric force tends to increase the capacitance as we will see in Chapter 2.

If the charge density distribution is known, the potential $\Phi(\mathbf{r})$ can be readily found as a solution of the Poisson’s equation. However, if the charge density $\rho$ is unknown a priori, as in most potential boundary value problems, the potential and electric field must be found first. Then the charge density is to be found from $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$, or in the case of surface charge on a conductor surface,

$$\sigma = \varepsilon_0 E_n, \quad (\text{C m}^{-2}),$$

where $E_n$ is the electric field normal to the conductor surface.

### 1.5 Formal Solution to the Poisson’s Equation

As shown in the preceding section, if the spatial distribution of the charge density $\rho(\mathbf{r})$ is given, the solution for the Poisson’s equation

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}, \quad (1.43)$$

can be written down as

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'. \quad (1.44)$$

This is understandable because the differential potential due to a point charge $dq = \rho dV$ is

$$d\Phi = \frac{1}{4\pi\varepsilon_0} \frac{\rho(\mathbf{r}')dV'}{|\mathbf{r} - \mathbf{r}'|},$$

and the solution in Eq. (1.44) is a result of superposition.

It is noted that the Poisson’s equation for the scalar potential still holds even for time varying charge density,

$$\nabla^2 \Phi_C(\mathbf{r}, t) = -\frac{\rho(\mathbf{r}, t)}{\varepsilon_0}, \quad (1.45)$$

if one employs the Coulomb gauge characterized by the absence of the longitudinal vector potential

$$\nabla \cdot \mathbf{A} = 0. \quad (1.46)$$
In the Coulomb gauge, the transverse vector potential satisfies the wave equation,

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) A_t = -\mu_0 J_t,$$

(1.47)

where $J_t$ is the transverse current. The solution for the scalar potential in the Coulomb gauge is non-retarded,

$$\Phi_C(r, t) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(r', t)}{|r - r'|} dV',$$

(1.48)

and so is the resultant longitudinal electric field,

$$E_l(r, t) = -\nabla \Phi_C = \frac{1}{4\pi\varepsilon_0} \int_V \frac{r - r'}{|r - r'|^3} \rho(r', t) dV'.$$

(1.49)

Such non-retarded (instantaneous) propagation of electromagnetic disturbance is clearly unphysical and should not exist. In fact, the non-retarded Coulomb electric field is exactly cancelled by a term contained in the retarded transverse electric field

$$E_t(r, t) = -\frac{\partial A_t}{\partial t} = -\frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \int \frac{J_t(t - \tau)}{|r - r'|^3} \rho(r', t) dV', \quad \tau = \frac{|r - r'|}{c},$$

as will be shown in Chapter 6.

The potential in Eq. (1.44) is in the form of convolution between the function

$$G(r, r') = \frac{1}{4\pi|r - r'|},$$

(1.50)
and the source function \( \rho(r')/\varepsilon_0 \). The function \( G(r, r') \) is called the Green’s function and introduced as a solution to the following singular Poisson’s equation

\[
\nabla^2 G = -\delta(r - r'),
\]

subject to the boundary condition that \( G = 0 \) at \( r = \infty \). \( G \) subject to this boundary condition is called the Green’s function in free space. It is a particular solution to the singular Poisson’s equation. Later, we will generalize the Green’s function so that it vanishes on a given closed surface by adding general solutions satisfying \( \nabla^2 G = 0 \). It is evident that the Green’s function is essentially the potential due to a point charge, for the charge density of an ideal point charge \( q \) located at \( r_0 \) can be written as

\[
\rho_c = q\delta(r - r').
\]

Here \( \delta(r - r') \) is an abbreviation for the three dimensional delta function. For example, in the cartesian coordinates,

\[
\delta(r - r') = \delta(x - x')\delta(y - y')\delta(z - z'),
\]

in the spherical coordinates \((r, \theta, \phi)\),

\[
\delta(r - r') = \delta(r - r')\delta[\theta - \theta']\delta[\phi - \phi']
\]

\[
= \frac{\delta(r - r')}{rr'\sin \theta}\delta(\theta - \theta')\delta(\phi - \phi')
\]

\[
= \frac{\delta(r - r')}{rr'}\cos \theta \cos \theta' \delta(\phi - \phi'),
\]

and in the cylindrical coordinates \((\rho, \phi, z)\),

\[
\delta(r - r') = \delta(\rho - \rho')\delta[\phi - \phi']\delta(z - z')
\]

\[
= \frac{\delta(\rho - \rho')}{\rho}\delta(\phi - \phi')\delta(z - z').
\]

The singular Poisson’s equation

\[
\nabla^2 G = -\delta(r - r'),
\]

can be solved formally as follows. Let the Fourier transform of \( G(r - r') \) be \( g(k) \), namely,

\[
g(k) = \int G(r - r')e^{-ik\cdot(r-r')}dr.
\]

Its inverse transform is

\[
G(r - r') = \frac{1}{(2\pi)^3} \int g(k)e^{ik\cdot(r-r')}d^3k.
\]
Eq. (1.59),
\[ g(k) = \frac{1}{k^2}. \] (1.62)

Substitution into Eq. (1.61) yields
\[
G(r - r') = \frac{1}{(2\pi)^3} \int \frac{1}{k^2} e^{i\mathbf{k} \cdot (r - r')} d^3k
\]
\[
= \frac{1}{(2\pi)^3} \int_0^\infty dk \int_0^\pi e^{ik|\mathbf{r} - \mathbf{r}'| \cos \theta} \sin \theta d\theta \int_0^{2\pi} d\phi
\]
\[
= \frac{1}{(2\pi)^2} \pi \int_0^\pi \delta(|\mathbf{r} - \mathbf{r}'| \cos \theta) \sin \theta d\theta
\]
\[
= \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|},
\] (1.63)

where the polar angle \(\theta\) in the \(k\)-space is measured from the direction of the relative position vector \(\mathbf{r} - \mathbf{r}'\), and
\[
d^3k = k^2 dk \sin \theta d\theta d\phi,
\] (1.64)
\[
\int_{-\infty}^\infty e^{ikx} dk = 2\pi \delta(x),
\] (1.65)
\[
\delta(ax) = \frac{1}{|a|} \delta(x),
\] (1.66)

are noted. (The same technique will be used in finding a Green’s function for the less trivial wave equation,
\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, \mathbf{r}'; t, t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t'),
\] (1.67)
in Chapter 6.)

If the charge density distribution is spatially confined within a small volume such that \(r \gg r'\), the inverse distance function \(1/|\mathbf{r} - \mathbf{r}'|\) may be Taylor expanded as follows:
\[
\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} - \nabla \left( \frac{1}{r} \right) \cdot \mathbf{r}' + \frac{1}{2!} \nabla \nabla \left( \frac{1}{r} \right) : \mathbf{r}' \mathbf{r}'' \cdots
\]
\[
= \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \frac{1}{2r^5} \sum_{i,j} (3r_i r_j - r^2 \delta_{ij}) r_i' r_j' \cdots
\] (1.68)

and so is the potential
\[
\Phi(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' = \frac{1}{4\pi \epsilon_0} \left( \frac{1}{r} \int_V \rho(\mathbf{r}') dV' + \frac{\mathbf{r}}{r^3} \cdot \int_V \mathbf{r}' \rho(\mathbf{r}') dV' + \frac{3\mathbf{r} \cdot \mathbf{r}}{2r^5} : \mathbf{Q} + \cdots \right),
\] (1.69)

where \(\mathbf{1}\) is the unit dyadic. The quantity \(q = \int \rho(\mathbf{r}') dV'\) is the total charge contained in the (small) volume \(V\) and the corresponding lowest order monopole potential is
\[
\Phi_{\text{monopole}} = \frac{1}{4\pi \epsilon_0} \frac{q}{r}.
\] (1.70)
The vector quantity
\[ \mathbf{p} = \int r \rho(r) dV, \]  
(1.71)
is the dipole moment and the corresponding dipole potential is
\[ \Phi_{\text{dipole}} = \frac{1}{4 \pi \epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}. \]  
(1.72)
The tensor
\[ \mathbf{Q} = Q_{ij} = \int r_i r_j \rho(r) dV, \]  
(1.73)
defines the quadrupole moment and the corresponding quadrupole potential is
\[ \Phi_{\text{quadrupole}} = \frac{1}{4 \pi \epsilon_0} \frac{3 \mathbf{r} - r^2 \mathbf{1}}{2 r^5} : \mathbf{Q} = \frac{1}{4 \pi \epsilon_0} \frac{1}{2 r^5} \sum_{i,j} (3 r_i r_j - r^2 \delta_{ij}) Q_{ij}. \]  
(1.74)

Example 1 *Electric Field Lines of a Dipole*

The potential due to a dipole moment directed in \( z \) direction \( \mathbf{p} = p \mathbf{e}_z \) at the origin is given by
\[ \Phi(r, \theta) = \frac{1}{4 \pi \epsilon_0} \frac{p}{r^2} \cos \theta. \]
The equipotential surfaces are described by
\[ \frac{\cos \theta}{r^2} = \text{const}. \]
The electric field can be found from
\[ \mathbf{E} = -\nabla \Phi = \frac{1}{4 \pi \epsilon_0} \left( \frac{2 p \cos \theta}{r^3} \mathbf{e}_r + \frac{p \sin \theta}{r^3} \mathbf{e}_\theta \right). \]  
(1.75)
The equation to describe an electric field line is, by definition,
\[ \frac{dr}{E_r} = \frac{rd\theta}{E_\theta}, \]  
(1.76)
which gives
\[ \frac{dr}{r} = 2 \cot \theta d\theta, \]
and thus
\[ r = \text{const.} \sin^2 \theta. \]  
(1.77)
Evidently, the electric field lines are normal to the equipotential surfaces. In Fig.1-5, one equipotential surface and three electric field lines are shown.

Example 2 *Linear quadrupole*
Figure 1-5: Equipotential surface and electric field lines of an electric dipole $p_z$. The $z$-axis is vertical.

A linear quadrupole consists of charges $q$ at $z = \pm a$ and $-2q$ at $z = 0$ as shown in Fig.1-6. The charge density is

$$\rho(r) = q[\delta(z - a) - 2\delta(z) + \delta(z + a)]\delta(x)\delta(y).$$

Therefore, only $Q_{zz}$ is nonvanishing,

$$Q_{zz} = \int z^2 \rho(r) dV = 2a^2 q,$$

and the quadrupole potential at $r \gg a$ is given by

$$\Phi(r) = \frac{1}{4\pi\epsilon_0} \frac{1}{2r^3} (3z^2 - r^2) Q_{zz} = \frac{1}{4\pi\epsilon_0} \frac{2qa^2}{r^3} \frac{3\cos^2 \theta - 1}{2} = \frac{1}{4\pi\epsilon_0} \frac{2qa^2}{r^3} P_2(\cos \theta),$$

where

$$P_2(\cos \theta) = \frac{3\cos^2 \theta - 1}{2},$$

is the Legendre function of order $l = 2$.

Alternatively, the potential can be found from the direct superposition of the potentials produced by each charge,

$$\Phi(r, \theta) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{\sqrt{r^2 + a^2 - 2ar \cos \theta}} - \frac{2}{r} + \frac{1}{\sqrt{r^2 + a^2 + 2ar \cos \theta}} \right].$$

For $r > a$, the functions $1/\sqrt{r^2 + a^2 \pm 2ar \cos \theta}$ can be expanded in terms of the Legendre functions
\[ P_l(\cos \theta) \]

\[
\frac{1}{\sqrt{r^2 + a^2 + 2ar \cos \theta}} = \frac{1}{r} \sum_{l=0}^{\infty} \left( \frac{a}{r} \right)^l (\pm 1)^l P_l(\cos \theta), \quad r > a.
\]

Retaining terms up to \( l = 2 \) (quadrupole), we recover

\[
\Phi(r, \theta) \approx \frac{1}{4\pi\epsilon_0} \frac{2qa^2}{r^3} P_2(\cos \theta), \quad r \gg a.
\]

At \( r \gg a \), the leading order potential is of quadrupole. The total charge is zero, thus no monopole potential at \( r \gg a \). Also, the charge system consists of two dipoles of equal magnitude oriented in opposite directions. Therefore, the dipole potential vanishes at \( r \gg a \) as well. Equipotential profile is shown in Fig. 1-6.

1.6 Potential due to a Ring Charge: Several Methods

A given potential problem can be solved in different coordinates systems. Of course, the solution is unique, and answers found by different methods should all agree. For the purpose of becoming familiar with several coordinates systems and some useful mathematical techniques, here we find the potential due to a ring charge having radius \( a \) and total charge \( q \) uniformly distributed using several independent methods.

**Method 1: Direct integration**

The potential is symmetric about the \( z \)-axis because of the uniform charge distribution. Therefore, we may evaluate the potential at arbitrary azimuthal angle \( \phi \) and here we choose \( \phi = \pi/2 \), that is, observing point in the \( y - z \) plane. The differential potential due to a “point charge”
Figure 1-7: Equipotential surface of a linear quadrupole. The z axis \((\theta = 0)\) is in the vertical direction. The thick line is at +1 unit potential, and the thin line is at −1.

\[ dq = q d\phi' / 2\pi \] located at angle \(\phi'\) is

\[ d\Phi = \frac{1}{4\pi\epsilon_0} \frac{dq}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\epsilon_0} \frac{q}{2\pi\sqrt{r^2 + a^2 - 2ar\cos\gamma}} d\phi', \tag{1.78} \]

where \(\gamma\) is the angle between the vectors \(\mathbf{r} = (r, \theta, \phi = \pi/2)\) and \(\mathbf{r}' = (a, \theta' = \pi/2, \phi').\) \(\cos\gamma\) reduces to

\[ \cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi') = \sin\theta \sin\phi'. \tag{1.79} \]

Integrating over \(\phi'\) from 0 to 2\(\pi\), we find

\[ \Phi(r, \theta) = \frac{1}{4\pi\epsilon_0} \frac{q}{2\pi} \int_0^{2\pi} \frac{d\phi'}{\sqrt{r^2 + a^2 - 2ar\sin\theta\sin\phi'}.} \tag{1.80} \]

Changing the variable from \(\phi'\) to \(\alpha\) through

\[ 2\alpha = \phi' + \frac{\pi}{2}, \]

the integral can be reduced to

\[ \frac{4}{\sqrt{r^2 + a^2 + 2ar\sin\theta}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2\alpha}} d\alpha = \frac{4}{\sqrt{r^2 + a^2 + 2ar\sin\theta}} K(k^2), \tag{1.81} \]
where

\[ k^2 = \frac{4ar \sin \theta}{r^2 + a^2 + 2ar \sin \theta}, \quad (1.82) \]
is the argument of the complete elliptic integral of the first kind defined by

$$K(k^2) \equiv \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \alpha}} d\alpha.$$  \hspace{1cm} (1.83)

The final form of the potential is

$$\Phi(r, \theta) = \frac{q}{2\pi^2 \epsilon_0} \frac{1}{\sqrt{r^2 + a^2 + 2ar \sin \theta}} K(k^2).$$  \hspace{1cm} (1.84)

$$f(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - x \sin^2 \alpha}} d\alpha$$

![Figure 1-9: $K(x)$ the complete elliptic integral of the first kind. It diverges logarithmically at $x \lesssim 1$, $K(x) \simeq \ln \left(\frac{4}{\sqrt{1-x}}\right)$.

The function $K(k^2)$ is shown in 1-9. It diverges as $x = k^2$ approaches unity, that is, near the ring itself, as expected. However, the divergence is only logarithmic,

$$\lim_{k^2 \to 1-} = \ln \left(\frac{4}{\sqrt{\epsilon}}\right).$$

**Example 3 Capacitance of a Thin Conductor Ring**

Let us assume a thin conducting ring with ring radius $a$ and wire radius $\rho \ll a$ shown in Fig.1-10. The potential on the ring surface can be found by letting $r = a - \rho$, $\theta = \pi/2$. In this limit,
the argument $k^2$ approaches unity,

$$k^2 = \frac{4a(a - \rho)}{(a - \rho)^2 + a^2 + 2a(a - \rho)} \approx 1 - \left(\frac{\rho}{2a}\right)^2,$$

and the ring potential becomes

$$\Phi_{\text{ring}} \simeq \frac{q}{4\pi^2 \varepsilon_0 a} \ln \left(\frac{8a}{\rho}\right). \quad (1.85)$$

Then the self-capacitance of the ring can be found from

$$C = \frac{q}{\Phi_{\text{ring}}} = \frac{4\pi^2 \varepsilon_0 a}{\ln \left(\frac{8a}{\rho}\right)}, \quad a \gg \rho. \quad (1.86)$$

This formula is fairly accurate even for a not-so-thin ring because of the mere logarithmic dependence on the aspect ratio, $a/\rho$. For example, when $a/\rho = 5$ (which is a fat torus rather than a thin ring), Eq. (1.86) still gives a capacitance within 2% of the correct value. An exact formula for the capacitance of a conducting torus will be worked out later in terms of the toroidal coordinates.

Figure 1-10: A thin conductor ring with major radius $a$ and minor radius $\rho$, $a \gg \rho$.

**Method 2: Multipole Expansion in the Spherical Coordinates**

As the second method, we directly solve the Poisson’s equation in the spherical coordinates,

$$\nabla^2 \Phi = -\frac{\rho(r)}{\varepsilon_0} = -\frac{1}{\varepsilon_0} \frac{q}{2\pi a^2} \delta(r - a) \delta(\cos \theta). \quad (1.87)$$

Except at the ring itself, the charge density vanishes. Therefore, in most part of space, the potential satisfies Laplace equation,

$$\nabla^2 \Phi = 0, \quad r \neq a, \theta \neq \frac{\pi}{2},$$

and we seek a potential in terms of elementary solutions to the Laplace equation. Since $\partial/\partial \phi = 0,$
the Laplace equation reduces to
\[
\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) = 0.
\] (1.88)

Assuming that the potential is separable in the form \(\Phi(r, \theta) = R(r)F(\theta)\), we find
\[
r^2 \left( \frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} \right) + \frac{1}{F} \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dF}{d\theta} \right) = 0.
\] (1.89)

Since the first term is a function of \(r\) only and the second term is a function of \(\theta\) only, each term must be constant cancelling each other. Introducing a separation constant \(l(l+1)\), we obtain two ordinary differential equations for \(R\) and \(F\),
\[
\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} - \frac{l(l+1)}{r^2} R = 0,
\] (1.90)
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dF}{d\theta} \right) + l(l+1)F = 0,
\] (1.91)

which can be written as
\[
\frac{d}{d\mu} \left( (1 - \mu^2) \frac{dF}{d\mu} \right) + l(l+1)F = 0,
\] (1.92)

where \(\mu = \cos \theta\). Eq. (1.92) is known as the Legendre’s differential equation. The solutions for \(R(r)\) are
\[
R(r) = r^l \text{ and } \frac{1}{r^{l+1}},
\] (1.93)

and the solutions for \(F(\theta)\) are the familiar Legendre functions,
\[
F(\theta) = P_l(\cos \theta) \text{ and } Q_l(\cos \theta),
\] (1.94)

where \(Q_l(\cos \theta)\) is the Legendre function of the second kind. Some low order Legendre functions for real \(x\) \((|x| \leq 1)\) are listed below.
\[
P_0(x) = 1, \ P_1(x) = x, \ P_2(x) = \frac{3x^2 - 1}{2}, \ P_3(x) = \frac{5x^3 - 3x}{2}, \cdots
\] (1.95)
\[
Q_l(x) = \frac{1}{2} P_l(x) \ln \frac{1+x}{1-x} - W_{l-1}(x), \ W_{-1}(x) = 0, \ W_0(x) = 1, \ W_1(x) = \frac{3}{2} x, \cdots
\] (1.96)

For a general complex variable \(z\), the definition for \(Q_l(z)\) is modified as follows:
\[
Q_l(z) = \frac{1}{2} P_l(z) \ln \frac{z+1}{z-1} - W_{l-1}(z), \ W_{-1}(z) = 0, \ W_0(z) = 1, \ W_1(z) = \frac{3}{2} z, \cdots
\] (1.97)

In the present problem, the potential should remain finite everywhere except at the ring. Since \(Q_l(\cos \theta)\) diverges at \(\theta = 0\) and \(\pi\), it should be discarded and we assume the following series solutions

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for the potential separately for interior \((r < a)\) and exterior \((r > a)\),

\[
\Phi(r, \theta) = \begin{cases} 
\sum_l A_l \left(\frac{r}{a}\right)^l P_l(\cos \theta), & r < a \\
\sum_l B_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos \theta), & r > a
\end{cases}
\]  

(1.98)

where \(A_l\) and \(B_l\) are constants to be determined. The potential should be continuous at \(r = a\) because there are no double layers to create a potential jump. Therefore, \(A_l = B_l\). To determine \(A_l\), we substitute the assumed potential into the Poisson’s equation,

\[
\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) = -\frac{1}{\epsilon_0 \frac{q}{2\pi a^2}} \delta(r - a) \delta(\cos \theta),
\]

to obtain

\[
\sum_l \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right) A_l R_l(r) P_l(\cos \theta) = -\frac{1}{\epsilon_0 \frac{q}{2\pi a^2}} \delta(r - a) \delta(\cos \theta),
\]  

(1.99)

where

\[
R_l(r) = \begin{cases} 
\left(\frac{r}{a}\right)^l, & r < a \\
\left(\frac{a}{r}\right)^{l+1}, & r > a
\end{cases}
\]  

(1.100)

If we multiply Eq. (1.99) by \(P_l(\cos \theta) \sin \theta\) and integrate over \(\theta\) from \(\theta = 0\) to \(\pi\), the summation over \(l\) disappears because of the orthogonality of the Legendre functions,

\[
\int_0^\pi P_l(\cos \theta) P_l'(\cos \theta) \sin \theta d\theta = \int_{-1}^1 P_l(\mu) P_l'(\mu) d\mu = \frac{2}{2l+1} \delta \mu.
\]  

(1.101)

We thus obtain

\[
A_l \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right) R_l(r) = -\frac{1}{\epsilon_0 \frac{q}{2\pi a^2}} \delta(r - a) \frac{2l+1}{2} P_l(0).
\]  

(1.102)

The LHS should contain a delta function \(\delta(r - a)\) to be compatible with that in the RHS. This can be seen as follows. The radial function \(R_l(r)\) has a discontinuity in its derivative at \(r = a\),

\[
\left. \frac{dR_l}{dr} \right|_{r=a+0} = -\frac{l+1}{a}, \quad \left. \frac{dR_l}{dr} \right|_{r=a-0} = +\frac{l}{a}.
\]  

(1.103)

Therefore, the second order derivative \(\frac{d^2 R_l}{dr^2}\) yields a delta function,

\[
\frac{d^2 R_l}{dr^2} = -\frac{2l+1}{a} \delta(r - a).
\]  

(1.104)
We thus finally find the expansion coefficient $A_l$,

$$A_l = \frac{q}{4\pi \varepsilon_0 a} P_l(0) = \frac{q}{4\pi \varepsilon_0 a} \frac{(-1)^{l/2} l!}{2^l [l(l+1)/2]^2}, \ (l = 0, 2, 4, 6, \cdots), \quad (1.105)$$

and the potential,

$$\Phi(r, \theta) = \frac{q}{4\pi \varepsilon_0 a} \sum_{l \text{ even}} \frac{(-1)^{l/2} l!}{2^l [l(l+1)/2]^2} R_l(r) P_l(\cos \theta). \quad (1.106)$$

![Figure 1-11: The derivative of the radial function $R(r)$ is discontinuous at $r = r'$ and thus its second order derivative yields a delta function.](image)

The disappearance of the odd harmonics is understandable because of the up-down symmetry of the problem. At $r \gg a$, the leading order term is the monopole potential,

$$\Phi_{l=0}(r) = \frac{q}{4\pi \varepsilon_0 a} \frac{a}{r},$$

followed by the quadrupole potential,

$$\Phi_{l=2}(r, \theta) = -\frac{q}{4\pi \varepsilon_0 a} \frac{1}{2} \left( \frac{a}{r} \right)^3 P_2(\cos \theta),$$

and so on. The dipole potential ($l = 1$) vanishes because of the symmetric charge distribution, $\rho(-\mathbf{r}) = \rho(\mathbf{r})$,

$$\mathbf{p} = \int \mathbf{r} \rho(\mathbf{r}) dV = 0.$$

For problems with axial symmetry ($\partial / \partial \phi = 0$) as in this example, knowing the potential along the axis $\Phi(z)$ is sufficient to find the potential at arbitrary point $\Phi(r, \theta)$ in the spherical coordinates. This is because for all of the Legendre functions, $P_l(1) = 1, P_l(-1) = (-1)^l$. In the case of the ring
charge, the axial potential can be readily found,

$$\Phi(z) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{z^2 + a^2}}. \quad (1.107)$$

For $|z| > a$, this can be expanded as

$$\Phi(z) = \frac{q}{4\pi\varepsilon_0 |z|} \left( 1 - \frac{1}{2} \left( \frac{a}{z} \right)^2 + \frac{3}{8} \left( \frac{a}{z} \right)^4 - \cdots \right). \quad (1.108)$$

Therefore, at arbitrary point $(r > a, \theta)$, the potential is

$$\Phi(r, \theta) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \left( 1 - \frac{1}{2} \left( \frac{a}{r} \right)^2 P_2(\cos \theta) + \frac{3}{8} \left( \frac{a}{r} \right)^4 P_4(\cos \theta) - \cdots \right), \quad r > a. \quad (1.109)$$

For interior region $(r < a)$, the potential becomes

$$\Phi(r, \theta) = \frac{q}{4\pi\varepsilon_0} \frac{1}{a} \left( 1 - \frac{1}{2} \left( \frac{r}{a} \right)^2 P_2(\cos \theta) + \frac{3}{8} \left( \frac{r}{a} \right)^4 P_4(\cos \theta) - \cdots \right), \quad r < a. \quad (1.110)$$

They agree with the potential given in Eq. (1.106).
Method 3: Cylindrical Coordinates

In the cylindrical coordinates \((\rho, \phi, z)\), the Poisson’s equation for the potential due to a ring charge becomes

\[
\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \Phi(\rho, z) = -\frac{q}{2\pi \varepsilon_0} \frac{\delta(\rho - a)}{a} \delta(z).
\]

(1.111)

Since the \(z\)-coordinate extends from \(-\infty\) to \(\infty\), we seek a solution in the form of Fourier transform,

\[
\Phi(\rho, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\rho, k) e^{ikz} dk,
\]

(1.112)

where \(\Phi(\rho, k)\) is the one-dimensional Fourier transform of the potential having dimensions of \(V\cdot m\).

Since \(\frac{\partial}{\partial z} \Phi(\rho, z)\),

is Fourier transformed as

\[
i k \Phi(\rho, k),
\]

and \(\delta(z)\) as unity, the equation for \(\Phi(\rho, k)\) becomes an ordinary differential equation,

\[
\left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - k^2 \right) \Phi(\rho, k) = -\frac{q}{2\pi \varepsilon_0} \frac{\delta(\rho - a)}{a}.
\]

(1.113)

Elementary solutions of the differential equation at \(\rho \neq a\),

\[
\left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - k^2 \right) \Phi(\rho, k) = 0,
\]

(1.114)

are the zero-th order modified Bessel functions,

\[
\Phi(\rho, k) = I_0(k\rho), \quad K_0(k\rho),
\]

(1.115)

shown in Fig.1-12.

The Fourier potential \(\Phi(\rho, k)\) which is continuous at \(\rho = a\) and remains bounded everywhere may be constructed in the following form,

\[
\Phi(\rho, k) = \begin{cases} 
A I_0(k\rho) K_0(ka), & \rho < a \\
A I_0(ka) K_0(k\rho), & \rho > a
\end{cases}
\]

(1.116)

The coefficient \(A\) can be determined readily from the discontinuity in the derivative of \(\Phi(\rho, k)\) at \(\rho = a\),

\[
\frac{d\Phi}{d\rho} \bigg|_{\rho=a+0} = Ak I_0(ka) K_0'(ka), \quad \frac{d\Phi}{d\rho} \bigg|_{\rho=a-0} = Ak I_0'(ka) K_0(ka),
\]

(1.117)
from which it follows that
\[
\frac{d^2 \phi}{d\rho^2} \bigg|_{\rho=a} = Ak \left( I_0(ka)K'_0(ka) - I'_0(ka)K_0(ka) \right) \delta(\rho - a) = -\frac{A}{a} \delta(\rho - a),
\]
where the Wronskian of the modified Bessel functions
\[
I_0(ka)K'_0(ka) - I'_0(ka)K_0(ka) = -\frac{1}{ak},
\]
has been exploited. From
\[
\frac{d^2 \phi}{d\rho^2} \bigg|_{\rho=a} = -\frac{A}{a} \delta(\rho - a) = -\frac{q}{2\pi\epsilon_0} \frac{\delta(\rho - a)}{a},
\]
we find
\[
A = \frac{q}{2\pi\epsilon_0},
\]
and the final form of the potential is
\[
\Phi(\rho, z) = \frac{q}{4\pi^2\epsilon_0} \int_{-\infty}^{\infty} \left\{ \begin{array}{ll} I_0(ka)K_0(ka) & \rho < a \\ I_0(ka)K_0(ka) \cos(kz) & \rho > a \end{array} \right\} e^{ikz}dk,
\]
Noting that the modified Bessel functions are even with respect to the argument, this may be further rewritten as
\[
\Phi(\rho, z) = \frac{q}{2\pi^2\epsilon_0} \int_{0}^{\infty} \left\{ \begin{array}{ll} I_0(ka)K_0(ka) & \rho < a \\ I_0(ka)K_0(ka) \cos(kz) & \rho > a \end{array} \right\} \cos(kz)dk,
\]
The convergence of the integral is rather poor since the function $I_0 K_0$ decreases with $k$ only algebraically. An alternative solution for the potential can be found in terms of Laplace transform rather than Fourier transform, 

$$
\Phi(\rho, z) = \int_0^\infty \Phi(\rho, k)e^{-|z|}dk.
$$

(1.124)

The Laplace’s equation thus reduces to 

$$
\left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + k^2 \right) \Phi(\rho, k) = 0,
$$

(1.125)

whose solution is the ordinary Bessel function $J_0(k\rho)$. We thus assume for $\Phi(\rho, k)$

$$
\Phi(\rho, k) = A(k)J_0(k\rho),
$$

(1.126)

where $A(k)$ may still be a function of $k$. The Poisson’s equation becomes 

$$
\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \int_0^\infty A(k)J_0(k\rho)e^{-|z|}dk = -\frac{q}{2\pi\epsilon_0} \frac{\delta(\rho - a)}{a} \delta(z).
$$

(1.127)

Noting 

$$
\frac{d^2}{dz^2}e^{-|z|} = k^2 e^{-|z|} - 2k\delta(z)
$$

(1.128)

we thus find 

$$
\int_0^\infty kA(k)J_0(k\rho)dk = \frac{q}{4\pi\epsilon_0} \frac{\delta(\rho - a)}{a}.
$$

(1.129)

However, the Bessel function forms an orthogonal set according to 

$$
\int_0^\infty k J_0(ka)J_0(k\rho)dk = \frac{\delta(\rho - a)}{a},
$$

(1.130)

which uniquely determines the function $A(k)$,

$$
A(k) = \frac{q}{4\pi\epsilon_0} J_0(ka).
$$

(1.131)

Therefore, the final form of the potential is 

$$
\Phi(\rho, z) = \frac{q}{4\pi\epsilon_0} \int_0^\infty J_0(ka)J_0(k\rho)e^{-|z|}dk.
$$

(1.132)

The convergence of this solution is much faster than the solution in Eq. (1.123) and thus more suitable for numerical evaluation.
Method 4: Toroidal Coordinates

The toroidal coordinates \((\eta, \theta, \phi)\) are related to the cartesian coordinates \((x, y, z)\) through the following transformation,

\[
\begin{align*}
x &= \frac{R \sinh \eta \cos \phi}{\cosh \eta - \cos \theta}, \\
y &= \frac{R \sinh \eta \sin \phi}{\cosh \eta - \cos \theta}, \\
z &= \frac{R \sin \theta}{\cosh \eta - \cos \theta}.
\end{align*}
\] (1.133)

This coordinate system is one example in which the Laplace equation is not separable, that is, the potential cannot be written as a product of independent functions, \(\Phi(\eta, \theta, \phi) \neq F_1(\eta)F_2(\theta)F_3(\phi)\). However, the potential is partially separable and can be sought in the form

\[
\Phi(\eta, \theta, \phi) = \sqrt{\cosh \eta - \cos \theta} F_1(\eta)F_2(\theta)F_3(\phi). \quad (1.134)
\]

Figure 1-13: Cross-section of the toroidal coordinates \((\eta, \theta, \phi)\). \(\eta \to 0\) corresponds to a thin ring of radius \(R\).

In the toroidal coordinates, the \(\eta = \text{constant}\) surfaces are toroids having a major radius \(R \coth \eta\) and minor radius \(R/\sinh \eta\). \(\eta \to \infty\) degenerates to a thin ring of radius \(R\). In the other limit, \(\eta \to 0\) describes a thin rod on the \(z\)-axis. The \(\theta = \text{constant}\) surfaces are spherical bowls as illustrated in Fig.1-13.

Assuming the partial separation for the potential in Eq. (1.134) leads to the following ordinary
equations for the functions $F_1(\eta), F_2(\theta)$ and $F_3(\phi)$,

\[
\frac{1}{\sinh \eta} \frac{d}{d\eta} \left( \sinh \eta \frac{dF_1}{d\eta} \right) + \left( \frac{1}{4} - \ell^2 - \frac{m^2}{\sinh^2 \eta} \right) F_1 = 0, \quad (1.135)
\]

\[
\left( \frac{d^2}{d\theta^2} + \ell^2 \right) F_2 = 0, \quad \left( \frac{d^2}{d\phi^2} + m^2 \right) F_3 = 0.
\]

Comparing Eq. (1.135) with the standard form of the differential equation for the associated Legendre function $P_{l-\frac{1}{2}}^m(\cos \theta), Q_{l-\frac{1}{2}}^m(\cos \theta),$

\[
\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) + \ell (\ell + 1) - \frac{m^2}{\sin^2 \theta} \right] (P_{l-\frac{1}{2}}^m, Q_{l-\frac{1}{2}}^m) = 0, \quad (1.136)
\]

we see that solutions for $F_1(\eta)$ are

\[
F_1(\eta) = P_{l-\frac{1}{2}}^m(\cosh \eta), \quad Q_{l-\frac{1}{2}}^m(\cosh \eta). \quad (1.137)
\]

The functions $F_2$ and $F_3$ are elementary,

\[
F_2(\theta) = e^{i\ell \theta}, \quad F_3(\phi) = e^{im\phi}, \quad (1.138)
\]

and the general solution for the potential may be written as

\[
\Phi(\eta, \theta, \phi) = \sqrt{\cosh \eta - \cos \theta} \sum_{l,m} \left( A_{lm} P_{l-\frac{1}{2}}^m(\cosh \eta) + B_{lm} Q_{l-\frac{1}{2}}^m(\cosh \eta) \right) e^{i\ell \theta + im\phi}. \quad (1.139)
\]

Of course, the potential is a real function. The solution above is an abbreviated form of a more cumbersome expression,

\[
\Phi(\eta, \theta, \phi) = \sqrt{\cosh \eta - \cos \theta} \sum_{l=0}^{\infty} \left( A_l P_{l-\frac{1}{2}}(\cosh \eta) + B_l Q_{l-\frac{1}{2}}(\cosh \eta) \right) \left( C_l \cos l\theta + D_l \sin l\theta \right) \\
\times \sum_{m=0}^{\infty} \left( E_l \cos m\phi + F_l \sin m\phi \right).
\]

We now consider a conducting toroid with a major radius $a$ and minor radius $b$. Its surface is described by $\eta = \text{const.}$ where

\[
a = R \coth \eta_0, \quad b = \frac{R}{\sinh \eta_0}. \quad (1.140)
\]

Because of axial symmetry, only the $m = 0$ term is present. If the toroid is at a potential $V$, the potential off the toroid can be written as

\[
\Phi(\eta, \theta) = \sqrt{\cosh \eta - \cos \theta} \sum_{l=0}^{\infty} A_l P_{l-\frac{1}{2}}(\cosh \eta) \cos l\theta, \quad (1.141)
\]
where the Legendre function of the second kind $Q_{l-\frac{1}{2}}(\cosh \eta)$ has been discarded because it diverges at $\eta \to 0$ which corresponds to the $z$–axis. (The potential along the $z$–axis should be bounded.) The expansion coefficients $A_l$ can be determined from the boundary condition, $\Phi = V$ at $\eta = \eta_0$,

$$V = \sqrt{\cosh \eta_0 - \cos \theta} \sum_{l=0}^{\infty} A_l P_{l-\frac{1}{2}}(\cosh \eta_0) \cos l\theta,$$

or

$$\sum_{l=0}^{\infty} A_l P_{l-\frac{1}{2}}(\cosh \eta_0) \cos l\theta = \frac{V}{\sqrt{\cosh \eta_0 - \cos \theta}}.$$  

Multiplying both sides by $\cos l'\theta$ and integrating over $\theta$ from $0$ to $\pi$, we obtain

$$A_l = V \frac{Q_{l-\frac{1}{2}}(\cosh \eta_0)}{P_{l-\frac{1}{2}}(\cosh \eta_0)} \frac{\sqrt{2}}{\pi} \epsilon_l,$$

where

$$\epsilon_0 = 1, \: \epsilon_l = 2 \: (l \geq 1),$$

and the following integral representation has been exploited,

$$\int_0^{\pi} \frac{\cos l\theta}{\sqrt{x - \cos \theta}} \cos l'\theta \, d\theta = \sqrt{2} Q_{l-\frac{1}{2}}(x).$$

Then, the final form of the potential is

$$\Phi(\eta, \theta) = \frac{\sqrt{2} V}{\pi} \sqrt{\cosh \eta - \cos \theta} \sum_{l=0}^{\infty} \frac{Q_{l-\frac{1}{2}}(\cosh \eta_0)}{P_{l-\frac{1}{2}}(\cosh \eta_0)} P_{l-\frac{1}{2}}(\cosh \eta) \cos(l\theta) \epsilon_l.$$  

The capacitance of a conducting torus can be found from the behavior of the potential at a large distance from the torus which should be in the form of monopole potential,

$$\Phi(r \to \infty) = \frac{q}{4\pi \epsilon_0 r}.$$  

At a large distance from the torus $r \gg R$, both $\eta$ and $\theta$ approach zero,

$$r^2 = x^2 + y^2 + z^2 = \frac{R^2(\sinh^2 \eta + \sin^2 \theta)}{(\cosh \eta - \cos \theta)^2} \to \frac{4R^2}{\eta^2 + \theta^2},$$

$$\sqrt{\cosh \eta - \cos \theta} \approx \frac{\sqrt{\eta^2 + \theta^2}}{\sqrt{2}} \approx \sqrt{2} \frac{R}{r},$$

$$P_{l-\frac{1}{2}}(\cosh \eta) \approx 1.$$
Therefore, the asymptotic potential is

\[ \Phi(r \gg R) = \frac{2VR}{r} \sum_{l=0}^{\infty} \frac{Q_{l-\frac{1}{2}}(\cosh \eta_0)}{P_{l-\frac{1}{2}}(\cosh \eta_0)} = \frac{q}{4\pi \epsilon_0 r}, \]  

(1.150)

from which the capacitance of the torus can be found,

\[ C \equiv \frac{q}{V} = 8\epsilon_0 R \sum_{l=0}^{\infty} \frac{Q_{l-\frac{1}{2}}(\cosh \eta_0)}{P_{l-\frac{1}{2}}(\cosh \eta_0)} \epsilon_l = 8\epsilon_0 a \tanh \eta_0 \sum_{l=0}^{\infty} \frac{Q_{l-\frac{1}{2}}(\cosh \eta_0)}{P_{l-\frac{1}{2}}(\cosh \eta_0)} \epsilon_l, \]  

(1.151)

where \( a \) is the major radius of the torus. The function \( Q_{l-\frac{1}{2}}(\cosh \eta_0) \) can be evaluated from Eq. (1.144) and \( P_{l-\frac{1}{2}}(\cosh \eta_0) \) from

\[ P_{l-\frac{1}{2}}(\cosh \eta_0) = \frac{1}{\pi} \int_0^{\pi} \frac{d\theta}{(\cosh \eta_0 + \sinh \eta_0 \cos \theta)^{l+\frac{1}{2}}}. \]  

(1.152)

**Example 4 Capacitance of a Fat Torus**

We wish to find the capacitance of a conducting torus having a major radius of \( a = 50 \) cm and minor radius of \( b = 10 \) cm. The torus is defined by \( \cosh \eta_0 = 5.0 \) and \( R = a \tanh \eta_0 = 49 \) cm. For \( \cosh \eta_0 = 5.0 \), the Legendre functions numerically evaluated are:

<table>
<thead>
<tr>
<th>( l )</th>
<th>( Q_{l-\frac{1}{2}}(5) )</th>
<th>( P_{l-\frac{1}{2}}(5) )</th>
<th>( \frac{Q_{l-\frac{1}{2}}(5)}{P_{l-\frac{1}{2}}(5)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00108</td>
<td>0.74575</td>
<td>1.34234</td>
</tr>
<tr>
<td>1</td>
<td>0.05063</td>
<td>2.03557</td>
<td>0.02487</td>
</tr>
<tr>
<td>2</td>
<td>0.00384</td>
<td>13.32184</td>
<td>0.00029</td>
</tr>
</tbody>
</table>

The ratio \( Q_{l-\frac{1}{2}}(5)/P_{l-\frac{1}{2}}(5) \) rapidly converges as \( l \) increases and it suffices to truncate the series at \( l = 2 \). The capacitance is

\[ C = 8\epsilon_0 R (1.3423 + 2 \times 0.0249 + 2 \times 0.0003 + \cdots) \simeq 8\epsilon_0 R \times 1.393 = 8\epsilon_0 a \times 1.36. \]

The approximate formula worked out earlier gives

\[ C \simeq \frac{4\pi^2 \epsilon_0 a}{\ln \left( \frac{8a}{b} \right)} = 8\epsilon_0 a \times 1.34, \]

which is in reasonable agreement with the numerical result.
1.7 Spherical Multipole Expansion of the Scalar Potential

The potential due to a prescribed charge distribution,
\[ \Phi(r) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(r')}{|r - r'|} dV', \]
can be expanded in terms of the general spherical harmonics as follows. For this purpose, it is sufficient to expand the Green’s function,
\[ G(r, r') = \frac{1}{4\pi |r - r'|}, \]  
(1.153)
which satisfies the singular Poisson’s equation,
\[ \nabla^2 G = -\delta(r - r'). \]  
(1.154)
The Green’s function \( G \) satisfies Laplace equation except at \( r = r' \),
\[ \nabla^2 G = 0, \quad r \neq r', \]  
(1.155)
and we first seek elementary solutions for Laplace equation. Assuming that the function \( G(r, \theta, \phi) \) is separable in the form
\[ G(r, \theta, \phi) = R(r)F_1(\theta)F_2(\phi), \]  
(1.156)
and substituting this into Laplace equation
\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) G(r, \theta, \phi) = 0, \]  
(1.157)
we obtain three ordinary equations,
\[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right) R(r) = 0, \]  
(1.158)
\[ \left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) + l(l+1) - \frac{m^2}{\sin^2 \theta} \right] F_1(\theta) = 0, \]  
(1.159)
\[ \left( \frac{d^2}{d\phi^2} + m^2 \right) F_2(\phi) = 0, \]  
(1.160)
where \( l(l+1) \) and \( m^2 \) are separation constants. Solutions of the radial function \( R(r) \) are as before,
\[ R(r) = r^l, \quad \frac{1}{r^{l+1}}. \]
In free space, the Green’s function must be periodic with respect to the azimuthal angle \( \phi \). Therefore, \( F_2(\phi) = e^{im\phi} \) with \( m \) being an integer.
Equation (1.159) is known as the modified Legendre equation and its solutions are

\[ F_1(\theta) = P_l^m(\cos \theta), \quad Q_l^m(\cos \theta), \quad (1.161) \]

where

\[ P_l^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad (1.162) \]
\[ Q_l^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} Q_l(x). \quad (1.163) \]

That \( P_l^m(\cos \theta) \) given in Eq. (1.162) satisfies the modified Legendre equation can be seen as follows. The ordinary Legendre function \( P_l(\cos \theta) = P_l^0(\cos \theta) \) satisfies

\[ \frac{d}{d\mu} \left( (1 - \mu^2) \frac{d}{d\mu} P_l(\mu) \right) + l(l+1)P_l(\mu) = 0. \]

Differentiating \( m \) times yields

\[ (1 - \mu^2) \frac{d^{m+2} P_l}{d\mu^{m+2}} - 2\mu(m+1) \frac{d^{m+1} P_l}{d\mu^{m+1}} + [l(l+1) - m(m+1)] \frac{d^m P_l}{d\mu^m} = 0. \]

Let us assume

\[ F_1(\theta) = (1 - \mu^2)^{m/2} f(\mu), \]

and substitute this into Eq. (1.159). \( f(\mu) \) satisfies the same equation as \( \frac{d^m P_l}{d\mu^m} \),

\[ (1 - \mu^2) \frac{d^2 f}{d\mu^2} - 2\mu(m+1) \frac{df}{d\mu} + [l(l+1) - m(m+1)] f = 0. \]

Therefore, \( f(\mu) = \frac{d^m P_l}{d\mu^m} \).

Since the Legendre function \( P_l(x) \) is a polynomial of order \( l \), the azimuthal mode number \( m \) is limited in the range \( 0 \leq m \leq l \). For later use, we combine the functions \( F_1(\theta) \) and \( F_2(\phi) \) and introduce the spherical harmonic function defined by

\[ Y_{lm}(\theta, \phi) = \text{const.} \ P_l^m(\cos \theta)e^{im\phi}, \quad (1.164) \]

where the constant is chosen from the following normalization,

\[ \int Y_{lm}(\theta, \phi)^* Y_{lm}(\theta, \phi) d\Omega = 1. \quad (1.165) \]

Noting

\[ \int_0^\pi [P_l^m(\cos \theta)]^2 \sin \theta d\theta = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}, \quad (1.166) \]

we find

\[ \text{const.} = \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!}. \]
\[ Y_{lm} = \sqrt{\frac{2l + 1}{4\pi}} \frac{(l - m)!}{(l + m)!} P_l^m(\cos \theta) e^{im\phi}. \] (1.167)

For historical reasons, it is customary to write \( Y_{lm}(\theta, \phi) \) in the form

\[ Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l + 1}{4\pi}} \frac{(l - m)!}{(l + m)!} P_l^m(\cos \theta) e^{im\phi}, \] for \( 0 \leq m \leq l \),

\[ Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi), \] for \( -l \leq m \leq 0 \),

or for arbitrary \( m \), positive or negative,

\[ Y_{lm}(\theta, \phi) = (-1)^{(m+|m|)/2} \sqrt{\frac{2l + 1}{4\pi}} \frac{(l - |m|)!}{(l + |m|)!} P_l^{|m|}(\cos \theta) e^{im\phi}. \] (1.168)

Some low order forms of \( Y_{lm}(\theta, \phi) \) are:

\[ l = 0 \quad m = 0 \quad Y_{00} = \frac{1}{\sqrt{4\pi}} \]

\[ l = 1 \quad m = 0 \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta \]

\[ m = \pm 1 \quad Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \]

\[ l = 2 \quad m = 0 \quad Y_{20} = \sqrt{\frac{5}{4\pi}} \frac{3\cos^2 \theta - 1}{2} \]

\[ m = \pm 1 \quad Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi} \]

\[ m = \pm 2 \quad Y_{2,\pm 2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{\pm 2i\phi} \]

Having found the general solutions of the Green’s function, we now assume the following expansion for \( G \),

\[ G(r, r') = \sum_{lm} A_{lm} g_l(r, r') Y_{lm}(\theta, \phi), \]

where the radial function \( g_l(r, r') \) is

\[ g_l(r, r') = \begin{cases} \frac{r'}{r^{l+1}}, & r < r', \\ \frac{r}{r'^{l+1}}, & r > r'. \end{cases} \]
In this form, the function \( g_l(r, r') \) remains bounded everywhere. The expansion coefficient \( A_{lm} \) can be determined by multiplying the both sides of

\[
\nabla^2 G = -\delta(r - r') = -\frac{\delta(r - r')}{rr'} \delta(\cos \theta - \cos \theta') \delta(\phi - \phi'),
\]

by \( Y^*_{l'm'}(\theta, \phi) \) and integrating the result over the entire solid angle,

\[
A_{lm} \frac{d^2}{dr^2} g_l(r, r') = -\frac{\delta(r - r')}{rr'} Y^*_{l'm'}(\theta', \phi'),
\]

where orthogonality of \( Y_{lm}(\theta, \phi) \),

\[
\int Y_{lm}(\theta, \phi) Y^*_{l'm'}(\theta, \phi) d\Omega = \delta_{ll'} \delta_{mm'},
\]

is exploited to remove summation over \( l \) and \( m \). Since the radial function \( g_l(r, r') \) has a discontinuity in its derivative at \( r = r' \), it follows that

\[
\frac{d^2}{dr^2} g_l(r, r') = -(2l + 1) \frac{\delta(r - r')}{r^2}.
\]

Therefore,

\[
A_{lm} = \frac{1}{2l + 1} Y^*_{l'm'}(\theta', \phi'),
\]

and the desired spherical harmonic expansion of the Green’s function is given by

\[
G(r, r') = \frac{1}{4\pi |r - r'|} = \sum_{lm} \frac{1}{2l + 1} g_l(r, r') Y_{lm}(\theta, \phi) Y^*_{l'm'}(\theta', \phi').
\]

For a given charge distribution \( \rho(r) \), this allows evaluation of the potential in the form of spherical harmonics,

\[
\Phi(r) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(r')}{|r - r'|} dV' = \frac{1}{4\pi \varepsilon_0} \sum_{lm} \frac{4\pi}{2l + 1} \frac{1}{r^{l+1}} Y_{lm}(\theta, \phi) \int r'^l Y^*_{l'm'}(\theta', \phi') \rho(r', \theta', \phi') dV'
\]

\[
= \frac{1}{4\pi \varepsilon_0} \sum_{lm} \frac{4\pi}{2l + 1} \frac{1}{r^{l+1}} Y_{lm}(\theta, \phi) q_{lm}, \text{ in the region } r > r',
\]

where

\[
q_{lm} = \int r'^l Y^*_{l'm'}(\theta', \phi') \rho(r', \theta', \phi') dV',
\]

is the electric multipole moment of order \((l, m)\).

**Example 5 Planar Quadrupole**

A planar quadrupole consists of charges +\( q \) and -\( q \) alternatively placed at each corner of a
square of side $a$ as shown in Fig.1-14. The charge density may be written as

$$\rho(r) = q \left( \delta(r) - \frac{\delta(r-a)}{a^2} \delta(\cos \theta) \delta(\phi) + \frac{\delta(r-\sqrt{2}a)}{2a^2} \delta(\cos \theta) \delta\left(\phi - \frac{\pi}{4}\right) - \frac{\delta(r-a)}{a^2} \delta(\cos \theta) \delta\left(\phi - \frac{\pi}{2}\right) \right).$$

The potential in the far field region $r \gg a$ is of quadrupole nature and given by

$$\Phi(r, \theta, \phi) = \frac{1}{\varepsilon_0} \frac{1}{3r^3} (Y_{2,2}q_{2,2} + Y_{2,-2}q_{2,-2}),$$

where

$$q_{2,\pm2} = \int r^2 Y_{2,\pm2}(\theta, \phi) \rho(r) dV = \frac{1}{2} \sqrt{\frac{15}{2\pi}} a^2 q e^{\mp i\pi/2}.$$

The potential reduces to

$$\Phi(r, \theta, \phi) \approx \frac{3}{8\pi\varepsilon_0} \frac{a^2}{r^3} \sin^2 \theta \sin(2\phi), \quad r \gg a.$$

The reader should check that this is consistent with the direct sum of four potentials in the limit $r \gg a$.

![Diagram](image_url)

Figure 1-14: Planar quadrupole in the $x - y$ plane.

1.8 Collection of Dipoles, Dielectric Properties

Dielectric properties of material media originate from dipole moments carried by atoms and molecules. Some molecules carry permanent dipole moments. For example, the water molecule has a dipole moment of about $6 \times 10^{-30}$ C-m due to deviation of the center of electron cloud from the center of proton charges. When water is placed in an external electric field, the dipoles tend to be
aligned in the direction of the electric field and a resultant electric field in water becomes smaller than the unperturbed external field by a factor \( \varepsilon / \varepsilon_0 \), where \( \varepsilon \) is the permittivity of water. Even a material composed of molecules having no permanent dipole moment exhibits dielectric property. For example, a dipole moment is induced in a hydrogen atom placed in an electric field through perturbation in the electron cloud distribution which is spatially symmetric in the absence of external electric field. In this section, the potential and electric field due to a collection of dipole moments will be analyzed.

The potential due to a single dipole \( \mathbf{p} \) located at \( \mathbf{r}' \) is

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{p}}{|\mathbf{r} - \mathbf{r}'|^3}.
\]  
(1.176)

Consider a continuous distribution of many dipoles. It is convenient to introduce a dipole moment density \( \mathbf{P} = n \mathbf{p} \) \((\text{C} \cdot \text{m}/\text{m}^3 = \text{C}/\text{m}^2)\) where \( n \) is the number density of dipoles. An incremental potential due to an incremental “point” dipole \( d\mathbf{p} = \mathbf{P}(\mathbf{r}')dV' \) is

\[
d\Phi = \frac{1}{4\pi\varepsilon_0} \frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'.
\]  
(1.177)

Integration of this yields

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'.
\]  
(1.178)

Since

\[
\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right),
\]  
(1.179)

where \( \nabla' \) means differentiation with respect to \( \mathbf{r}' \), the integral in Eq. (1.178) can be rewritten as

\[
\int \frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' = \int \nabla' \cdot \left( \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) dV' - \int \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'.
\]

The first term in the right can be converted into a surface integral through the Gauss theorem,

\[
\int_V \nabla' \cdot \left( \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) dV' = \oint_S \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \cdot d\mathbf{S}',
\]

which vanishes on a closed surface with infinite extent on which all sources should be absent. Therefore, the potential due to distributed dipole moments is

\[
\Phi_{\text{dipole}}(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0} \int \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'.
\]  
(1.180)

Comparing with the standard form of the potential,

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV',
\]
we see that

$$\rho_{\text{eff}} = -\nabla \cdot \mathbf{P},$$

can be regarded as an effective charge density. To distinguish it from the free charge density, $\rho_{\text{eff}}$ defined above is called bound charge density. In general, $\rho_{\text{eff}}$ cannot be controlled by external means. In a dielectric body, the bound charge appears as a surface charge density. Adding the monopole potential due to a free charge density $\rho_{\text{free}}(\mathbf{r})$, we find the total potential

$$\Phi(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho_{\text{free}}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' - \frac{1}{4\pi \varepsilon_0} \int \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'. \quad (1.181)$$

Noting

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi \delta(\mathbf{r} - \mathbf{r}'),$$

we find

$$\nabla \cdot \mathbf{E} = -\nabla^2 \Phi = \frac{\rho_{\text{free}}}{\varepsilon_0} - \frac{\nabla \cdot \mathbf{P}}{\varepsilon_0}. \quad (1.182)$$

In this form too, it is evident that $-\nabla \cdot \mathbf{P}$ can be regarded as an effective charge density. Eq. (1.181) can be rearranged as

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{\text{free}}. \quad (1.183)$$

Introducing a new vector $\mathbf{D}$ (displacement vector) by

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad (1.184)$$

we write Eq. (1.183) as

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}}, \quad (1.185)$$

which is equivalent to the original Maxwell’s equation

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{all}}}{\varepsilon_0}, \quad (1.186)$$

provided that the total charge density $\rho_{\text{all}}$ consists of free charges and dipole charges and that higher order charge distributions (quadrupole, octupole, etc.) are ignorable. The vector $\mathbf{D}$ was named the displacement vector by Maxwell. Its fundamental importance in electrodynamics will be appreciated later in time varying fields because it was with this displacement vector that Maxwell was able to predict propagation of electromagnetic waves in vacuum. As briefly discussed in Section 2, Maxwell’s equations would not be consistent with the charge conservation principle if the displacement current,

$$\frac{\partial (\varepsilon \mathbf{E})}{\partial t} = \frac{\partial \mathbf{D}}{\partial t},$$

were absent.

In usual linear insulators, both $\mathbf{D}$ and $\mathbf{P}$ are proportional to the local electric field $\mathbf{E}$ which
allows us to introduce an effective permittivity $\varepsilon$,

$$D = \varepsilon_0 E + P = \varepsilon E. \quad (1.187)$$

In some solid and liquid crystals, the permittivity takes a tensor form,

$$D = \varepsilon \cdot E \text{ or } D_i = \varepsilon_{ij} E_j, \quad (1.188)$$

because of anisotropic polarizability. Double refraction phenomenon in optics already known in the 17th century is due to the tensorial nature of permittivity in some crystals as we will study in Chapter 5.

Figure 1-15: In an electric field, the hydrogen atom exhibits a dipole moment due to the displacement of the center of electron cloud from the proton.

The origin of field induced dipole moment may be seen qualitatively for the case of hydrogen atom as follows. In an electric field, the distribution of the electron cloud around the proton becomes asymmetric because of the electric force acting on the electron. The center of electron cloud is displaced from the proton by $x$ given by

$$m\omega_0^2 x = eE,$$

where $\omega_0$ is the frequency of bound harmonic motion of the electron and $m$ is the electron mass. ($\hbar\omega_0$ is of the order of the ionization potential energy.) Then, the dipole moment induced by the electric field is

$$p = ex = \frac{e^2}{m\omega_0^2} E.$$

If the atomic number density is $n$, the dipole moment density $P$ is

$$P = np = \frac{ne^2}{m\omega_0^2} E, \quad (1.189)$$
and the displacement vector is given by

\[
D = \varepsilon_0 E + P = \varepsilon_0 \left(1 + \frac{ne^2}{\varepsilon_0 m \omega_0^2}\right) E = \varepsilon E.
\]  

(1.190)

The permittivity can therefore be defined by

\[
\varepsilon = \varepsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2}\right),
\]

(1.191)

where

\[
\omega_p = \sqrt{\frac{ne^2}{\varepsilon_0 m}}.
\]

(1.192)

is an effective plasma frequency. (The plasma frequency pertains to free electrons in a plasma. \(\omega_p\) introduced here is so called only for dimensional convenience.)

In the incremental contribution to the permittivity normalized by \(\varepsilon_0\),

\[
\frac{\omega_p^2}{\omega_0^2} = n \frac{e^2}{m \varepsilon_0 \omega_0^2},
\]

the quantity

\[
\frac{e^2}{4\pi \varepsilon_0 m \omega_0^2},
\]

is of the order of \(r_0^3\) where \(r_0\) is the Bohr radius. This can be seen from the force balance for the electron,

\[
mr_0 \omega_0^2 = \frac{e^2}{4\pi \varepsilon_0 r_0^2},
\]

which indeed gives

\[
\frac{e^2}{4\pi \varepsilon_0 m \omega_0^2} = r_0^3.
\]

The quantity

\[
\alpha = 4\pi \varepsilon_0 r_0^3 = \frac{e^2}{m \omega_0^2},
\]

(1.193)

is called the atomic polarizability. A more rigorous quantum mechanical calculation yields for this quantity

\[
\alpha = \frac{9}{2} \times 4\pi \varepsilon_0 r_0^3,
\]

(1.194)

indicating a substantial correction (by a factor 4.5) to the classical picture based on the assumption of rigid shift of the electron cloud. For molecular hydrogen \(H_2\), the polarizability at \(T = 273\) K is approximately given by

\[
\alpha \simeq 5.4 \times 4\pi \varepsilon_0 r_0^3.
\]
Satisfactory quantum mechanical calculation was performed by Kolos and Wolniewicz relatively recently (1967). This is in a fair agreement with the permittivity experimentally measured in the standard condition 20 C, 1 atmospheric pressure,

\[ \varepsilon \simeq (1 + 2.7 \times 10^{-4}) \varepsilon_0. \]

The atomic polarizability \( \alpha \) and the macroscopic relative permittivity \( \varepsilon_r \) are related through a simple relationship known as the Clausius-Mossoti equation,

\[ \frac{n \alpha}{\varepsilon_0} = \frac{3}{\varepsilon_r - 1} \frac{\varepsilon_r - 1}{\varepsilon_r + 2}, \]

(1.195)

where \( n \) is the number density of atoms. This may qualitatively be seen as follows. A dipole moment induced by an atom,

\[ p = e x = \frac{e^2}{m \omega_0^2} E = \alpha E_{\text{ext}}, \]

(1.196)

produces an electric field

\[ E' = -\frac{p}{4\pi \varepsilon_0 r^3}, \]

(1.197)

at a distance \( r \). If the number density of atoms is \( n \), the average inter-atom distance \( R \) can be found from

\[ \frac{4\pi}{3} R^3 = \frac{1}{n}. \]

(1.198)

Then,

\[ E' = -\frac{\alpha}{4\pi \varepsilon_0 R^3} E_{\text{ext}}, \]

(1.199)

and the total electric field is

\[
E = \left(1 - \frac{\alpha}{4\pi \varepsilon_0 R^3}\right) E_{\text{ext}} = \left(1 - \frac{n \alpha}{3 \varepsilon_0}\right) E_{\text{ext}},
\]

(1.200)

(1.201)

Therefore, using this total field in calculation of the polarization field \( P \), we find

\[ P = \frac{n \alpha}{1 - \frac{n \alpha}{3 \varepsilon_0}} E = \frac{n \alpha / \varepsilon_0}{1 - \frac{n \alpha}{3 \varepsilon_0}} \varepsilon_0 E, \]

(1.202)

which defines the relative permittivity \( \varepsilon_r \) as

\[ \varepsilon_r - 1 = \frac{n \alpha / \varepsilon_0}{1 - \frac{n \alpha}{3 \varepsilon_0}}. \]

(1.203)

Solving for \( n \alpha / \varepsilon_0 \), we obtain Eq.(1.195). Note that the relative permittivity is an easily measurable quantity and it reflects, somewhat surprisingly, microscopic atomic polarizability through a simple
relationship.

In an oscillating electric field, the displacement $x$ is to be found from the equation of motion,

$$m \left( \frac{\partial^2}{\partial t^2} + \omega_0^2 \right) x = -eE_0 e^{-i\omega t}, \quad (1.204)$$

which yields

$$x(t) = \frac{eE_0 e^{-i\omega t}}{\omega^2 - \omega_0^2}. \quad (1.205)$$

A resultant permittivity is

$$\varepsilon(\omega) = \varepsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2} \right). \quad (1.206)$$

This exhibits a resonance at the frequency $\omega_0$. The resonance plays an important role in calculation of energy loss of charged particles moving in a dielectric medium as will be shown in Chapter 8.

Most molecules have permanent electric dipole moments. In the absence of electric fields, dipoles are randomly oriented due to thermal agitation. When an electric field is present, each dipole acquires a potential energy,

$$U = -\mathbf{p} \cdot \mathbf{E} = -pE \cos \theta, \quad (1.207)$$

where $\theta$ is the angle between the dipole moment $\mathbf{p}$ and electric field $\mathbf{E}$. Dipoles are randomly oriented but should obey the Boltzmann distribution in thermal equilibrium,

$$n(\cos \theta) = n_0 \exp \left( -\frac{U}{k_B T} \right) = n_0 \exp \left( \frac{pE \cos \theta}{k_B T} \right), \quad (1.208)$$

where $n(\cos \theta)/n_0$ is the probability of dipole orientation in $\theta$ direction, $n_0$ is the number density of the dipoles and $k_B = 1.38 \times 10^{-23}$ J/K is the Boltzmann constant. The average dipole number density oriented in the direction of the electric field can be calculated from

$$\bar{n} = \frac{\int n(\cos \theta) \cos \theta d\Omega}{\int \exp \left( \frac{pE \cos \theta}{k_B T} \right) d\Omega}. \quad (1.209)$$

In practice, the potential energy is much smaller than the thermal energy $pE \ll k_B T$. Therefore, the exponential function can be approximated by

$$\exp \left( \frac{pE \cos \theta}{k_B T} \right) \simeq 1 + \frac{pE \cos \theta}{k_B T}, \quad (1.210)$$

and we obtain

$$\bar{n} \simeq \frac{n_0 p}{3k_B T} E. \quad (1.211)$$
Resultant dipole moment density is

\[ P = \tilde{n}p = \frac{n_0p^2}{3k_BT}E. \]  (1.212)

This defines a permittivity in a medium consisting of molecules having a permanent dipole moment \( p \),

\[ \varepsilon = \varepsilon_0 \left( 1 + \frac{n_0p^2}{3\varepsilon_0k_BT} \right). \]  (1.213)

Note that the correction term

\[ \Delta \varepsilon = \frac{n_0p^2}{3k_BT}, \]  (1.214)

is inversely proportional to the temperature. This property can be exploited in separating contributions from atomic and molecular polarizability in a given medium. In liquids and solids, the additional permittivity given above can be comparable with \( \varepsilon_0 \) primarily because of the large number density \( n_0 \). For example, a water molecule has a dipole moment of \( p = 6 \times 10^{-30} \) C m. At room temperature, the additional permittivity due to molecular polarizability of water is approximately \( \Delta \varepsilon \approx 11\varepsilon_0 \). The measured permittivity of water at room temperature is \( 81\varepsilon_0 \). The atomic polarizability is thus dominant.

### 1.9 Boundary Conditions for \( E \) and \( D \)

In electrostatics, the electric field \( E \) obeys the Maxwell’s equations,

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0}, \]

and

\[ \nabla \times E = 0. \]

In problems involving dielectrics, it is more convenient to introduce the displacement vector \( D \),

\[ \nabla \cdot D = \rho_{\text{free}}, \]  (1.215)

where \( \rho_{\text{free}} \) is the free charge density that can be controlled by external means.

Let us consider a boundary of two dielectrics having permittivities \( \varepsilon_1 \) and \( \varepsilon_2 \), respectively. Integration of \( \nabla \cdot D = \rho_{\text{free}} \) over the volume of a pancake \( dV = dS \cdot dn \) (\( dn \) is the thickness of the pancake in the direction perpendicular to the surface \( dS \)) on the boundary yields

\[ \int \nabla \cdot D dV = \oint D \cdot dS = \int \rho_{\text{free}} dV. \]  (1.216)

In the limit of infinitesimally thin pancake, this reduces to

\[ (D_{n1} - D_{n2})dS = \rho_{\text{free}} dndS = \sigma_{\text{free}} dS, \]  (1.217)
where $\sigma_{\text{free}} = \rho_{\text{free}} \, dn$ is the free surface charge density residing on the boundary. Note that the volume charge density in the presence of surface charge can be written as

$$\rho = \sigma \delta(n - n').$$

(1.218)

General boundary condition for the normal component of the displacement vector is thus given by

$$D_{1n} - D_{2n} = \sigma_{\text{free}}.$$

(1.219)

In the absence of free surface charge, the normal component of the displacement vector should be continuous,

$$D_{n1} = D_{n2}, \quad \text{no free charge.}$$

(1.220)

Figure 1-17: $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ applied to a rectangle at the boundary of two dielectrics. The tangential component of the electric field $\mathbf{E}_t$ is continuous. This holds in general for time varying fields as well.
The equation $\nabla \times \mathbf{E} = 0$ demands that the tangential component of electric field be continuous across a boundary of dielectrics. This can be seen by integrating $\nabla \times \mathbf{E} = 0$ over a small rectangular area on the boundary,

$$
\int \nabla \times \mathbf{E} \cdot d\mathbf{S} = \oint \mathbf{E} \cdot d\mathbf{l} = 0,
$$

which yields

$$
(E_1 - E_2) \cdot d\mathbf{l} = 0, \quad E_{1t} = E_{2t}.
$$

(1.221)

In fact, the continuity of the tangential component of the electric field holds for general time varying case,

$$
\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t},
$$

(1.222)

for the area integral

$$
\int_S \mathbf{B} \cdot d\mathbf{S},
$$

vanishes in the limit of infinitesimally thin rectangle, $S \to 0$. In contrast to charge and current densities, a singular magnetic field involving a delta function is not physically realizable because then the magnetic energy simply diverges. Note that the square of a delta function is not integrable,

$$
\int_0^\infty \delta^2(x) dx = \delta(0) = \infty.
$$

In a conductor, there can be no static electric fields and $\mathbf{E} = 0$ must hold inside conductors. Therefore, at a conductor surface, the tangential component of the electric field should vanish and the electric field lines fall normal to the surface. The normal component of the electric field and the surface charge density $\sigma$ are related through

$$
E_n = \frac{\sigma}{\varepsilon_0}, \text{ at conductor surface.}
$$

(1.224)

The potential of a conducting body is constant. The surface electric field depends on the amount of charge carried by the body and also curvature of the surface. A trivial case is the electric field at the surface of charged conducting sphere of radius $a$,

$$
E = \frac{1}{4\pi \varepsilon_0} \frac{q}{a^2}.
$$

In general, the field increases as the curvature radius decreases. The electric field at the tip of needle can be strong enough to allow emission of electrons as in tunneling electron microscopes. In power engineering, conductor surfaces should be made smooth and round as much as possible to avoid breakdown.

**Example 6 Dielectric Sphere in an Electric Field**

Let us consider an uncharged dielectric sphere having a uniform permittivity $\varepsilon$ placed in an uniform external electric field $\mathbf{E}_0 = E_0 \mathbf{e}_z$. The sphere perturbs the external field because a bound
surface charge \( \rho = -\nabla \cdot \mathbf{P} \) will be induced on the sphere. The interior and exterior potentials may be expanded in the spherical harmonics,

\[
\Phi(r, \theta) = \Phi_0 + \sum_l A_l \left( \frac{r}{a} \right)^l P_l(\cos \theta), \quad r < a \text{ (interior)},
\]

\[
\Phi(r, \theta) = \Phi_0 + \sum_l B_l \left( \frac{a}{r} \right)^{l+1} P_l(\cos \theta), \quad r > a \text{ (exterior)},
\]

where

\[
\Phi_0(r, \theta) = -E_0 z = -E_0 r \cos \theta,
\]

is the potential associated with the external electric field \( \mathbf{E}_0 = E_0 \mathbf{e}_z \). Since there is no double layer to cause potential jump at the sphere surface, the potential is continuous at the surface \( r = a \), from which it follows \( A_l = B_l \). This also follows from the continuity of the tangential component (\( \theta \) component) of the electric field,

\[
E_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta}.
\]

The continuity of the normal (radial) component of the displacement vector requires

\[
\epsilon \left( -E_0 \cos \theta + \sum_l A_l \frac{1}{a} P_l(\cos \theta) \right) = -\epsilon_0 \left( E_0 \cos \theta + \sum_l A_l \frac{l+1}{a} P_l(\cos \theta) \right).
\]

Since \( \cos \theta = P_1(\cos \theta) \), only the \( l = 1 \) terms are non-vanishing and we readily find

\[
A_1 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a E_0.
\]

Then the solutions for the potentials are

\[
\Phi(r, \theta) = \begin{cases} 
\Phi_0(r, \theta) + \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 r \cos \theta, & r < a \\
\Phi_0(r, \theta) + \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 \frac{a^3}{r^2} \cos \theta, & r > a
\end{cases}
\]

The electric field in the dielectric sphere is uniform and given by

\[
E_{iz} = \frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0, \quad r < a.
\]

This is smaller than the external field \( E_0 \) since \( \epsilon > \epsilon_0 \). The interior displacement vector is

\[
D_{iz} = \varepsilon E_{iz} = \frac{3\varepsilon}{\varepsilon + 2\epsilon_0} \epsilon_0 E_0,
\]
which is larger than $D_0 = \varepsilon_0 E_0$. The perturbed exterior potential,

\[ \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \frac{a^3}{r^2} E_0 \cos \theta, \] (1.234)

is of dipole form with an effective dipole moment

\[ p_z = 4\pi \varepsilon_0 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} a^3 E_0, \] (1.235)

located at the center of the sphere.

In the limit of $\varepsilon \gg \varepsilon_0$, the potential reduces to

\[ \Phi(r, \theta) = \begin{cases} 0, & r < a \\ \Phi_0(r, \theta) + \frac{a^3}{r^2} E_0 \cos \theta, & r > a \end{cases} \] (1.236)

which describes the potential when a conducting sphere is placed in an external electric field. A sphere of infinite permittivity is mathematically identical to a conducting sphere.

The polarization vector $\mathbf{P}$ can be found from

\[ D_z = \varepsilon_0 E_z + P_z, \] (1.237)

\[ P_z = \frac{3(\varepsilon - \varepsilon_0)}{\varepsilon + 2\varepsilon_0} \varepsilon_0 E_0, \quad r < a. \] (1.238)

Since $\mathbf{P} = 0$ outside the sphere, the divergence of the polarization vector yields the bound charge density induced on the sphere surface,

\[ \rho_{\text{eff}} = -\nabla \cdot \mathbf{P} = \frac{2(\varepsilon - \varepsilon_0)}{\varepsilon + 2\varepsilon_0} \varepsilon_0 E_0 \cos \theta \delta(r - a). \] (1.239)

The total dipole moment carried by the sphere is

\[ p_z = \frac{4\pi a^3}{3} P_z = \frac{4\pi(\varepsilon - \varepsilon_0) a^3}{\varepsilon + 2\varepsilon_0} \varepsilon_0 E_0, \]

which agrees with that deduced from the exterior potential.
1.10 Electric Force

The energy density associated with an electric field

$$\frac{1}{2} \varepsilon_0 E^2, \quad (\text{J/m}^3)$$

(1.240)

manifests itself as either pressure or tensile stress depending on the direction of the force relative to the electric field. A charge density $\rho$ placed in an electric field $E$ experiences a force per unit volume,

$$f = \rho E$$

$$= \varepsilon_0 (\nabla \cdot E) E$$

$$= \nabla \cdot (\varepsilon_0 E E) - \varepsilon_0 E \cdot \nabla E.$$  (1.241)

In electrostatics, $\nabla \times E = 0$, and thus the curvature $E \cdot \nabla E$ and gradient $E \nabla E = \frac{1}{2} \nabla E^2$ of the electric field are identical,

$$\nabla (E \cdot E) = 2E \times \nabla \times E + 2E \cdot \nabla E$$

$$\nabla E^2 = 2E \cdot \nabla E.$$  (1.242)

Therefore,

$$f = \nabla \cdot \left( \varepsilon_0 EE - \frac{1}{2} \varepsilon_0 E^2 \mathbf{1} \right),$$  (1.243)

where $\mathbf{1}$ is the unit tensor. The tensor

$$T_{ij} = \varepsilon_0 E_i E_j - \frac{1}{2} \varepsilon_0 E^2 \delta_{ij},$$  (1.244)

is called the Maxwell’s stress tensor associated with the electric field. For a linear dielectric, $\varepsilon_0$ may be replaced with its permittivity $\varepsilon$. The force vector is

$$f = \nabla \cdot T, \quad \text{or } f_i = \frac{\partial}{\partial x_j} T_{ij}.$$  (1.245)

Let us consider a simple case: an electric field in the $z$-direction, $E = E_z \mathbf{e}_z$. The tensor $T_{ij}$ is diagonal with the following components,

$$T = \begin{pmatrix}
-\frac{1}{2} \varepsilon_0 E_z^2 & 0 & 0 \\
0 & -\frac{1}{2} \varepsilon_0 E_z^2 & 0 \\
0 & 0 & +\frac{1}{2} \varepsilon_0 E_z^2
\end{pmatrix}.$$  (1.246)
The force in the directions perpendicular to the field are

\[ f_x = -\frac{\partial}{\partial x} \left( \frac{1}{2} \varepsilon_0 E_z^2 \right), \]

\[ f_y = -\frac{\partial}{\partial y} \left( \frac{1}{2} \varepsilon_0 E_z^2 \right), \]

which appear as a pressure acting from a higher energy density region to lower energy density region. The force in the direction of the electric field is

\[ f_z = +\frac{\partial}{\partial z} \left( \frac{1}{2} \varepsilon_0 E_z^2 \right), \]

which appears as tension acting from a lower energy density region to higher energy density region.

The force to act on a volume \( V \) can be found from the integral

\[ F = \int_V f \, dV = \oint_S \mathbf{T} \cdot d\mathbf{S} \]

\[ = \oint_S \left( \varepsilon_0 \mathbf{E} \cdot \mathbf{n} - \frac{1}{2} \varepsilon_0 E^2 \mathbf{n} \right) dS, \]

where \( \mathbf{n} \) is the unit normal vector on the closed surface \( S \), \( d\mathbf{S} = n \, dS \). For a charged conducting sphere, the electric field is radially outward everywhere. At the surface, the force per unit area is

\[ \frac{1}{2} \varepsilon_0 E^2, \quad (N/m^2), \]

acting radially outward. Since \( \mathbf{E} = 0 \) inside a conducting sphere, the force acts from lower energy density region to higher energy density region as expected of tensile stress. Of course, no net force acts on the sphere.

**Example 7 Dielectric Hemispheres in an External Electric Field**

As a less trivial example, we consider a dielectric sphere consisting of identical hemispheres with a narrow gap at an equatorial plane. It is placed in an electric field with the gap plane perpendicular to the field. We wish to find a force to act between the hemisphere. The force, if any, should be axial in the direction perpendicular to the gap. The \( z \)-component of the integral,

\[ F = \oint \left( \varepsilon \mathbf{E} (\mathbf{E} \cdot \mathbf{n}) - \frac{1}{2} \varepsilon E^2 \mathbf{n} \right) dS \]

is

\[ F_z = \oint \left( \varepsilon E_z E_n - \frac{1}{2} \varepsilon E^2 n_z \right) dS, \]

where on the spherical surface of one of the hemispheres,

\[ E_z = E_0 \frac{3}{\varepsilon + 2\varepsilon_0} \left( \varepsilon \cos^2 \theta + \varepsilon_0 \sin^2 \theta \right), \]
\[ E_r = E_0 \frac{3\varepsilon}{\varepsilon + 2\varepsilon_0} \cos \theta, \quad (1.253) \]
\[ E_\theta = -E_0 \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} \sin \theta, \quad (1.254) \]

\[ E^2 = E_r^2 + E_\theta^2 \]
\[ = E_0^2 \frac{9}{(\varepsilon + 2\varepsilon_0)^2} (\varepsilon^2 \cos^2 \theta + \varepsilon_0^2 \sin^2 \theta). \quad (1.255) \]

In the gap, the field is uniform,

Figure 1-18: A dielectric sphere with a negligible gap at an equator placed in an electric field normal to the gap surface. The hemispheres attract each other. The D-field lines (not E field lines) shown are relevant to the preceding Example as well.

\[ E_{\text{gap}} = \frac{\varepsilon}{\varepsilon_0} E_{iz} = \frac{3\varepsilon}{\varepsilon + 2\varepsilon_0} E_0, \quad (1.256) \]

where
\[ E_{iz} = \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0, \quad (1.257) \]

is the internal electric field in the dielectric sphere given in Eq. (1.232). The integral over the hemispherical surface is

\[ F_{\pm 1} = 2\pi a^2 \int_0^{\pi/2} \left( \varepsilon_0 E_z E_\tau - \frac{1}{2} \varepsilon_0 E^2 \cos \theta \right) \sin \theta d\theta \]
\[ = 2\pi a^2 \frac{9\varepsilon_0 E_0^2}{(\varepsilon + 2\varepsilon_0)^2} \int_0^{\pi/2} \left[ \varepsilon^2 \cos^2 \theta + \varepsilon_0 \sin^2 \theta - \frac{1}{2} (\varepsilon^2 \cos^2 \theta + \varepsilon_0 \sin^2 \theta) \right] \cos \theta \sin \theta d\theta \]
\[ = \frac{\pi a^2}{(\varepsilon + 2\varepsilon_0)^2} \frac{1}{4} (\varepsilon^2 + 2\varepsilon_0 - \varepsilon_0^2). \quad (1.258) \]
This force is repelling. The contribution from the flat surface in the gap is

\[ F_{z2} = -\pi a^2 \frac{9\varepsilon_0 E_0^2 \varepsilon^2}{(\varepsilon + 2\varepsilon_0)^2} \frac{1}{2}, \]  

(1.259)

and the net force is

\[ F_z = -\pi a^2 \frac{9(\varepsilon - \varepsilon_0)^2}{4(\varepsilon + 2\varepsilon_0)^2} \varepsilon_0 E_0^2. \]  

(1.260)

The minus sign indicates an attractive force between the hemispheres. In the limit \( \varepsilon \gg \varepsilon_0 \), the force becomes

\[ F_z = -\frac{9}{4} \pi a^2 \varepsilon_0 E_0^2. \]

This corresponds to the case of solid or closed conducting hemispheres.

### 1.11 Force in Capacitor

For two electrode systems, the potential difference between the electrodes \( V \) and the charges \( \pm Q \) residing on each electrode are related through the capacitance \( C \),

\[ Q = CV. \]  

(1.261)

An incremental energy required to increase the charge by \( dQ \) is

\[ dU = VdQ = \frac{1}{C}QdQ = CVdV. \]  

(1.262)

Therefore the amount of energy stored in a capacitor is

\[ U = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2. \]  

(1.263)

This is applicable for a single electrode as well if the second electrode is at infinity (self-capacitance). For example, a conducting sphere of radius \( a \) has a self-capacitance of

\[ C = 4\pi \varepsilon_0 a. \]  

(1.264)

If it carries a charge \( Q \), the amount of potential energy associated with it is

\[ U = \frac{Q^2}{8\pi \varepsilon_0 a}. \]  

(1.265)

Of course, the energy is stored in the space surrounding the sphere, and the energy can alternatively be calculated from the integral,

\[ U = \int_a^\infty \frac{1}{2} \varepsilon_0 E_r^2 4\pi r^2 dr = \frac{Q^2}{8\pi \varepsilon_0 a}, \]  

(1.266)
where
\[
E_r = \frac{Q}{4\pi \varepsilon_0 r^2}, \quad r > a
\] (1.267)
is the electric field.

If a capacitance is described as a function of geometrical factor \( \xi \), such as electrode separation distance and electrode size, the electric force tends to act in such a way to increase the capacitance,

\[
F_\xi = \frac{1}{2} V^2 \frac{\partial C(\xi)}{\partial \xi}, \quad \text{if the voltage is held constant}, \quad (1.268)
\]
\[
F_\xi = -\frac{1}{2} Q^2 \frac{\partial}{\partial \xi} \left( \frac{1}{C(\xi)} \right), \quad \text{if the charge is held constant}. \quad (1.269)
\]

For example, the capacitance of parallel plates capacitor consisting of circular disks is

\[
C(d, a) = \frac{\pi a^2}{d},
\] (1.270)

where \( a \) is the disk radius and \( d \) is the separation distance. The disks evidently attract each other so as to reduce the distance \( d \) or increase the capacitance with a force

\[
F_d = -\frac{1}{2} \varepsilon_0 \frac{\pi a^2}{d^2} V^2 = -\frac{1}{2} \frac{Q^2}{\varepsilon_0 \pi a^2}. \quad (1.271)
\]

The radial force is

\[
F_a = \varepsilon_0 \frac{\pi a}{d} V^2 = \frac{d}{\varepsilon_0 \pi a^3} Q^2, \quad (1.272)
\]

which acts so as to increase the radius of the disks. Note that when the charge is fixed, the force acts to reduce the energy stored in the capacitor, while when the voltage is fixed, the force acts to increase the energy. A power supply is a large reservoir of energy and a capacitor connected to a power supply (the case of fixed voltage) is not a closed system. The earlier statement based on the variational principle that an electrostatic equilibrium is the minimum energy state of course pertains to closed systems.
Problems

1.1 Verify that for each of the expressions of the three dimensional delta functions,

\[ \delta(r - r') = \delta(x - x')\delta(y - y')\delta(z - z'), \]

\[ \delta(r - r') = \frac{\delta(\rho - \rho')}{\rho'} \delta(\phi - \phi')\delta(z - z'), \]

\[ \delta(r - r') = \frac{\delta(r - r')}{rr'} \delta(\cos \theta - \cos \theta')\delta(\phi - \phi'), \]

the volume integral is unity,

\[ \int \delta(r - r')dV = 1. \]

1.2 The electron cloud in a hydrogen atom is described by the charge density distribution

\[ \rho(r) = \frac{-e}{\pi a^3} \exp\left(\frac{-2r}{a}\right), \]

where \( a = 5.3 \times 10^{-11} \) m is the Bohr radius. Show that the potential energy of a hydrogen atom is

\[ U = -\frac{1}{4\pi \varepsilon_0} \frac{e^2}{a}. \]

1.3 A planar quadrupole in the \( x - y \) plane consists of four charges, \( +q \) and \( -q \) alternatively placed at the corners of a square of side \( a \). What is the potential energy of the quadrupole?

1.4 Four charges \( e, -e, -e \) and \( -e \) are placed at the corners of a tetrahedron having side \( a \). Find the electric dipole and quadrupole moments.

1.5 Show that for \( l = 1 \),

\[ \sum_{m=-1}^{1} rY_{1,m}(\theta, \phi)q_{1,m} = \frac{3}{4\pi} \mathbf{r} \cdot \mathbf{p}, \]
and a resultant dipole potential is consistent with the direct expansion of the potential, 

\[ \Phi_{\text{dipole}} = \frac{1}{4\pi \varepsilon_0} \frac{\mathbf{r} \cdot \mathbf{p}}{r^3}. \]

1.6 An octupole consists of eight charges (four \(+q\) and four \(-q\)) alternatively placed at the corners of a cube of side \(a\). Determine the far field potential \(\Phi(r, \theta, \phi)\) at \(r \gg a\) and also the potential energy of the octupole.

1.7 Equal octants on a spherical surface of radius \(a\) are maintained alternatively at potentials \(V\) and \(-V\). Determine the lowest order far field potential \(\Phi(r, \theta, \phi)\) at \(r \gg a\). Useful expansion is

\[ \Phi(r) = \sum_{lm} \left( \frac{a}{r} \right)^{l+1} Y_{lm}(\theta, \phi) \int \Phi_s(\theta', \phi') Y_{lm}^*(\theta', \phi') d\Omega', \quad r > a, \]

where \(\Phi_s(\theta, \phi)\) is the surface potential and \(d\Omega' = \sin \theta' d\theta' d\phi'\).

1.8 The potential due to a long line charge \(\lambda\) (C/m) is

\[ \Phi(\rho) = -\frac{\lambda}{2\pi \varepsilon_0} \ln \rho + \text{constant}, \]

where \(\rho\) is the distance from the line charge. Verify this from the basic formula,

\[ \Phi(r) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(r')}{|r - r'|} dV', \]

\[ = \frac{1}{4\pi \varepsilon_0} \int_{-\infty}^{\infty} \frac{\lambda}{\sqrt{\rho^2 + z'^2}} dz'. \]

Then, show that the capacitance per unit length of a parallel-wire transmission line with wire separation distance \(d\) and common wire radii \(a\) (\(\ll d\)) is approximately given by

\[ \frac{C}{l} \approx \frac{\pi \varepsilon_0}{\ln \left( \frac{d}{a} \right)}, \quad \text{(F/m)}. \]

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What is the inductance per unit length of the transmission line? (Hint: The product

\[
\frac{C}{l} \frac{L}{l},
\]

is constant and equal to \( \varepsilon_0 \mu_0 \). For a coaxial cable filled with insulating material having permittivity \( \varepsilon \), the capacitance is

\[
\frac{C}{l} = \frac{2\pi \varepsilon}{\ln(a/b)},
\]

and the product \( \frac{C}{l} \frac{L}{l} \) is equal to \( \varepsilon \mu_0 \).

1.9 An insulating circular disk of radius \( a \) carries a total charge \( q \) uniformly distributed over its area \( \pi a^2 \). By first finding the potential on the axis of the disk, \( \Phi(z) \), generalize it to a potential at arbitrary position \( \Phi(r, \theta) \) in terms of the spherical harmonic functions \( P_l(\cos \theta) \). (The case of charged conducting disk will be analyzed in Chapter 2.) Check whether the Poisson’s equation,

\[
\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0},
\]

is satisfied by your solution. (It should be.)

1.10 Solve the preceding problem using the cylindrical coordinates \( (\rho, z) \).

1.11 The Legendre polynomial \( P_l(x) \) can be generated from

\[
P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l, \quad \text{(Rodrigues’ formula)}.
\]

Using this repeatedly, verify the orthogonality of Legendre functions,

\[
\int_{-1}^{1} P_l(x) P_{l'}(x) dx = \frac{2}{2l+1} \delta_{ll'}.
\]

1.12 Evaluate numerically the capacitance of a conducting torus having a major radius of 10 cm and minor radius of 3 cm.

1.13 Two concentric circular rings of radii \( a \) and \( b \) in the same plane carry charges \( q \) and \( -q \), respectively. Show that the far field potential is of quadrupole nature. What is the potential energy of the system?

1.14 Two coaxial conductor rings of radii \( a \) and \( b \) and axial separation distance \( c \) carry charges \( q \) and \( -q \). What is the dominant far field potential? Find the mutual capacitance.

1.15 The permittivity of a molecular hydrogen (H\(_2\)) gas under the standard condition, 0 C, one atmospheric pressure, is

\[
\varepsilon = \varepsilon_0 + \Delta \varepsilon = (1 + 2.7 \times 10^{-4}) \varepsilon_0.
\]
When \( \Delta \varepsilon \) is written in the form

\[
\Delta \varepsilon = \text{const.} \frac{\omega_0^2}{\omega_0^2} \varepsilon_0 = \text{const.} \frac{ne^2}{m_e \omega_0^2},
\]

where \( n \) is the molecule density, \( m_e = 9.1 \times 10^{-31} \) kg is the electron mass, and \( \omega_0 \) is the frequency of bound harmonic motion of electron \( \hbar \omega_0 = 27.2 \) eV, what should the constant be? For atomic hydrogen, the constant is 4.5 as predicted by quantum mechanics.

1.16 Two dipole moments \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) are a distance \( \mathbf{r} \) apart. Show that the potential energy of the two dipole system is given by

\[
U = \frac{1}{4\pi \varepsilon_0} \frac{1}{r^3} \left( \mathbf{p}_1 \cdot \mathbf{p}_2 - \frac{3}{r^2} (\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r}) \right).
\]

1.17 Two parallel charge sheets of opposite polarity \( \pm \sigma \) (C/m\(^2\)) separated by a small distance \( \delta \) form a double layer. (a) Show that the potential jump across the double layer is

\[
\Delta \Phi = \frac{\sigma \delta}{\varepsilon_0} = \frac{\kappa}{\varepsilon},
\]

where \( \kappa = \sigma \delta \) (C m\(^{-1}\)) is called the moment of double layer. (b) A circular double layer of radius \( a \) and moment \( \kappa \) is on the \( x - y \) plane. Find the potential \( \Phi(r, \theta) \).

1.18 Concentric spherical capacitor of radii \( a \) (inner) and \( b \) (outer) is filled with a nonuniform dielectric material whose permittivity depends on the radius \( r, \varepsilon(r) \). Determine \( \varepsilon(r) \) if the radial electric field between the electrodes is to be constant. Assume that the permittivity at \( r = b \) is \( \varepsilon_0 \).

1.19 The problem of ring charge can alternatively be solved by the method of Laplace transform in the cylindrical coordinates as shown in Chapter 1,

\[
\Phi(\rho, z) = \frac{q}{4\pi \varepsilon_0} \int_0^\infty J_0(ka)J_0(k\rho)e^{-k|z|}dk.
\]

Show that the solution above satisfies the Poisson’s equation,

\[
\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \Phi(\rho, z) = -\frac{q}{2\pi \varepsilon_0 a} \delta(\rho - a) \delta(z).
\]

1.20 A conducting sphere of radius \( a \) is placed in a uniform electric field \( \mathbf{E}_0 = E_0 \mathbf{e}_z \) whose potential is \( \Phi_0 = -E_0 z \). The dipole moment induced on the sphere is

\[
p_z = 4\pi \varepsilon_0 a^3 E_0,
\]
which produces a potential

\[ \Phi' = \frac{1}{4\pi \varepsilon_0} \frac{p_z}{r^2} \cos \theta. \]

(a) The exterior electric field is

\[
E(r, \theta) = -\nabla \Phi_0 - \nabla \Phi' = E_0 e_z + \frac{p_z}{4\pi \varepsilon_0 r^3} (2 \cos \theta e_r + \sin \theta e_\theta), \ r > a.
\]

Find a change in the total electric energy and interpret your result. Note that \( E = 0 \) in the sphere.

(b) A dipole \( \mathbf{p} \) in an electric field \( \mathbf{E} \) has a potential energy

\[ U = -\mathbf{p} \cdot \mathbf{E} = -4\pi \varepsilon_0 a^3 E_0^2. \]

Recover this result from the direct integral,

\[ U = \frac{1}{2} \oint \sigma \Phi_0 dS, \]

over the sphere surface. Note that at a conductor surface, \( \sigma = \varepsilon_0 E_r (r = a) \).

1.21 A charge \( q_1 \) is placed at a distance \( z = a \) from a dielectric plate having a permittivity \( \varepsilon \). Another charge \( q_2 \) is at the mirror position \( z = -a \). Find the force on each charge. As you will find, the forces are not equal in magnitude. Explain.

Hint: If a charge \( q \) is placed at a distance \( z = a \) from the surface of a dielectric plate, the potential in the air region is

\[ \Phi_{\text{air}} = \frac{1}{4\pi \varepsilon_0} \left( \frac{q}{|\mathbf{r} - \mathbf{a}|} - \frac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0} \frac{q}{|\mathbf{r} + \mathbf{a}|} \right), \ z > 0 \]
and that in the dielectric is

$$\Phi_{\text{dielectric}} = \frac{1}{4\pi \varepsilon} \frac{2\varepsilon}{\varepsilon + \varepsilon_0} \frac{q}{|r - a|}, \quad z < 0,$$

where $a = ae_z$.

1.22 The permittivity of an unmagnetized plasma is

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right),$$

where $\omega_{pe} = \sqrt{ne^2/m_e\varepsilon_0}$ is the plasma frequency. Show that a spherical plasma exhibits a dipole ($l = 1$) oscillation at a frequency

$$\omega = \frac{\omega_{pe}}{\sqrt{3}}.$$

What are the frequencies of higher multipole modes?