Chapter 8

Radiation by Moving Charges

8.1 Introduction

The problem of radiation of electromagnetic waves by a single charged particle moving at an arbitrary velocity had correctly been formulated independently by Lienard and Wiechert before the advent of the special relativity theory. This is because once emitted from a charged particle, electromagnetic waves propagate at the speed $c$ irrespective of the velocity of the charged particle, just as sound waves propagate at a speed independent of the source velocity. (The major finding made by Einstein was that electromagnetic waves still propagate at the speed $c$ regardless of the observer’s velocity in contrast to the case of sound waves.)

The scalar and vector potentials due to a moving charge can be found rigorously using the Green’s function for the wave equation. Then radiation electromagnetic fields can readily be calculated. In nonrelativistic regime, the radiation power only depends on the acceleration of charged particles. As the velocity approaches $c$, however, significant increase in the radiation power occurs. Furthermore, in highly relativistic limit, radiation occurs primarily along the direction of velocity within an angular spread of order $\theta \simeq 1/\gamma$ about the velocity irrespective of the direction of acceleration. Here $\gamma = 1/\sqrt{1 - \beta^2}$ is the relativity factor with $\beta = v/c$. Hence radiation frequency is subject to strong Doppler shift. For example, in synchrotron radiation due to highly relativistic electron beam bent or undulated by a magnetic field, radiation even in hard x-ray regime can be created.

In material medium, radiation processes without acceleration on charged particles are possible. If the velocity of a charged particle exceeds the velocity of electromagnetic waves in the medium

$$v > \frac{1}{\sqrt{\varepsilon \mu_0}},$$

where $\varepsilon$ is the permittivity, Cherenkov radiation occurs. Furthermore, if a charged particle crosses a boundary of two dielectric media, the transition radiation occurs even if the condition for Cherenkov radiation is not met. Transition radiation is due to sudden change in the normalized velocity from $\beta_1 = v \sqrt{\varepsilon_1 \mu_0}$ to $\beta_2 = v \sqrt{\varepsilon_2 \mu_0}$ which may be regarded as an effective acceleration even though the
8.2 Lienard-Wiechert Potentials

The charge and current densities of a moving point charge are singular and described by

\[ \rho(\mathbf{r}, t) = e \delta[\mathbf{r} - \mathbf{r}_p(t)], \] 
\[ \mathbf{J}(\mathbf{r}, t) = e \mathbf{v}(t) \delta[\mathbf{r} - \mathbf{r}_p(t)], \] 

where \( e \) is the charge, \( \mathbf{r}_p(t) \) is the instantaneous location of the charge and \( \mathbf{v}(t) = d\mathbf{r}_p(t)/dt \) is the instantaneous velocity of the charge which may be changing with time. Exploiting the Green’s function for the wave equation,

\[ G(\mathbf{r} - \mathbf{r}', t - t') = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \delta \left( t - t' - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right), \] 

we can write down solutions for the inhomogeneous wave equations,

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Phi = -\frac{\rho}{\varepsilon_0}, \] 
\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\mu_0 \mathbf{J}, \]

in the form

\[ \Phi(\mathbf{r}, t) = \frac{e}{4\pi \varepsilon_0} \int dV' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta[\mathbf{r}' - \mathbf{r}_p(t')] \delta[f(t')], \] 
\[ \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 e}{4\pi} \int dV' \int dt' \frac{\mathbf{v}(t')}{|\mathbf{r} - \mathbf{r}'|} \delta[\mathbf{r}' - \mathbf{r}_p(t')] \delta[f(t')], \]

where

\[ f(t') = t' - t + \frac{|\mathbf{r} - \mathbf{r}'|}{c}. \] 

The volume integrations can be carried out immediately with the results

\[ \Phi(\mathbf{r}, t) = \frac{e}{4\pi \varepsilon_0} \int \frac{1}{|\mathbf{r} - \mathbf{r}_p(t')|} \delta[f(t')] dt', \] 
\[ \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 e}{4\pi} \int \frac{\mathbf{v}(t')}{|\mathbf{r} - \mathbf{r}_p(t')|} \delta[f(t')] dt', \]

where \( f(t') \) is now

\[ f(t') = t' - t + \frac{|\mathbf{r} - \mathbf{r}_p(t')|}{c}. \]
The integral involving the delta function can be simplified as
\[
\int g(t') \delta[f(t')] dt' = \int g(t') \delta[f(t')] \frac{1}{|df/dt'|} dt'
= \left. \frac{g(t')}{|df/dt'|} \right|_{t'=0},
\] (8.11)
where \( t' \) is now understood as a solution for \( t' \) satisfying
\[
f(t') = 0
\] or
\[
t' - t + \frac{|r - r_p(t')|}{c} = 0.
\] (8.12)
This is in general an implicit equation for \( t' \). The time derivative of \( f(t') \) is
\[
\frac{df}{dt'} = 1 - \frac{n(t') \cdot v_p(t')}{c} = 1 - n(t') \cdot \beta(t'),
\] (8.13)
where \( n(t') \) is the unit vector along the relative distance \( r - r_p(t') \) and \( \beta = v/c \). The observing time \( t \) and time \( t' \) are related through
\[
[1 - n(t') \cdot \beta(t')] dt' = dt,
\]
or
\[
\frac{dt}{dt'} = 1 - n(t') \cdot \beta(t').
\]
After performing time integration, we finally obtain
\[
\Phi(r, t) = e \frac{1}{4\pi\varepsilon_0} \frac{1}{1 - n(t') \cdot \beta(t')} \frac{1}{|r - r_p(t')|} = e \frac{1}{4\pi\varepsilon_0} \frac{1}{\kappa(t') \cdot |r - r_p(t')|},
\] (8.14)
\[
A(r, t) = \frac{\mu_0}{4\pi} \frac{1}{1 - n(t') \cdot \beta(t')} \frac{v(t')}{|r - r_p(t')|} = \frac{\mu_0}{4\pi} \frac{v(t')}{\kappa(t') \cdot |r - r_p(t')|},
\] (8.15)
where
\[
\kappa(t') = 1 - n(t') \cdot \beta(t').
\] (8.16)
These retarded potentials, called Lienard-Wiechert potentials, had been formulated in 1898. They are applicable to arbitrary velocity of the charged particle. Retarded nature of the potentials clearly appears in the condition that all time varying quantities, \( r_p(t'), v(t'), n(t') \), must be evaluated at \( t' \), not at the observing time \( t \) because of finite propagation speed of electromagnetic disturbance.

Having found the retarded potentials, we are now ready to calculate the electromagnetic fields due to a moving point charge. The electric field is to be found from
\[
E(r, t) = -\nabla \Phi - \frac{\partial A}{\partial t},
\] (8.17)
where spatial and time derivatives pertain to \( r \) (the coordinates of the observing location) and
t (observing time). Since the potentials $\Phi$ and $\mathbf{A}$ are implicit functions of $\mathbf{r}$ and $t$, it is more convenient to use the original integral representations, Eqs. (8.8) and (8.9), respectively, for proper differentiation with respect to $\mathbf{r}$ and $t$. For example, the spatial derivative of the scalar potential can be performed as follows. Letting $R(t') = |\mathbf{r} - \mathbf{r}_p(t')|$, and introducing a unit vector in the direction $\mathbf{r}_p(t')$,

$$
\mathbf{n}(t') = \frac{\mathbf{r} - \mathbf{r}_p(t')}{|\mathbf{r} - \mathbf{r}_p(t')|},
$$

we find

$$
\nabla \Phi = \frac{e}{4\pi \varepsilon_0} \int \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}_p(t')|} \delta[\mathbf{f}(t')] \right) dt',
$$

$$
= \frac{e}{4\pi \varepsilon_0} \int \mathbf{n} \frac{\partial}{\partial R} \left( \frac{1}{R} \delta[\mathbf{f}(t')] \right) dt',
$$

$$
= -\frac{e}{4\pi \varepsilon_0} \int \left[ \frac{n}{R^2} \delta[\mathbf{f}(t')] - \frac{n}{cR} \frac{d}{df} \delta[\mathbf{f}(t')] \right] dt',
$$

$$
= -\frac{e}{4\pi \varepsilon_0} \left[ \frac{n}{\kappa R^2} + \frac{1}{c\kappa} \frac{d}{dt'} \left( \frac{n}{\kappa R} \right) \right],
$$

(8.19)

where use is made of the integration by parts,

$$
\int g(t') \frac{d}{df} \delta[\mathbf{f}(t')] dt'
$$

$$
= \int g(t') \frac{1}{df} \frac{d}{dt'} \delta[\mathbf{f}(t')] dt'
$$

$$
= -\left[ \frac{1}{df/dt'} \frac{df}{df/dt'} \frac{d}{dt'} \left( \frac{g(t')}{\kappa} \right) \right]_{f(t')=0}
$$

$$
= -\left[ \frac{1}{\kappa} \frac{d}{dt'} \left( \frac{g(t')}{\kappa} \right) \right]_{f(t')=0}.
$$

(8.21)

Similarly,

$$
\frac{\partial \mathbf{A}}{\partial t} = \frac{e}{4\pi \varepsilon_0} \frac{1}{c\kappa} \frac{d}{dt'} \left( \frac{\beta}{\kappa R} \right),
$$

(8.23)

and the electric field becomes

$$
\mathbf{E}(\mathbf{r}, t) = \frac{e}{4\pi \varepsilon_0} \left[ \frac{n}{\kappa R^2} + \frac{1}{c\kappa} \frac{d}{dt'} \left( \frac{n - \beta}{\kappa R} \right) \right]_{f(t')=0}.
$$

(8.24)

To proceed further, we need concrete expressions for the derivatives,

$$
\frac{dn}{dt'} \text{ and } \frac{d}{dt'} \left( \frac{1}{\kappa R} \right).
$$

The unit vector $\mathbf{n}$ along the distance vector $\mathbf{R} = \mathbf{r} - \mathbf{r}_p(t')$ changes its direction only through the
velocity component perpendicular to \( \mathbf{R} \),
\[
d\mathbf{n} = -\frac{\mathbf{v}_\perp}{R} dt',
\] (8.25)
as can be seen in Fig. 8-1. This yields

![Diagram showing the change in the unit vector \( \mathbf{n} \) caused by the perpendicular velocity \( \mathbf{v}_\perp \).](image)

**Figure 8-1:** The change in the unit vector \( \mathbf{n} \) is caused by the perpendicular velocity \( \mathbf{v}_\perp \).

\[
\frac{d\mathbf{n}}{dt'} = -\frac{\mathbf{v}_\perp}{R}.
\] (8.26)

Also,
\[
\frac{d}{dt'} \left( \frac{1}{\kappa R} \right) = -\frac{1}{(\kappa R)^2} \frac{d}{dt'} \left[ (1 - \mathbf{n} \cdot \mathbf{\beta}) R \right]
\]
\[
= -\frac{1}{(\kappa R)^2} \left[ \mathbf{v}_\perp \cdot \mathbf{\beta} - (\mathbf{n} \cdot \mathbf{\beta}) R - c(1 - \mathbf{n} \cdot \mathbf{\beta}) \mathbf{n} \cdot \mathbf{\beta} \right]
\]
\[
= -\frac{c}{(\kappa R)^2} \left( \beta^2 - \mathbf{n} \cdot \mathbf{\beta} - \frac{1}{c} R (\mathbf{n} \cdot \mathbf{\beta}) \right).
\] (8.27)

Substitution of Eqs. (8.26) and (8.27) to Eq. (8.24) gives
\[
\mathbf{E}(\mathbf{r}, t) = \frac{e}{4\pi \varepsilon_0} \left[ \frac{\mathbf{n} - \mathbf{\beta}}{(\kappa R)^2} - \frac{\mathbf{n} - \mathbf{\beta}}{\kappa(\kappa R)^2} \left( \beta^2 - \mathbf{n} \cdot \mathbf{\beta} - \frac{1}{c} R (\mathbf{n} \cdot \mathbf{\beta}) \right) - \frac{\dot{\mathbf{\beta}}}{c \kappa^2 R} \right]
\]
\[
= \frac{e}{4\pi \varepsilon_0} \left[ \frac{1 - \beta^2}{\kappa^3 R^2} (\mathbf{n} - \mathbf{\beta}) + \frac{1}{c \kappa^3 R} \mathbf{n} \times [(\mathbf{n} - \mathbf{\beta}) \times \mathbf{\beta}] \right]_{f(t') = 0}.
\] (8.28)

The first term in the RHS,
\[
\mathbf{E}_{\text{Coulomb}} = \frac{e}{4\pi \varepsilon_0} \left[ \frac{1 - \beta^2}{\kappa^3 R^2} (\mathbf{n} - \mathbf{\beta}) \right]_{f(t') = 0},
\] (8.29)
is the Coulomb field corrected for relativistic effects. It is proportional to $1/R^2$ and thus does not contribute to radiation of energy. The second term,

$$E_{rad} = \frac{e}{4\pi\varepsilon_0 c} \left[ \frac{1}{\kappa^3 R} \mathbf{n} \times \left( (\mathbf{n} - \beta) \times \dot{\beta} \right) \right]_{f(t')=0},$$

contains acceleration $\dot{\beta}$ and is proportional to $1/R$. This is the desired radiation electric field due to a moving charged particle.

The magnetic field can be calculated in a similar manner from

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$= -\frac{1}{c} [\mathbf{n} \times E]_{f(t')=0}.$$  \hspace{1cm} (8.31)

Derivation of this result is left for an exercise.

### 8.3 Radiation from a Charge under Linear Acceleration

If the acceleration is parallel (or anti-parallel) to the velocity, $\dot{\beta} \times \beta = 0$, the radiation electric field reduces to

$$E_{rad} = \frac{e}{4\pi\varepsilon_0} \left[ \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\beta})}{c\kappa^3 R} \right]_{f(t')=0}. \hspace{1cm} (8.32)$$

The angular distribution of radiation power at the observing time $t$ is

$$\frac{dP(t)}{d\Omega} = c\varepsilon_0 |E_{rad}|^2 \\ R^2$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{e^2}{4\pi c} \left[ \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\beta})}{c\kappa^3 R} \right]_{f(t')=0}^2. \hspace{1cm} (8.33)$$

Denoting the angle between $\dot{\beta}$ and $\mathbf{n}$ by $\theta$, we have $|\mathbf{n} \times (\mathbf{n} \times \dot{\beta})|^2 = \beta^2 \sin^2 \theta$, and thus

$$\frac{dP(t)}{d\Omega} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{4\pi c} \left[ \frac{\dot{\beta}^2 \sin^2 \theta}{(1 - \beta \cos \theta)^6} \right]_{f(t')=0}. \hspace{1cm} (8.34)$$

However, the power $P(t)$ in the above formulation is the rate of energy radiation at $t$, the observing time, which is not necessarily equal to the energy loss rate of the charge at the retarded time $t'$ determined from $f(t') = 0$. To find the radiation power at the retarded time $P(t')$, let us consider the amount of differential energy $d\mathcal{E}/d\Omega$ radiated during the time interval between $t'$ and $t' + dt'$. By definition $d\mathcal{E} = P(t') dt'$. The radiation energy $d\mathcal{E}$ is sandwiched between two eccentric spherical surfaces with a volume

$$dV = R^2 c dt' (1 - \beta \cos \theta).$$
Therefore, the differential radiation energy is

\[
\frac{d\mathcal{E}}{d\Omega} = \frac{1}{2} \varepsilon_0 |\mathbf{E}_{\text{rad}}|^2 R^2 c dt'(1 - \beta \cos \theta),
\]

and

\[
\frac{dP(t')}{d\Omega} = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{4\pi c} \left[ \frac{\dot{\beta}^2 \sin^2 \theta}{(1 - \beta \cos \theta)^5} \times (1 - \beta \cos \theta) \right]_{f(t')=0}.
\]

(8.35)

In nonrelativistic limit \(|\beta| < 1\), the radiation occurs predominantly in the direction perpendicular to the acceleration \(\theta = \pi/2\). The total radiation power in this case is

\[
P \simeq \frac{1}{4\pi \varepsilon_0} \frac{e^2 \dot{\beta}^2}{4\pi c} \int \sin^2 \theta d\Omega
= \frac{1}{4\pi \varepsilon_0} \frac{2e^2 \dot{\beta}^2}{3c}, \quad \beta \ll 1.
\]

(8.36)

This is the well known Larmor’s formula for radiation power due to a nonrelativistic charge. Since in nonrelativistic limit,

\[
\mathbf{n} \times [\mathbf{n} - \mathbf{\dot{\beta}}] \times \mathbf{\dot{\beta}} \simeq \mathbf{n} \times (\mathbf{n} \times \mathbf{\dot{\beta}}),
\]

(8.37)

Larmor’s formula is applicable for acceleration in arbitrary direction relative to the velocity.

For arbitrary magnitude of the velocity \(\beta\), the radiation power can be found from

\[
P(t') = \frac{1}{4\pi \varepsilon_0} \frac{e^2 \dot{\beta}^2}{4\pi c} \int \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5} d\Omega
= \frac{1}{4\pi \varepsilon_0} \frac{e^2 \dot{\beta}^2}{4\pi c} \frac{1}{2\pi} \int_0^\pi \frac{\sin^3 \theta}{(1 - \beta \cos \theta)^5} d\theta
= \frac{1}{4\pi \varepsilon_0} \frac{2e^2 \dot{\beta}^2}{3c} \gamma^6,
\]

(8.38)

where

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}},
\]

(8.39)

is the relativity factor and the following integral is used,

\[
\int_0^\pi \frac{\sin^3 \theta}{(1 - \beta \cos \theta)^5} d\theta = \int_{-1}^1 \frac{1 - x^2}{(1 - \beta x)^5} dx = \frac{4}{3} \frac{1}{(1 - \beta^2)^3}.
\]

(8.40)
The angular dependence of the radiation intensity,

$$\frac{\sin^2 \theta}{(1 - \beta \cos \theta)^6},$$  \hspace{1cm} (8.41)

peaks at angle $\theta_0$ where

$$\cos \theta_0 = \frac{\sqrt{1 + 24\beta^2} - 1}{4\beta}. \hspace{1cm} (8.42)$$

In highly relativistic limit $\gamma \gg 1$, the angle $\theta_0$ becomes of order

$$\theta_0 \approx \frac{1}{\gamma} \ll 1, \hspace{1cm} (8.43)$$

which indicates a very sharp pencil or beam of radiation along the direction of the velocity $\beta$. (This is also the case for acceleration perpendicular to the velocity as shown in the following section.) Angular distribution of radiation intensity $I(\theta)$ for $\beta = 0, 0.2, 0.9$ and 0.999 is shown below for a common acceleration. Note that the radiation intensity rapidly increases with $\gamma = 1/\sqrt{1 - \beta^2}$.

\begin{itemize}
  \item $\beta = 0 \ (\gamma = 1.0)$.
  \item $\beta = 0.2 \ (\gamma = 1.02)$.
  \item $\beta = 0.9 \ (\gamma = 2.3)$.
  \item $\beta = 0.99 \ (\gamma = 7.09)$.
\end{itemize}
Polar plot of $I(\theta)$ for several $\beta$ factors but with common parallel acceleration.

For linear acceleration, the momentum change of the charged particle is

$$\frac{dp}{dt} = \frac{d}{dt} \left( \frac{mv}{\sqrt{1 - \beta^2}} \right) = m\gamma^3 \dot{\gamma}. \quad (8.44)$$

Therefore, the radiation power can be rewritten as

$$P(t') = \frac{1}{4\pi\varepsilon_0} \frac{2e^2}{3c^3m^2} \left( \frac{dp}{dt} \right)^2 = \frac{1}{4\pi\varepsilon_0} \frac{2e^4}{3c^3m^2} E_{ac}^2$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2e^2}{3c^3m^2} \left( \frac{dE}{dx'} \right)^2, \quad (8.45)$$

where $E_{ac}$ is an external acceleration electric field and

$$\frac{dE}{dx'},$$

is the energy gradient of a linear accelerator which is at most of the order of 100 MeV/m in practice. The radiation loss in linear accelerators is negligibly small compared with energy gain. This is one of the advantages of high energy linear accelerators. Note that the radiation power due to linear acceleration is independent of the particle energy or the relativity factor $\gamma$.

### 8.4 Radiation from a Charge in Circular Motion

In circular motion, the acceleration is perpendicular to the velocity. Here we consider highly relativistic motion of a charged particle with $\beta \approx 1$, for nonrelativistic case has already been discussed in Chapter 4. A geometry convenient for analysis to follow is shown in Fig.8-2. A particle undergoes circular motion with an orbit radius $\rho$ in the $x-z$ plane and it passes the origin at $t' = 0$. At that instant, the acceleration is in the $x$ direction while the velocity is in the $z$ direction,

$$\dot{\beta} = |\dot{\beta}| e_x, \quad \beta = \beta e_z.$$ 

The radiation electric field can then be written down in terms of cartesian components,

$$E(r, t) = \frac{e}{4\pi\varepsilon_0 \kappa^3 R} \left[ (n - \beta) \times \dot{\beta} \right]$$

$$= \frac{e}{4\pi\varepsilon_0 \kappa^3 R} \left[ |\dot{\beta}| \left[ (\beta - \cos \theta) \cos \phi e_\theta + (1 - \beta \cos \theta) \sin \phi e_\phi \right] \right], \quad (8.46)$$
Figure 8-2: Particle undergoing circular motion in the $x - z$ plane with radius $\rho$ and frequency $\omega_0$. At $t = 0$, the particle passes the origin.

and the angular distribution of radiation power is given by

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0} \frac{e^2 |\dot{\beta}|^2}{4\pi c} \frac{1}{(1 - \beta \cos \theta)^5} \left[ (1 - \beta \cos \theta)^2 - \frac{1}{\gamma^2} \sin^2 \theta \cos^2 \phi \right]. \quad (8.47)$$

The total radiation power is

$$P(t') = \frac{1}{4\pi\varepsilon_0} \frac{e^2 |\dot{\beta}|^2}{4\pi c} \int \frac{1}{(1 - \beta \cos \theta)^5} \left[ (1 - \beta \cos \theta)^2 - \frac{1}{\gamma^2} \sin^2 \theta \cos^2 \phi \right] d\Omega$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{e^2 |\dot{\beta}|^2}{4\pi c} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \frac{1}{(1 - \beta \cos \theta)^5} \left[ (1 - \beta \cos \theta)^2 - \frac{1}{\gamma^2} \sin^2 \theta \cos^2 \phi \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2e^2 |\dot{\beta}|^2}{3c^3} \gamma^4. \quad (8.48)$$

Relevant integrals are:

$$\int_{-1}^1 \frac{dx}{(1 - \beta x)^3} = \frac{2}{(1 - \beta^2)^2} = 2\gamma^4,$$

$$\int_{-1}^1 \frac{1 - x^2}{(1 - \beta x)^5} dx = \frac{4}{3} \frac{1}{(1 - \beta^2)^3} = \frac{4}{3} \gamma^6.$$

In highly relativistic case, the acceleration may be approximated by

$$|\dot{\gamma}| = \frac{v^2}{\rho} \approx \frac{c^2}{\rho}. \quad (8.49)$$
Then, the radiation power in terms of the orbit radius $\rho$ is

$$P(t') \simeq \frac{1}{4\pi \varepsilon_0} \frac{2e^2 c \gamma^4}{3\rho^2}.$$  \quad (8.50)

To maintain the radiation loss in a circular accelerator at a tolerable level, the orbit radius must be increased as the particle energy $\gamma mc^2$ increases. Note that the radiation power is a sensitive function of the particle energy in contrast to the case of linear acceleration.

As an example, let us consider the Betatron, the well known inductive electron accelerator invented by Kerst. In the Betatron, the electron cyclotron orbit is maintained constant,

$$\frac{\gamma mc^2}{\rho} = ecB(t) = ecB_0 \sin \omega t \simeq ecB_0 \omega t,$$

where only the initial phase of the sinusoidal magnetic field is useful for acceleration, $\omega t \ll 1$. Then the rate of electron energy gain is

$$\frac{d}{dt} (\gamma mc^2) = ecB_0 \omega = \text{const.}$$

Equating this to the radiation energy loss,

$$P(t') = \frac{1}{4\pi \varepsilon_0} \frac{2}{3c^3} \left( \frac{e^2}{\rho} \right)^2 \gamma^4,$$

we find

$$\gamma_{\text{max}} \simeq \left( \frac{3\omega e B_0 \rho^3}{2mc^2 r_e} \right)^{1/4},$$  \quad (8.51)

where

$$r_e = \frac{e^2}{4\pi \varepsilon_0 mc^2} = 2.8 \times 10^{-15} \text{ m},$$  \quad (8.52)

is the classical radius of electron. If $\rho = 50 \text{ cm}$, $\omega = 2\pi \times 60 \text{ rad/s}$, and $B_0 = 0.5 \text{ T (5 kG)}$, the upper limit of $\gamma$ is about 400 and the maximum electron energy attainable is approximately 200 MeV.

### 8.5 Fourier Spectrum of Radiation Fields

The formulae such as Eqs. (8.45) and (8.50) only tell us the total radiation power integrated over the frequency. Radiation field emitted by highly relativistic particle is hardly monochromatic but consists of broad frequency spectrum. Knowing such frequency spectrum of radiation is of practical importance for identifying radiation source. A typical example is synchrotron radiation due to highly relativistic electrons bent by, or trapped in, a magnetic field. In nonrelativistic limit, the radiation fields all have a single frequency component corresponding to the classical electron cyclotron frequency $\omega_c = eB/m$. However, as the relativity factor $\gamma$ increases, the radiation fields
consist of harmonics of the fundamental frequency \( \omega_c = eB/\gamma m \). In highly relativistic case \( \gamma \gg 1 \), the frequency spectrum becomes almost continuous peaking at the frequency \( \omega \simeq \gamma^3 \omega_c = \gamma^2 eB/m \).

Frequency spectrum of the radiation electric field can be formulated by directly applying Fourier transformation on the field in Eq. (8.24),

\[
E(r, \omega) = \int_{-\infty}^{\infty} E(r, t)e^{i\omega t} dt
= \frac{1}{4\pi\varepsilon_0} \frac{e}{c} \int_{-\infty}^{\infty} \left[ \frac{n \times [(n - \beta) \times \beta]}{\kappa^2 R} \right] e^{i\omega t} dt.
\] (8.53)

Changing the variable from \( t \) to \( t' \) and assuming \( r \gg r_p \), we obtain

\[
E(r, \omega) = \frac{1}{4\pi\varepsilon_0} \frac{e}{c} \frac{r}{r_p} \int_{-\infty}^{\infty} \left[ \frac{n \times [(n - \beta) \times \beta]}{\kappa^2} \right] \exp\left[i\omega (t' - \frac{1}{c} n \cdot r_p(t'))\right] dt'.
\] (8.54)

The unit vector \( n(t') \) may be regarded constant since \( r \gg r_p \) and thus approximated by \( n \simeq r/r \). Then,

\[
\frac{n \times [(n - \beta) \times \beta]}{\kappa^2} \simeq \frac{d}{dt'} \left( \frac{n \times (n \times \beta)}{\kappa} \right),
\] (8.55)

and Eq. (8.53) can be integrated by parts,

\[
E(r, \omega) = -\frac{i\omega}{4\pi\varepsilon_0} \frac{e}{c} \frac{r}{r_p} \int_{-\infty}^{\infty} n \times (n \times \beta) \exp\left[i\omega (t' - \frac{1}{c} n \cdot r_p(t'))\right] dt'.
\] (8.56)

Any time derivatives contained in the physical electric field are merely multiplied by \(-i\omega\) in the Fourier space and the disappearance of the acceleration \( \beta \) is not surprising. Amazing fact about Eq. (8.56) is that it is applicable to radiation fields which do not require particle acceleration such as Cherenkov and transition radiation provided a proper velocity of electromagnetic waves in dielectrics is substituted for \( c \).

The radiation energy (not power) associated with the electric field is

\[
c\varepsilon_0 r^2 \int \int |E(r, t)|^2 dt d\Omega, \ (J).
\] (8.57)

However, since the electric field \( E(r, t) \) and its Fourier transform \( E(r, \omega) \) are related through Parseval’s theorem:

\[
\int |E(r, t)|^2 dt = \frac{1}{2\pi} \int |E(r, \omega)|^2 d\omega,
\] (8.58)

the radiation energy can be written in terms of the Fourier transform \( E(r, \omega) \) as

\[
\frac{1}{2\pi} c\varepsilon_0 r^2 \int \int |E(r, \omega)|^2 d\Omega d\omega.
\] (8.59)
The quantity
\[ \frac{dI(\omega)}{d\Omega} \equiv \frac{1}{2\pi} c \varepsilon_0 r^2 |E(\mathbf{r}, \omega)|^2, \quad (8.60) \]
can therefore be identified as the radiation energy per unit solid angle per unit frequency. Substituting Eq. (8.56), we find
\[ \frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{8\pi^2 c^2} \omega^2 \left| \int \mathbf{n} \times (\mathbf{n} \times \mathbf{\beta}) \exp \left[ i\omega \left( t' - \frac{1}{c} \mathbf{n} \cdot \mathbf{r}_p(t') \right) \right] dt' \right|^2, -\infty < \omega < \infty, \quad (8.61) \]
or
\[ \frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{4\pi^2 c^2} \omega^2 \left| \int \mathbf{n} \times (\mathbf{n} \times \mathbf{\beta}) \exp \left[ i\omega \left( t' - \frac{1}{c} \mathbf{n} \cdot \mathbf{r}_p(t') \right) \right] dt' \right|^2, 0 < \omega < \infty. \quad (8.62) \]

Let us work on a few examples.

### 8.6 Synchrotron Radiation I

Synchrotron radiation is due to highly relativistic electrons trapped in a magnetic field. The radiation beam rotates together with an electron and is directed along the direction of the velocity with an angular spread of order \( \Delta \theta \simeq 1/\gamma \ll 1 \). If the orbiting frequency is \( \omega_0 = eB/\gamma m_e \), the radiation beam shines a detector for a duration
\[ \Delta t' \simeq \frac{\Delta \theta}{\omega_0} = \frac{1}{\gamma \omega_0}, \quad (8.63) \]
as seen by the electron at the retarded time \( t' \). Since
\[ \frac{\Delta t}{\Delta t'} = 1 - \mathbf{n} \cdot \mathbf{\beta} = 1 - \beta = \frac{1}{2\gamma^2}, \quad (8.64) \]
which is entirely due to Doppler effect, the pulse width detected is of order
\[ \Delta t \simeq \frac{\Delta t'}{2\gamma^2} = \frac{1}{2\gamma^3 \omega_0}. \quad (8.65) \]

Therefore, synchrotron radiation is dominated by frequency components in the range
\[ \omega \simeq \gamma^3 \omega_0 = \gamma^2 \frac{eB}{m_e}. \quad (8.66) \]

In ultrarelativistic case, the frequency spectrum of synchrotron radiation can extend to very high frequencies even for a modest magnetic field.

We calculate the amount of energy radiated in one period of cyclotron motion \( T = 2\pi/\omega_0 \). Since the radiation power is constant, the radiated energy is
\[ \mathcal{E} = \frac{1}{T} \int_0^\infty I(\omega) d\omega = \frac{\omega_0}{2\pi} \int_0^\infty I(\omega) d\omega. \quad (8.67) \]
The time integration in the energy spectrum,

\[
\frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{4\pi^2c^2} \omega^2 \left| \int_{-T/2}^{T/2} \mathbf{n} \times (\mathbf{n} \times \beta) \exp \left[ i\omega \left( t' - \frac{1}{c} \mathbf{n} \cdot \mathbf{r}_p(t') \right) \right] dt' \right|^2, \quad 0 < \omega < \infty, \quad (8.68)
\]

can be extended from \(-\infty\) to \(\infty\) since the characteristic frequency of the radiation field is much higher than the fundamental frequency \(\omega \gg \omega_0\),

\[
\frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{4\pi^2c^2} \omega^2 \left| \int_{-\infty}^{\infty} \mathbf{n} \times (\mathbf{n} \times \beta) \exp \left[ i\omega \left( t' - \frac{1}{c} \mathbf{n} \cdot \mathbf{r}_p(t') \right) \right] dt' \right|^2, \quad 0 < \omega < \infty. \quad (8.69)
\]

To perform the integration, we assume the trajectory shown in Fig. (8-2) in which an electron passes the origin at \(t = 0\). The vector \(\mathbf{n}\) is assumed to be in the \(y-z\) plane since the radiation profile is essentially symmetric about the \(z\) axis. The trajectory is described by

\[
\mathbf{r}_p(t') = \rho \left[ (1 - \cos \omega_0 t') \mathbf{e}_x + \sin \omega_0 t \mathbf{e}_z \right], \quad (8.70)
\]

and the velocity is

\[
\beta = \frac{\omega_0 \rho}{c} (\sin \omega_0 t \mathbf{e}_x + \cos \omega_0 t \mathbf{e}_z). \quad (8.71)
\]

Then,

\[
\frac{1}{c} \mathbf{n} \cdot \mathbf{r}_p(t') = \frac{\rho}{c} \sin (\omega_0 t) \cos \theta. \quad (8.72)
\]

Since

\[
\mathbf{n} \times (\mathbf{n} \times \beta) = -\beta_\perp = -\sin \omega_0 t \mathbf{e}_x + \cos \omega_0 t \sin \theta \mathbf{e}_\perp, \quad (8.73)
\]

where

\[
\mathbf{e}_\perp = \mathbf{n} \times \mathbf{e}_x, \quad (8.74)
\]

is a unit vector perpendicular to both \(\mathbf{n}\) and \(x\) axis, and the radiation lasts for a very short time and is limited within a small angle \(\theta\), Eq. (8.73) reduces to

\[
-\beta_\perp \simeq -\omega_0 t \mathbf{e}_x + \theta \mathbf{e}_\perp. \quad (8.75)
\]

Within the same order of accuracy, the phase function \(\omega(t - \mathbf{n} \cdot \mathbf{r}_p/c)\) can be approximated by

\[
\omega(t - \frac{\mathbf{n} \cdot \mathbf{r}_p}{c}) = \omega(t - \frac{\rho \sin \omega_0 t \cos \theta}{c}) \simeq \frac{\omega}{2} \left[ \left( \frac{1}{\gamma^2} + \theta^2 \right) t + \frac{c^2}{3\rho^2 \gamma^3} \right], \quad (8.76)
\]

where \(v\) has been approximated by \(c\) but \(1 - \beta\) by

\[
1 - \beta \simeq \frac{1}{2\gamma^2}. \quad (8.77)
\]
Then,
\[
\int_{-\infty}^{\infty} \beta_\perp \exp \left[ i\omega \left( t - \frac{n \cdot r_p}{c} \right) \right] dt \\
= \int_{-\infty}^{\infty} (\omega_0 t e_x - \theta e_\perp) \exp \left\{ \frac{i\omega}{2} \left[ \left( \frac{1}{\gamma^2} + \theta^2 \right) t + \frac{c^2}{\sqrt{3} \rho^2} t^3 \right] \right\} dt \\
= 2i\omega_0 \int_{0}^{\infty} t \sin \left\{ \frac{\omega}{2} \left[ \left( \frac{1}{\gamma^2} + \theta^2 \right) t + \frac{c^2}{\sqrt{3} \rho^2} t^3 \right] \right\} dt \, e_x \\
- 2\theta \int_{0}^{\infty} \cos \left\{ \frac{\omega}{2} \left[ \left( \frac{1}{\gamma^2} + \theta^2 \right) t + \frac{c^2}{\sqrt{3} \rho^2} t^3 \right] \right\} dt \, e_\perp. \\
\text{(8.77)}
\]

The integrals reduce to the modified Bessel functions of fractional orders (or the Airy’s functions),
\[
\int_{0}^{\infty} \cos \left[ \frac{3}{2} x \left( \tau + \frac{1}{3} \tau^3 \right) \right] d\tau = \frac{1}{\sqrt{3}} K_{1/3} (x), \tag{8.78}
\]
\[
\int_{0}^{\infty} \tau \sin \left[ \frac{3}{2} x \left( \tau + \frac{1}{3} \tau^3 \right) \right] d\tau = \frac{1}{\sqrt{3}} K_{2/3} (x), \tag{8.79}
\]
where
\[
\tau = \frac{\gamma\omega_0 t}{\sqrt{1 + \gamma^2 \theta^2}}, \quad x = \frac{\omega}{3\gamma^3 \omega_0} (1 + \gamma^2 \theta^2)^{3/2}. \tag{8.80}
\]

Then
\[
2i\omega_0 \int_{0}^{\infty} t \sin \left\{ \frac{\omega}{2} \left[ \left( \frac{1}{\gamma^2} + \theta^2 \right) t + \frac{c^2}{\sqrt{3} \rho^2} t^3 \right] \right\} dt = 2i \frac{1}{\gamma^2 + \theta^2} \frac{1}{\omega_0} \frac{1}{\sqrt{3}} K_{2/3} (x), \\
2\theta \int_{0}^{\infty} \cos \left\{ \frac{\omega}{2} \left[ \left( \frac{1}{\gamma^2} + \theta^2 \right) t + \frac{c^2}{\sqrt{3} \rho^2} t^3 \right] \right\} dt = 2\theta \frac{\gamma^2 + \theta^2}{\omega_0} \frac{1}{\sqrt{3}} K_{2/3} (x),
\]

and for \( I(\omega)/d\Omega \), we obtain
\[
\frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi \varepsilon_0} \frac{e^2 \omega^2}{3\pi^2 c \omega_0^2} \left( \frac{1}{\gamma^2} + \theta^2 \right) \left[ \left( \frac{1}{\gamma^2} + \theta^2 \right) K_{2/3} (x) + \theta^2 K_{1/3} (x) \right] \\
= \frac{1}{4\pi \varepsilon_0} \frac{3e^2 \gamma^2 \dot{\omega}^2}{\pi^2 c} (1 + \Theta^2) \left\{ (1 + \Theta^2) K_{2/3} \hat{\omega} (1 + \Theta^2)^{3/2} + \Theta^2 K_{1/3} \hat{\omega} (1 + \Theta^2)^{3/2} \right\},
\]
where \( \Theta = \gamma \theta \) and \( \hat{\omega} \) is the normalized frequency,
\[
\hat{\omega} = \frac{\omega}{3\gamma^3 \omega_0}. \tag{8.81}
\]

The modified Bessel functions \( K_{2/3} (x) \) and \( K_{1/3} (x) \) both diverge at \( x \to 0 \). However, \( xK_{2/3} (x) \) and \( xK_{1/3} (x) \) are well behaving and vanish at small \( x \). Fig. (8-3) shows \( 2x^2 K_{2/3} (x) \) and \( x^2 K_{1/3} (x) \) which represent radiation intensities associated with electric field polarization along \( e_x \) (that is, in the particle orbit plane) and \( e_\perp \), respectively.

The energy spectrum \( I(\omega) \) emitted during one revolution \( (T = 2\pi/\omega_0) \) can be found by inte-
Figure 8-3: $2x^2K_{2/3}^2(x)$ (solid line) and $x^2K_{1/3}^2(x)$ (dotted line). The factor of 2 in $2x^2K_{2/3}^2(x)$ assumes $\theta \simeq 1/\gamma$.

grating $dI(\omega)/d\Omega$ over the solid angle,

$$I(\omega) = \frac{1}{4\pi \varepsilon_0} \frac{3e^2\gamma^2}{\pi^2 c^2} \hat{\omega}^2 \int_0^\pi \sin \theta' d\theta' \int_0^{2\pi} d\phi' (1 + \Theta^2) \times \left\{ (1 + \Theta^2) K_{2/3}^2[\hat{\omega} (1 + \Theta^2)^{3/2}] + \Theta^2 K_{1/3}^2[\hat{\omega} (1 + \Theta^2)^{3/2}] \right\}$$

$$= \frac{1}{4\pi \varepsilon_0} \frac{6e^2\gamma^2}{\pi c} \hat{\omega}^2 \int_{-\pi/2}^{\pi/2} \cos \theta d\theta (1 + \Theta^2) \left\{ (1 + \Theta^2) K_{2/3}^2[\hat{\omega} (1 + \Theta^2)^{3/2}] + \Theta^2 K_{1/3}^2[\hat{\omega} (1 + \Theta^2)^{3/2}] \right\}$$

$$\simeq \frac{1}{4\pi \varepsilon_0} \frac{6e^2\gamma^2}{\pi c} \hat{\omega}^2 \int_{-\infty}^{\infty} (1 + \Theta^2) \left\{ (1 + \Theta^2) K_{2/3}^2[\hat{\omega} (1 + \Theta^2)^{3/2}] + \Theta^2 K_{1/3}^2[\hat{\omega} (1 + \Theta^2)^{3/2}] \right\} d\Theta$$

$$= \frac{1}{4\pi \varepsilon_0} \frac{6e^2\gamma^2}{\pi c} f(\hat{\omega}), \quad (8.82)$$

where $\theta' = \frac{\pi}{2} - \theta$ is the polar angle from the axis of electron revolution (y axis), $\phi'$ is the azimuthal angle about the y axis, and the function $f(\hat{\omega})$ is defined by

$$f(\hat{\omega}) = 2\hat{\omega}^2 \int_0^\infty (1 + \Theta^2) \left\{ (1 + \Theta^2) K_{2/3}^2[\hat{\omega} (1 + \Theta^2)^{3/2}] + \Theta^2 K_{1/3}^2[\hat{\omega} (1 + \Theta^2)^{3/2}] \right\} d\Theta, \quad (8.83)$$

and shown in Fig. (8-4) (linear scale) and Fig. (8-5) (log-log scale). In Fig. (8-5), the straight line in the low frequency regime ($\omega \ll \gamma^3\omega_0$) has a slope of 1/3, and indicates $I(\omega) \propto \omega^{1/3}$. $f(\hat{\omega})$ peaks at $\hat{\omega} \simeq 0.14$, or $\omega \simeq 0.42\gamma^3\omega_0$, and its peak value is about 0.83. In high frequency regime $\omega \gg \gamma^3\omega_0$, the spectrum decays exponentially. The energy radiated per revolution can be calculated as
Figure 8-4: The function \( f(x) = f(\omega/3\gamma^3\omega_0) \) plotted in linear scale.

\[
E = \int_0^\infty I(\omega)\,d\omega \simeq \frac{1}{4\pi\varepsilon_0} \frac{6e^2\gamma}{\pi c} 3\gamma^3\omega_0 \int_0^\infty f(\tilde{\omega})\,d\tilde{\omega}, \quad (8.84)
\]

where the integral numerically evaluated is approximately

\[
\int_0^\infty f(\tilde{\omega})\,d\tilde{\omega} \simeq 0.713.
\]

Then

\[
E = \int_0^\infty I(\omega)\,d\omega = \frac{1}{4\pi\varepsilon_0} \frac{4.08e^2\gamma^4\omega_0}{c}, \quad (J)
\]

and the radiation power is

\[
P = \frac{E\omega_0}{2\pi} = \frac{1}{4\pi\varepsilon_0} \frac{0.65e^2c\gamma^4}{\rho^2}, \quad (8.85)
\]

which agrees reasonably well with Eq. (8.50),

\[
P = \frac{1}{4\pi\varepsilon_0} \frac{2e^2c\gamma^4}{3\rho^2} \simeq \frac{1}{4\pi\varepsilon_0} \frac{0.667e^2c\gamma^4}{\rho^2}, \quad (W).
\]

The discrepancy may be attributed to the various approximations made in the analysis.

8.7 Synchrotron Radiation II

An alternative approach to finding the frequency spectrum of synchrotron radiation is to apply discrete Fourier analysis in terms of harmonics of the fundamental frequency \( \omega_0 \) directly to radiation.
fields. If a charge \( e \) is in circular motion with a constant angular frequency \( \omega_0 \), the radiation field contains higher harmonics of \( \omega_0 \) and can be Fourier decomposed as follows. Let us recall the Lienard-Wiechert vector potential,

\[
A(r, t) = \frac{\mu_0}{4\pi} \left. \frac{e v(t')}{(1 - \mathbf{n} \cdot \beta) R(t')} \right|_{f(t') = 0}, \tag{8.86}
\]

where \( R = |r - r_p(t')| \) and the subscript \( f(t') = 0 \) indicates that all time dependent quantities should be evaluated at the retarded time \( t' \) determined from the implicit equation for \( t' \),

\[
f(t') = t' - t + \frac{|r - r_p(t')|}{c} = 0. \tag{8.87}
\]

The vector potential can be Fourier decomposed as

\[
A(r, t) = \sum_I A_I(r) e^{-i\omega_0 t}, \tag{8.88}
\]

where

\[
A_I(r) = \frac{\mu_0 e}{4\pi} \int_0^T \frac{e v(t')}{(1 - \mathbf{n} \cdot \beta) R} e^{i\omega_0 t'} dt', \tag{8.89}
\]

with \( T = 2\pi/\omega_0 \) being the period of the circular motion. Changing the integration variable from \( t \) to \( t' \) by noting

\[
\frac{dt}{dt'} = 1 - \mathbf{n} \cdot \beta, \tag{8.90}
\]
leads to

\[ A_l(r) \simeq \frac{\mu_0 c}{4\pi r} e^{ikr} \int_0^T v(t') e^{i(\omega_0 t' - k_l n \cdot r_p)} dt', \]  

(8.91)

where \( k_l = \omega_0/c \). Note that the period \( T \) remains unchanged through the transformation. Let the particle trajectory be

\[ r_p(t') = \rho (\cos \omega_0 t' e_x + \sin \omega_0 t' e_y), \]

(8.92)

\[ v(t') = \rho \omega_0 (-\sin \omega_0 t' e_x + \cos \omega_0 t' e_y). \]

(8.93)

Since all radiation fields rotate with the charge, the observing point can be chosen at arbitrary azimuthal angle \( \phi \) and we choose \( \phi = \frac{\pi}{2} \), so that \( n = (1, \theta, \phi = \frac{\pi}{2}) \). Then

\[ n \cdot r_p(t') = \rho \sin \theta \sin \omega_0 t'. \]

(8.94)

The velocity in the spherical coordinates is

\[ v(t') = \rho \omega_0 (\sin \theta \cos \omega_0 t' e_r + \cos \theta \cos \omega_0 t' e_\theta + \sin \omega_0 t' e_\phi). \]

(8.95)

Thus, the \( \theta \) component of \( A_l(r) \) is given by

\[ A_{l\theta} = \frac{\mu_0 c}{4\pi r} e^{ikr} \frac{\rho \omega_0}{T} \cos \theta \int_0^T \cos \omega_0 t' e^{i\omega_0 (t' - \rho \sin \theta \sin \omega_0 t'/c)} dt'. \]

(8.96)

Letting \( x = \omega_0 t' \), and noting \( \beta = \rho \omega_0/c \), we can rewrite this as

\[ A_{l\theta} = e^{ikr} \frac{\mu_0 c \rho \omega_0}{4\pi r} \cos \theta \int_0^{2\pi} \cos x e^{i(x - \beta \sin \theta \sin x)} dx. \]

(8.97)

The integral reduces to

\[ \int_0^{2\pi} \cos x e^{i(x - \beta \sin \theta \sin x)} dx = \pi [J_{l+1}(l\beta \sin \theta) + J_{l-1}(l\beta \sin \theta)] = \frac{2\pi}{\beta \sin \theta} J_l(l\beta \sin \theta), \]

and thus finally,

\[ A_{l\theta} = \frac{\mu_0 c \rho \omega_0}{4\pi r} e^{ikr} \cot \theta J_l(l\beta \sin \theta). \]

(8.98)

Similarly,

\[ A_{l\phi} = i \frac{\mu_0 c \rho \omega_0}{4\pi r} e^{ikr} J_l'(l\beta \sin \theta), \]

(8.99)

where use has been made of the recurrence formula of the Bessel functions,

\[ J_{l-1}(x) - J_{l+1}(x) = 2J_l'(x). \]

(8.100)
The far-field radiation magnetic field can be found from

\[ \mathbf{H}_l \simeq \frac{i}{\mu_0} \mathbf{k}_l \times \mathbf{A}_l, \]  

which yields

\[ H_{l\theta} = \frac{\epsilon \rho \omega_0 k_l}{4\pi r} e^{ik_lr} J'_l(l\beta \sin \theta), \]  

\[ H_{l\phi} = \frac{i \epsilon k_l}{4\pi r} e^{ik_lr} \cot \theta J_l(l\beta \sin \theta). \]  

The radiation power associated with the \( l \)-th harmonic is

\[ P_l = e \mu_0 r^2 \int |H_l|^2 d\Omega = \frac{1}{8\pi \epsilon_0} \frac{(\epsilon \omega_0)^2}{c} \int_0^\pi \left[ \beta^2 J_l^2(l\beta \sin \theta) + \cot^2 \theta J_l^2(l\beta \sin \theta) \right] \sin \theta d\theta. \]  

Since \( P_l = P_{-l} \), the total power is

\[ P = \sum_{l=1}^{\infty} P_l, \]  

where \( P_l \) is now

\[ P_l = \frac{1}{4\pi \epsilon_0} \frac{(\epsilon \omega_0)^2}{c} \int_0^\pi \left[ \beta^2 J_l^2(l\beta \sin \theta) + \cot^2 \theta J_l^2(l\beta \sin \theta) \right] \sin \theta d\theta, \quad l \geq 1. \]  

In nonrelativistic limit \( \beta \ll 1 \), the \( l = 1 \) term is dominant. For \( x \ll 1 \),

\[ J_1(x) \simeq \frac{1}{2} x, \quad J'_1(x) \simeq \frac{1}{2}. \]  

Then the lowest order radiation power agrees with the Larmor’s formula,

\[ P_1 \simeq \frac{1}{4\pi \epsilon_0} \frac{(\epsilon \omega_0)^2}{c} \frac{\beta^2}{4} \int_0^\pi (1 + \cos^2 \theta) \sin \theta d\theta = \frac{1}{4\pi \epsilon_0} \frac{2 \epsilon^2 a^2}{3} \frac{1}{c^2}. \]  

where \( a = v^2/\rho \) is the acceleration.

The integral in Eq. (8.105) cannot be reduced to elementary functions. However, the total power given in Eq. (8.105) should reduce to Eq. (8.50),

\[ P = \frac{1}{4\pi \epsilon_0} \frac{2 \epsilon^2 a^2}{3 c^2} \frac{1}{\gamma^4} = \frac{1}{4\pi \epsilon_0} \frac{2 \epsilon^2 a^2}{3 c^3} \frac{1}{(1 - \beta^2)^2}. \]  

To show this, we modify the integral by noting

\[ J^2_l(x) = \frac{1}{\pi} \int_0^\pi J_0(2x \sin \theta) \cos(2l\theta) d\theta. \]  

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\[ J_l^2(x) = \frac{1}{\pi} \int_0^\pi J_0(2x \sin \theta) \left( \cos 2\theta - \frac{t^2}{\pi^2} \right) \cos(2l) d\theta. \quad (8.111) \]

Then

\[ \beta^2 J_l^2(\beta l \sin \theta) + \cot^2 \theta J_l^2(\beta l \sin \theta) = \frac{1}{\pi} \int_0^\pi J_0(2\beta l \sin \theta \sin x) (\beta^2 \cos 2x - 1) \cos(2l) dx. \quad (8.112) \]

Furthermore,

\[ \int_0^\pi J_0(2\beta l \sin \theta \sin x) \sin \theta d\theta = \frac{\sin(2\beta l \sin x)}{\beta l \sin x}, \quad (8.113) \]

and the power \( P_l \) reduces to

\[ P_l = \frac{1}{4\pi\epsilon_0} \frac{(e\omega_0 l)^2}{\pi c} \int_0^\pi \frac{\sin(2\beta l \sin x)}{\beta l \sin x} \left( \beta^2 \cos 2x - 1 \right) \cos(2l) dx. \quad (8.114) \]

Noting

\[
\begin{align*}
\int_0^\pi \sin x \sin(2\beta l \sin x) \cos(2l) dx &= \frac{1}{2} \int_0^\pi \left[ \sin(2l + 1)x + \sin(1 - 2l)x \right] \sin(2\beta l \sin x) dx \\
&= \frac{\pi}{2} [J_{2l+1}(2\beta l) - J_{2l-1}(2\beta l)] \\
&= -\pi J_{2l}(2\beta l) \quad (8.115)
\end{align*}
\]

and

\[
\begin{align*}
\int_0^\pi \sin(2\beta l \sin x) \cos(2l) dx &= 2l \int_0^\beta \int_0^\pi \cos(2\beta \sin x) \cos(2l) dx d\beta \\
&= 2\pi l \int_0^\beta J_2(2\beta \beta) d\beta, \quad (8.116)
\end{align*}
\]

the power \( P_l \) can be rewritten as

\[ P_l = \frac{1}{4\pi\epsilon_0} \frac{2(e\omega_0 l)^2}{c\beta l} \left[ \beta^2 J_2'(2\beta l) - l(1 - \beta^2) \int_0^\beta J_2(2\beta l) d\beta \right], \quad (8.117) \]

and the total power is

\[ P = \sum_{l=1}^\infty P_l. \]

Relevant sum formula of the Bessel functions is

\[ \sum_{l=1}^\infty J_{2l}(2lx) = \frac{x^2}{2(1 - x^2)}. \quad (8.118) \]
Differentiating by $x$,

$$\sum_l 2lj_l'(2lx) = \frac{x}{(1-x^2)^2}. \quad (8.119)$$

Also,

$$\sum_l l^2 \int_0^\beta j_l(2lx)dx = \frac{\beta^3}{6(1-\beta^2)^3}. \quad (8.120)$$

This is one of Kapteyn series formulae. (See for example, Mathematical Formulae (in Japanese), (Iwanami, Tokyo, 1960), vol. 3, p. 212.) Then, finally, the total radiation power becomes

$$P = \frac{1}{4\pi \varepsilon_0} \frac{2e^2a^2}{3c^3} \gamma^4, \quad a = \frac{v^2}{\rho} \quad (8.121)$$

which is consistent with the known radiation power from a charge undergoing circular motion.

Analytic expression for the radiation power $P_l$ can be found by exploiting following approximation,

$$j_l(2\beta l) \simeq \frac{1}{\sqrt{\pi l^{1/3}}} \text{Ai} \left( \frac{2^{2/3}}{x} \right), \quad (8.122)$$

where $\text{Ai}(x)$ is the Airy function defined by

$$\text{Ai}(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos \left( \frac{1}{3}t^3 + xt \right) dt. \quad (8.123)$$

$$f(x) = \frac{1}{\sqrt{\pi}} \int_0^{20} \cos \left( \frac{1}{3}t^3 + xt \right) dt$$

For large $x \gg 1$, the function takes the form

$$\text{Ai}(x) \simeq \frac{1}{2x^{1/4}} \exp \left( -\frac{2}{3}x^{3/2} \right), \quad x \gg 1. \quad (8.124)$$

Also, $\text{Ai}'(0) = -0.4587$. Using these approximations, we find the following approximate formulae,

$$P_l \simeq \frac{1}{4\pi \varepsilon_0} \times 0.5175 \frac{e^2\omega_0^2}{c} l^{1/3}, \quad l \ll \gamma^3, \quad (8.125)$$

$$P_l \simeq \frac{1}{4\pi \varepsilon_0} \frac{e^2\omega_0^2}{2\sqrt{\pi}c} \sqrt{\frac{l}{\gamma}} \exp \left( -\frac{2}{3} \frac{l}{\gamma^3} \right), \quad l \gg \gamma^3. \quad (8.126)$$

Note that the radiation power increases with $l$ in the manner $P_l \propto l^{1/3}$ up to $l \simeq \gamma^3$ beyond which $P_l$ decays exponentially. This is consistent with the analysis in the preceding section.

### 8.8 Free Electron Laser

In a synchrotron radiation source, many beamlines can be installed by bending an electron beam. In straight sections of race track, no radiation occurs. However, by inserting a device called wiggler,
Figure 8-6: In a wiggler, an electron beam is modulated by a periodic magnetic field. Electrons acquire spatially oscillating perpendicular displacement $x(z)$ and velocity $v_x(z)$ which together with the radiation magnetic field $B_{Ry}$ produces a ponderomotive force $v_x(z) \times B_{Ry}(z)$ directed in the $z$ direction. The force acts to cause electron bunching required for amplification of coherent radiation.

high intensity radiation can be extracted. A wiggler consists of periodically alternating magnets and gives an electron beam periodic kick perpendicular to both the beam velocity and magnetic field. Electrons receive kicks at an interval

$$\Delta t' = \frac{\lambda_w}{c},$$

(8.127)

where $\lambda_w$ is the “wavelength” of the periodic wiggler structure. Because of Doppler shift, this time interval is shortened by a factor $\frac{1}{1 - \beta} \simeq 2\gamma^2$ for a stationary detector in front of the beam,

$$\Delta t \simeq \frac{\lambda_w}{2\gamma^2 c}.$$

(8.128)

Therefore, the wavelength of resultant radiation is approximately given by

$$\lambda \simeq \frac{\lambda_w}{2\gamma^2} \ll \lambda_w,$$

(8.129)

and the frequency by

$$\omega \simeq \frac{4\pi \gamma^2 c}{\lambda_w}.$$

(8.130)

The intensity of free electron laser can be orders of magnitude higher than that of synchrotron radiation because of coherent amplification through the periodic structure. Electrons tend to be bunched in the wiggler as the electron beam travels through the periodic structure. In contrast, no collective interaction between electrons and electromagnetic waves exists in synchrotron radiation.

In electron bunching, the magnetic ponderomotive force plays a major role. Let us assume a
periodic wiggler magnetic field in \( y \) direction,

\[
B_y = B_0 \cos \left( \frac{2\pi}{\lambda_w} z \right) = B_0 \cos k_w z, \quad k_w = \frac{2\pi}{\lambda_w}.
\]  

(8.131)

The Lorentz force is

\[
\mathbf{F} = e\mathbf{v} \times \mathbf{B},
\]

or

\[
F_x = -ev_z B_0 \cos k_w z.
\]

Then electron acquires a velocity \( v_x \) in \( x \) direction,

\[
v_x = \frac{eB_0}{m\gamma k_w} \sin k_w z,
\]

(8.132)

and a resultant ponderomotive force is

\[
-ev_x \times \mathbf{B}_R,
\]

(8.133)

where \( \mathbf{B}_R \) is the radiation magnetic field propagating in the form \( e^{i(kz-\omega t)} \) along the beam. Since the acceleration due to the wiggler magnetic field is in \( x \) direction, the radiation electric field is predominantly in \( x \) direction and radiation magnetic field \( \mathbf{B}_R \) is in \( y \) direction. Then the ponderomotive force directed in \( z \) direction is proportional to \( e^{i[(k+k_w)z-\omega t]} \) and propagates at a velocity

\[
\frac{\omega}{k+k_w}.
\]

When this propagation velocity matches the electron beam velocity \( \beta c \), strong interaction between the radiation field and electron motion takes place and electrons tend to be bunched. This results in positive feedback for wave amplification. From the condition

\[
\frac{\omega}{k+k_w} \simeq \beta c, \quad \text{or} \quad \frac{k}{k+k_w} = \beta,
\]

we readily recover

\[
k = \frac{\beta}{1-\beta} k_w \simeq 2\gamma^2 k_w.
\]

8.9 Radiation Accompanying \( \beta \) Decay

Equation (8.61) for the angular distribution of radiation energy can be applied to cases in which particle acceleration is not involved explicitly. In \( \beta \) decay, an energetic electron (or positron) is suddenly released from a nucleus together with neutrino. The situation is equivalent to sudden acceleration of an electron. The duration of acceleration \( \Delta t \) is limited by the uncertainty principle
\[ \omega_{\text{max}} \simeq \frac{\gamma mc^2}{\hbar}. \]  

(8.134)

In $\beta$ decay, the maximum value of $\gamma$ is of order of 30.

Let us assume that an electron suddenly acquires a velocity $\beta = v/c$ and then travels at a constant velocity. The integration in Eq. (8.61) is limited from $t = 0$ to $\infty$,

\[ \frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi \varepsilon_0} \frac{\varepsilon^2}{4\pi^2 c^2} \omega^2 \beta^2 \sin^2 \theta \left| \int_0^\infty e^{i\omega(1-\beta \cos \theta) t} dt \right|^2 \]

\[ = \frac{1}{4\pi \varepsilon_0} \frac{\varepsilon^2}{4\pi^2 c (1 - \beta \cos \theta)^2}, \]

(8.135)

where $\theta$ is the angle between the velocity $\beta$ and the unit vector $\mathbf{n}$. Integration over the solid angle yields

\[ I(\omega) = \frac{1}{4\pi \varepsilon_0} \frac{\varepsilon^2}{\pi c} \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] = \text{const.,} \quad \omega \lesssim \omega_{\text{max}} = \frac{\gamma mc^2}{\hbar}. \]  

(8.136)

The frequency spectrum is flat up to $\omega_{\text{max}}$. Therefore, the total energy radiated through $\beta$ decay is approximately given by

\[ \mathcal{E} \simeq \frac{1}{4\pi \varepsilon_0} \frac{\varepsilon^2}{\pi c} \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \frac{\gamma mc^2}{\hbar} \simeq \frac{1}{\pi} \left[ \ln(4\gamma^2) - 2 \right] \alpha \gamma mc^2, \]  

(8.137)

where the dimensionless quantity $\alpha$,

\[ \alpha = \frac{1}{4\pi \varepsilon_0} \frac{\varepsilon^2}{\pi c h} \simeq \frac{1}{137}, \]  

(8.138)

is the fine structure constant. The energy emitted as radiation through $\beta$ decay is a small fraction of the electron energy.

### 8.10 Cherenkov Radiation

Cherenkov radiation occurs when a charged particle travels faster than electromagnetic waves in a material medium. It does not require acceleration of charges and the basic mechanism is very similar to that of sound shock waves in gases. As in the case of $\beta$ decay, we assume a charge travelling along a straight line at a velocity $\beta = v/c(\omega)$, where

\[ c(\omega) = \frac{1}{\sqrt{\varepsilon(\omega) \mu_0}}, \]
is the velocity of electromagnetic waves in a dielectric having a permittivity \( \varepsilon(\omega) \). Eq. (8.61) should be modified as follows after taking into account the proper definition of \( c(\omega) \),

\[
\frac{dI(\omega)}{d\Omega} = \frac{\mu_0 \, e^2}{4\pi \, 4\pi^2 c(\omega)^2} \omega^2 \beta^2 \sin^2 \theta \left| \int_{-\infty}^{\infty} e^{i\omega(1-\beta \cos \theta)t} \, dt \right|^2. \tag{8.139}
\]

Note that the integration limits are from \(-\infty\) to \( \infty \). The time integral is singular,

\[
\int_{-\infty}^{\infty} e^{i\omega(1-\beta \cos \theta)t} \, dt = 2\pi \delta[\omega(1- \beta \cos \theta)],
\]

and the condition for radiation is

\[
\cos \theta = \frac{1}{\beta} < 1,
\]

or

\[
\beta(\omega) = \frac{\nu}{c(\omega)} > 1. \tag{8.140}
\]

Then,

\[
\frac{dI(\omega)}{d\Omega} = \mu_0 \, \frac{e^2}{4\pi \, 4\pi^2 c(\omega)^2} \omega^2 \beta^2 \sin^2 \theta \delta^2[\omega(1- \beta \cos \theta)]. \tag{8.141}
\]

Square of a delta function is not integrable and the radiation energy simply diverges. This is merely due to the assumption that the charge is radiating forever from \( t = -\infty \) to \( \infty \) which is of course unphysical. It is more appropriate to consider a radiation power rather than energy. For this purpose, we consider a thin slab of the dielectric of thickness \( dz \). The transit time over the distance \( dz \) is \( \Delta T = \frac{dz}{v} \) and we calculate energy radiated during that time,

\[
\frac{dI(\omega)}{d\Omega} = \mu_0 \, \frac{e^2}{4\pi \, 4\pi^2 c(\omega)^2} \omega^2 \beta^2 \sin^2 \theta \left| \int_{-\Delta T/2}^{\Delta T/2} e^{i\omega(1-\beta \cos \theta)t} \, dt \right|^2. \tag{8.142}
\]

The integral can be carried out easily,

\[
\int_{-\Delta T/2}^{\Delta T/2} e^{i\omega(1-\beta \cos \theta)t} \, dt = \frac{2}{\omega(1- \beta \cos \theta)} \sin \left[ \frac{(1- \beta \cos \theta) \Delta T \omega}{2} \right].
\]

Thus

\[
\frac{dI(\omega)}{d\Omega} = \mu_0 \, \frac{e^2 \omega^2}{4\pi \, 4\pi^2 c(\omega)^2} \sin^2 \theta \left( \frac{\sin \alpha}{\alpha} \right)^2 (dz)^2, \tag{8.143}
\]

where

\[
\alpha = \frac{1}{2}(1- \beta \cos \theta) \omega \Delta T. \tag{8.144}
\]
In high frequency regime $\omega \Delta T \gg 1$, the function $(\sin \alpha/\alpha)^2$ may be approximated by a delta function $\pi \delta(\alpha)$. Integration over the solid angle yields

$$\frac{dI(\omega)}{dz} = \frac{1}{4\pi \varepsilon_0} \left( \frac{e}{c_0} \right)^2 \omega \left( 1 - \frac{1}{\beta^2} \right)$$

$$= \frac{1}{4\pi \varepsilon_0} \left( \frac{e}{c_0} \right)^2 \omega \left( 1 - \frac{1}{\varepsilon(\omega)\mu_0 v^2} \right), \quad (8.145)$$

where $c_0 = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light in vacuum. The rate of energy loss due to Cherenkov emission is given by

$$-\frac{d\mathcal{E}}{dz} = \frac{1}{4\pi \varepsilon_0} \left( \frac{e}{c_0} \right)^2 \int_0^\infty \omega \left( 1 - \frac{1}{\varepsilon(\omega)\mu_0 v^2} \right) d\omega, \quad 0 < \omega < \infty. \quad (8.146)$$

The result obtained is meaningful only if

$$\beta > 1, \quad \text{or} \quad v > c(\omega) = \frac{1}{\sqrt{\varepsilon(\omega)\mu_0}}, \quad (8.147)$$

which is the condition for Cherenkov radiation. The instantaneous radiation power can be estimated from

$$P = \frac{d}{dt} \int I(\omega) d\omega$$

$$= \frac{1}{4\pi \varepsilon_0} \left( \frac{e}{c_0} \right)^2 v \int \omega \left( 1 - \frac{1}{\varepsilon(\omega)\mu_0 v^2} \right) d\omega. \quad (8.148)$$

In order to find the field profiles emitted through Cherenkov radiation, we start from the wave equations for the potentials in a material medium,

$$\left( \nabla^2 - \mu_0 \tilde{\varepsilon} \frac{\partial^2}{\partial t^2} \right) \tilde{\Phi}(\mathbf{r}, t) = -\rho_{\text{free}}, \quad (8.149)$$

$$\left( \nabla^2 - \mu_0 \tilde{\varepsilon} \frac{\partial^2}{\partial t^2} \right) \mathbf{A}(\mathbf{r}, t) = -\mu_0 \mathbf{J}, \quad (8.150)$$

where $\tilde{\varepsilon}$ is the dielectric operator containing time derivative,

$$\tilde{\varepsilon} = \varepsilon \left( \frac{\partial}{\partial t} \right). \quad (8.151)$$

For a charged particle $e$ travelling at a constant velocity $\mathbf{v}$, the charge density and current density are described by

$$\rho = e\delta(\mathbf{r} - \mathbf{vt}), \quad (8.152)$$

$$\mathbf{J} = e\mathbf{v}\delta(\mathbf{r} - \mathbf{vt}). \quad (8.153)$$
Then, after Fourier-Laplace transformation, the Fourier potentials can readily be found,

\[ \Phi(k, \omega) = \frac{2\pi e}{\varepsilon(\omega) \left( k^2 - \frac{\omega^2}{c^2(\omega)} \right)} \delta(\omega - k \cdot v), \quad (8.154) \]

\[ A(k, \omega) = \frac{2\pi \mu_0 v}{k^2 - \frac{\omega^2}{c^2(\omega)}} \delta(\omega - k \cdot v), \quad (8.155) \]

where, as before,

\[ c^2(\omega) = \frac{1}{\varepsilon(\omega) \mu_0}, \quad (8.156) \]

and the transformation

\[ \int \int \delta(r - vt) e^{i(\omega - k \cdot r)} dV dt = 2\pi \delta(\omega - k \cdot v), \quad (8.157) \]

is substituted. Since the physical electric field is

\[ E(r, t) = -\nabla \Phi(r, t) - \frac{\partial}{\partial t} A(r, t), \quad (8.158) \]

the Fourier component of the electric field is given by

\[ E(k, \omega) = -ik\Phi(k, \omega) + i\omega A(k, \omega) \\
= -2\pi ie \frac{k - \frac{\omega}{c^2(\omega)} v}{\varepsilon(\omega) \left( k^2 - \frac{\omega^2}{c^2(\omega)} \right)} \delta(\omega - k \cdot v). \quad (8.159) \]

Similarly, the Fourier-Laplace component of the magnetic field is

\[ B(k, \omega) = i k \times A(k, \omega) \\
= 2\pi i \mu_0 \frac{k \times v}{k^2 - \frac{\omega^2}{c^2(\omega)}} \delta(\omega - k \cdot v). \quad (8.160) \]

The physical electromagnetic fields can then be found through inverse transformations,

\[ E(r, t) = \frac{1}{(2\pi)^4} \int d^3k \int d\omega E(k, \omega) e^{i(k \cdot r - \omega t)}, \quad (8.161) \]

\[ B(r, t) = \frac{1}{(2\pi)^4} \int d^3k \int d\omega B(k, \omega) e^{i(k \cdot r - \omega t)}. \quad (8.162) \]

To proceed further, we assume that the charged particle is travelling along the z axis at a constant velocity \( v \). The system is symmetric about the axis and we may assume an observing point in the \( x - z \) plane without loss of generality. We denote the cylindrical coordinates of the
observing point by \((\rho, \phi = 0, z)\) and spherical Fourier coordinates by \(\mathbf{k} = (k, \theta, \phi)\). Then,

\[
\mathbf{k} \cdot \mathbf{r} = k(z \cos \theta + \rho \sin \theta \cos \phi).
\]

Because of the cylindrical symmetry, radiation of energy is expected in the radial direction \(\rho\), and the relevant Poynting vector is

\[
S_\rho = (\mathbf{E} \times \mathbf{H}^*)_\rho = -E_z(r, t)H_\phi^*(r, t).
\]

The energy radiated per unit length along the particle trajectory (\(z\) axis) can be calculated from

\[
-\frac{dE}{dz} = -2\pi \rho \int E_z(r, t)H_\phi^*(r, t) dt = -\rho \int E_z(\mathbf{r}, \omega)H_\phi^*(\mathbf{r}, \omega) d\omega,
\]

where \(E_z(\mathbf{r}, \omega)\) is the Laplace transform of the electric field,

\[
E_z(\mathbf{r}, \omega) = \frac{1}{(2\pi)^3} \int d^3k E_z(\mathbf{k}, \omega) e^{ik \cdot \mathbf{r}},
\]

and \(H_\phi(\mathbf{r}, \omega)\) is the Laplace transform of the magnetic field,

\[
H_\phi(\mathbf{r}, \omega) = \frac{1}{(2\pi)^3} \int d^3k H_\phi(\mathbf{k}, \omega) e^{ik \cdot \mathbf{r}}.
\]
The Laplace transform of the axial electric field \( E_z(r, \omega) \) can be calculated as follows:

\[
E_z(r, \omega) = \frac{1}{(2\pi)^2} \int d^3k E_z(k, \omega) e^{ik \cdot r}
\]

\[
= -\frac{ie}{(2\pi)^2} \int_0^\infty k^2 dk \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \frac{k \cos \theta - \frac{\omega n}{c(\omega)}}{\varepsilon(\omega) \left( k^2 - \frac{\omega^2}{c^2(\omega)} \right)} \delta(\omega - k \cos \theta) e^{i k (\cos \theta + \rho \sin \theta \cos \phi)}
\]

\[
= -\frac{i \mu_0 \omega}{2\pi} \exp \left( \frac{i \omega z}{\beta c(\omega)} \right) \int_0^\infty \kappa d\kappa \frac{\beta^2 - 1}{\kappa^2 - 1} J_0 \left( \frac{\omega \rho}{c(\omega)} \sqrt{\kappa^2 - \frac{1}{\beta^2}} \right)
\]

\[
= -\frac{i \mu_0 \omega}{2\pi} \exp \left( \frac{i \omega z}{\beta c(\omega)} \right) \int_0^\infty \kappa d\kappa \frac{\beta^2 - 1}{\kappa^2 - 1} J_0 \left( \frac{\omega \rho}{c(\omega)} \sqrt{\kappa^2 - \frac{1}{\beta^2}} \right), \quad (8.167)
\]

where

\[
\kappa = \frac{k c(\omega)}{\omega}, \quad \beta = \frac{\nu}{c(\omega)}. \quad (8.168)
\]

Letting

\[
\zeta^2 = \kappa^2 - \frac{1}{\beta^2}, \quad (8.169)
\]

we finally obtain

\[
E_z(r, \omega) = \frac{i \mu_0 \omega}{2\pi} \exp \left( \frac{i \omega z}{\beta c(\omega)} \right) \left( 1 - \frac{1}{\beta^2} \right) \int_0^\infty \frac{\zeta J_0 \left( \frac{\omega \rho}{c(\omega)} \right)}{\zeta^2 - 1 + \frac{1}{\beta^2}} d\zeta
\]

\[
= \frac{ie \mu_0}{2\pi} \omega \exp \left( \frac{i \omega z}{\beta c(\omega)} \right) \left( 1 - \frac{1}{\beta^2} \right) K_0(\alpha \rho), \quad (8.170)
\]

where

\[
\alpha = \pm \frac{\omega}{c(\omega)} \sqrt{\frac{1}{\beta^2} - 1}, \quad (8.171)
\]

and use is made of the integral representation of the modified Bessel function \( K_0(ax) \),

\[
K_0(ax) = \int_0^\infty \frac{t J_0(at)}{t^2 + x^2} dt. \quad (8.172)
\]

In the asymptotic regime \(|\alpha| \rho \gg 1\), \( K_0(\alpha \rho) \) approaches

\[
\sqrt{\frac{\pi}{2\alpha \rho}} e^{-\alpha \rho}. \quad (8.173)
\]

For this to be propagating radially outward in the form \( e^{ik \cdot \rho} \), we must choose

\[
\alpha = -\frac{\omega}{c(\omega)} \sqrt{\frac{1}{\beta^2} - 1}
\]

\[
= -i \frac{\omega}{c(\omega)} \sqrt{1 - \frac{1}{\beta^2}}, \quad \beta > 1. \quad (8.174)
\]
With this choice for \( \alpha \), the Laplace transform of the axial electric field becomes proportional to

\[
\exp \left[ \frac{i\omega}{c(\omega)} \left( \sqrt{1 - \frac{1}{\beta^2}} \rho + \frac{z}{\beta} \right) \right],
\]

and the radial and axial wavenumbers can be identified as

\[
k_\rho = \frac{\omega}{c(\omega)} \sqrt{1 - \frac{1}{\beta^2}}, \quad k_z = \frac{\omega}{v},
\]

respectively. Cherenkov radiation is confined in a cone characterized by an angle \( \theta \),

\[
\sin \theta = \frac{c(\omega)}{v},
\]

as shown in Fig. 8-8.

The Laplace transform of the azimuthal magnetic field is given by

\[
H_\phi(x, \omega) = \frac{1}{(2\pi)^3} \int d^3k H_\phi(k, \omega)e^{ik\cdot x} = -\frac{ie}{(2\pi)^2} \int k^2 dk \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \frac{k v \sin \theta}{k^2 - \frac{\omega^2}{c^2(\omega)}} \delta(\omega - k v \cos \theta)
\]

\[
\times e^{ik(z \cos \theta + \rho \sin \theta \cos \phi)} = e^{\frac{e}{2\pi} \alpha K_1(\alpha \rho) \exp \left( \frac{i\omega}{c(\omega)} z \right)},
\]

Figure 8-8: Cherenkov cone. Radiation fields are confined in the cone.
where the following integral
\[ \int_0^\infty \frac{x^2 J_1(bx)}{x^2 + a^2} \, dx = aK_1(ab), \quad (8.179) \]
is noted. (Calculation steps are left for an exercise.) Substituting \( E_z(r, \omega) \) and \( H_\phi(r, \omega) \) into Eq. (8.164), we find
\[ -\frac{d\mathcal{E}}{dz} = -\frac{i\mu_0 e^2}{(2\pi)^2 \rho} \int_{-\infty}^\infty \omega \left(1 - \frac{1}{\beta^2}\right) \alpha^* K_0(\alpha \rho) K_1^*(\alpha \rho) \, d\omega. \quad (8.180) \]
In the asymptotic region \( |\alpha| \rho \gg 1 \), this reduces to
\[ -\frac{d\mathcal{E}}{dz} = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{r_0} \int_{-\infty}^\infty \omega \left(1 - \frac{1}{\beta^2}\right) \, d\omega, \quad (8.181) \]
in agreement with the earlier result, Eq. (8.146). Eq. (8.180) can be used even when Cherenkov condition is not satisfied. In this case, energy loss is through near field Coulomb interaction between a charged particle and ions and electrons in molecules in the dielectric media. We will return to this problem in Section 8.12.

### 8.11 Transition Radiation

Transition radiation occurs when a charge crosses a boundary of two dielectric media. No acceleration is required, nor is it necessary for charge to move faster than the speed of light as in Cherenkov radiation. In this respect, transition radiation is a least demanding radiation mechanism. Radiation emitted from a charge approaching a conductor is an extreme case of transition radiation with an infinite permittivity, and may be regarded as the inverse process of radiation accompanying \( \beta \) decay. Disappearance, rather than creation, of charge is responsible for transition radiation.

We first consider a simple case: a charge \( e \) approaching normally a conducting plate at a velocity \( v \ (> 0) \). On impact, the charge is assumed to come to rest. A conducting plate is mathematically equivalent to an infinitely permissive dielectric plate. An image charge \(-e\) moving in the opposite direction in the conducting plate can be introduced so that the current density is
\[ J_z(r, t) = -ev [\delta(z + vt) + \delta(z - vt)] \delta(x) \delta(y), \quad -\infty < t < 0, \quad (8.182) \]
where at \( t = 0 \) (or \( z = 0 \)) the particle is brought to rest. Its Laplace transform is
\[ J_z(r, \omega) = \int_{-\infty}^0 J_z(r, t) e^{i\omega t} \, dt = -e \exp \left(-i \frac{\omega}{v} |z| \right) \delta(x) \delta(y). \quad (8.183) \]
Let the observing point be at \( P \) located at \((r, \theta)\). (The system is symmetric about the axis and thus
Figure 8-9: Radiation from a charge impinging on a metal surface. Sudden deceleration at the metal surface is the inverse process of radiation accompanying beta decay.

\( \phi \) is ignorable. The vector potential at \( P \) can be calculated in the usual manner,

\[
A_z(r, \omega) = \frac{\mu_0}{4\pi} \int \frac{J_z(r', \omega)}{|r - r'|} e^{i k |r - r'|} dV' \approx \frac{\mu_0}{4\pi r} e^{i k r} \int J_z(r', \omega) e^{-i k r'} dV' \\
= -\frac{\mu_0 c}{4\pi r} e^{i k r} \int_0^\infty \left[ \exp \left( -i \frac{\omega}{v} z' \right) e^{-i k z' \cos \theta} + \exp \left( -i \frac{\omega}{v} z' \right) e^{i k z' \cos \theta} \right] dz' \\
= i \frac{\mu_0 c \beta}{4\pi r} e^{i k r} \frac{1}{k} \left( \frac{1}{1 + \beta \cos \theta} + \frac{1}{1 - \beta \cos \theta} \right), \quad z > 0, \tag{8.184}
\]

and the magnetic field from

\[
H_\phi(r, \omega) \simeq -i k A_z \sin \theta / \mu_0 \\
= \frac{\epsilon \beta}{4\pi r} e^{i k r} \left( \frac{1}{1 + \beta \cos \theta} + \frac{1}{1 - \beta \cos \theta} \right) \sin \theta. \tag{8.185}
\]
The angular distribution of radiation energy is thus given by

\[
\frac{dI(\omega)}{d\Omega} = \frac{1}{2\pi} r^2 c \mu_0 |H_{\phi}(r, \omega)|^2 = \frac{e^2 \beta^2 c \mu_0}{32\pi^3} \left( \frac{1}{1 + \beta \cos \theta} + \frac{1}{1 - \beta \cos \theta} \right)^2 \sin^2 \theta = \frac{1}{4\pi \varepsilon_0} \frac{e^2 v^2}{2\pi^2 \varepsilon^3} \frac{\sin^2 \theta}{(1 - \beta^2 \cos^2 \theta)^2}, \quad -\infty < \omega < \infty.
\]

Integration over the solid angle in the region \(z > 0\) \((0 < \theta < \pi/2)\) yields

\[
I(\omega) = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{2\pi c} \frac{1}{\beta} \left( (1 + \beta^2) \ln \frac{1 + \beta}{1 - \beta} - 2\beta \right), \quad \omega > 0.
\]

Evidently, the radiation energy \(\int_0^\infty I(\omega) d\omega\) diverges. This is due to the assumption of a perfect conductor. In practice, metals cannot be regarded as perfect conductor. The condition that the surface impedance of metal

\[
Z = \sqrt{-\frac{i\omega \mu_0}{-i\omega \varepsilon_0 + \sigma}},
\]

be sufficiently small compared with the free space impedance \(Z_0 = \sqrt{\mu_0/\varepsilon_0}\) imposes an upper limit of the frequency, \(\omega \varepsilon_0 \ll \sigma\) and a cutoff emerges in the integral \(\int_0^{\omega_{\text{max}}} I(\omega) d\omega\).

![Figure 8-10: Transition radiation emitted by a charge passing through a dielectric boundary.](image)

We now analyze the case of a dielectric slab having a relative permittivity \(\varepsilon_r = \varepsilon/\varepsilon_0\). In this
case, the particle continues to travel after passing the boundary and the current density is now

\[ J_z(r, t) = -ev\delta(x)\delta(y)\delta(z + vt), \quad -\infty < t < \infty. \] (8.187)

Its Laplace transform is

\[ J_z(r, \omega) = -ev\delta(x)\delta(y)e^{-i\omega t}. \] (8.188)

The contribution to the radiation fields from the region \( z > 0 \) consists of two parts, one directly from the charge (as in free space) and the other via reflection at the dielectric boundary. Denoting the magnetic reflection coefficient by \( \Gamma \) in the Fresnel’s formulae,

\[ \Gamma = \frac{\varepsilon_r \cos \theta - \sqrt{\varepsilon_r - \sin^2 \theta}}{\varepsilon_r \cos \theta + \sqrt{\varepsilon_r - \sin^2 \theta}}, \] (8.189)

and following the same procedure as in the case of conductor plate, we find

\[ H_{\phi 1}(r, \omega) \simeq \frac{ev}{4\pi cr} e^{ikr} \left( \frac{1}{1 + \beta \cos \theta} + \frac{\Gamma}{1 - \beta \cos \theta} \right) \sin \theta, \quad z > 0. \] (8.190)

The contribution from the region \( z < 0 \) involves refraction at the boundary, and thus additional retardation because of the longer path length. In Fig.8-10, for \( z < 0 \), we observe

\[ l = r + z' \tan \theta'' \sin \theta, \quad z' < 0 \]
\[ l' = -z' / \cos \theta'', \]
\[ \frac{\sin \theta'}{\sin \theta''} = \sqrt{\varepsilon}, \]

and thus

\[ kl + \sqrt{\varepsilon}k'l' = kr - k\sqrt{\varepsilon - \sin^2 \theta}. \] (8.191)

Then the contribution from the region \( z < 0 \) to the integral becomes

\[ H_{\phi 2}(r, \omega) = -\frac{ev}{4\pi cr} (1 - \Gamma) \frac{1}{1 + \beta \sqrt{\varepsilon_r - \sin^2 \theta}} e^{ikr} \sin \theta \] (8.192)

Note that the factor \( 1 - \Gamma \) here indicates the field amplitude transmitted into the air region. The total magnetic field is \( H_{\phi 1} + H_{\phi 2} \), and the angular distribution of the radiation energy is given by

\[ \frac{dI(\omega, \theta)}{d\Omega} = \frac{\nu^2 e^2 \mu_0}{2\pi} |H_\phi(r, \omega)|^2 = \frac{\mu_0 e^2 v^2}{32\pi^3 c} A^2 \sin^2 \theta, \] (8.193)

where

\[ A = \frac{1}{1 + \beta \cos \theta} + \frac{\Gamma}{1 - \beta \cos \theta} - \frac{1 - \Gamma}{1 + \beta \sqrt{\varepsilon_r - \sin^2 \theta}}. \] (8.194)

In the case of ideal conductor \( \varepsilon_r \to \infty, \Gamma = 1 \), we recover

\[ \frac{dI(\omega, \theta)}{d\Omega} = \frac{1}{4\pi \varepsilon_0} \frac{e^2 v^2}{2\pi^2 c^3} \left( \frac{\sin \theta}{1 - \beta^2 \cos^2 \theta} \right)^2. \]
When $\varepsilon_r = 1$, $\Gamma = 0$, radiation evidently disappears.

The factor $A$ in Eq. (8.194) does not fully agree with that in the original work by Frank and Ginzburg,

$$A' = \frac{1}{1 + \beta \cos \theta} + \frac{\Gamma}{1 - \beta \cos \theta} - \frac{1 + \Gamma}{\varepsilon_r (1 + \beta \sqrt{\varepsilon_r - \sin^2 \theta})},$$

although this too vanishes when $\varepsilon_r \to 1$ and reduces to the case of conducting medium when $\varepsilon_r \to \infty$.

### 8.12 Energy Loss of Charged Particles Moving in Dielectrics

The formula derived in Eq. (8.180) yields a physically meaning energy loss rate even when the Cherenkov condition is not satisfied, $\beta < 1$. A charged particle moving in a dielectric medium “collides” with atoms and lose its energy through Coulomb interaction with electrons in atoms. Electrons in an atom are bounded. However, they do respond to electromagnetic disturbance and absorb energy through the resonance $\varepsilon(\omega) = 0$, where

$$\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2}\right).$$

Resonance of the type

$$\frac{1}{x - x_0},$$

can be handled mathematically by introducing an imaginary part,

$$\frac{1}{x - x_0} = P\frac{1}{x - x_0} - i\pi \delta(x - x_0),$$

where $P$ stands for the principal part. This is justifiable because the imaginary part of the function

$$\frac{1}{x - x_0 + i\varepsilon} = \frac{x - x_0}{(x - x_0)^2 + \varepsilon^2} - i \frac{\varepsilon}{(x - x_0)^2 + \varepsilon^2},$$

remains finite even in the limit $\varepsilon \to 0$,

$$\lim_{\varepsilon \to 0} \int \frac{\varepsilon}{(x - x_0)^2 + \varepsilon^2} dx = \pi.$$

Physically, the resonance leads to absorption of wave energy by charged particles in a material medium dielectrics, plasmas, etc.

The characteristic scale length of interaction between a charged particle and atoms in a dielectric is evidently of the order of atomic size which indicates that the interaction is of near-field, nonradiating nature dominated by longitudinal (electrostatic) fields. As we will see, the major contribution to the energy loss occurs through the pole of the dielectric function, $\varepsilon(\omega) = 0$.

In the near-field region $\alpha \rho \ll 1$, the modified Bessel functions $K_0(\alpha \rho), K_1(\alpha \rho)$ may be approx-
imated by

\[ K_0(\alpha \rho) \simeq -\ln \left( \frac{\alpha \rho}{2} \right) - \gamma_E, \quad (8.201) \]

\[ K_1(\alpha \rho) \simeq \frac{1}{\alpha \rho}, \quad (8.202) \]

where \( \gamma_E = 0.5772 \cdots \) is the Euler’s constant. The real part of Eq. (8.180) becomes

\[ \frac{d \mathcal{E}}{dz} \simeq \frac{\mu_0 e^2}{(2\pi)^2} \text{Re} \int_{-\infty}^{\infty} i\omega \left( 1 - \frac{1}{\varepsilon(\omega) \mu_0 v^2} \right) \left[ \ln \left( \frac{\alpha \rho}{2} \right) + \gamma_E \right] d\omega, \quad (8.203) \]

with

\[ \alpha = \frac{\omega}{\varepsilon(\omega)} \sqrt{\frac{1}{\beta^2} - 1}, \beta < 1. \quad (8.204) \]

The dielectric function \( \varepsilon(\omega) \) is in the form

\[ \varepsilon(\omega) = \varepsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2} \right). \]

Therefore, the integration can be carried out by evaluating the pole contribution at \( \varepsilon(\omega) = 0 \), which occurs at

\[ \omega = \pm \sqrt{\omega_p^2 + \omega_0^2}, \quad (8.205) \]

and exploiting Plemelj’s formula,

\[ \lim_{\nu \to 0} \frac{1}{x - a + i\nu} = \text{P} \frac{1}{x - a} - i\pi \delta(x - a), \]

where \( \text{P} \) indicates the principal part of the singular function \( 1/(x - a) \). The result is

\[ -\frac{d \mathcal{E}}{dz} \simeq \frac{1}{4\pi \varepsilon_0} \left( \frac{e\omega_p}{v} \right)^2 \ln \left( \frac{1.12v}{\rho_\text{min} \omega_p} \right). \quad (8.206) \]

As the minimum distance \( \rho \), the intermolecular distance may be substituted because the shell electrons effectively shield the electric field of the charge well inside the atom.

In a plasma, \( \omega_0 \) is evidently zero (because electrons in a plasma are free). Then,

\[ \frac{d \mathcal{E}}{dz} \simeq \frac{1}{4\pi \varepsilon_0} \left( \frac{e\omega_p}{v} \right)^2 \ln \left( \frac{v}{\rho_\text{min} \omega_p} \right), \quad (8.207) \]

where

\[ \rho_\text{min} \simeq \left( \frac{3}{4\pi n} \right)^{1/3}, \]

is the average distance between ions. For a Maxwellian electron distribution with a temperature

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the average energy loss rate may be estimated from

$$\frac{d\mathcal{E}}{dt} \approx \frac{1}{4\pi\varepsilon_0} \frac{(e\omega_p)^2}{3v_Tc} \ln \left( \frac{8\sqrt{2}\pi}{3} n\lambda_D^3 \right), \quad (8.208)$$

where

$$\lambda_D = \frac{vT_e}{\omega_p}, \quad (8.209)$$

is the Debye shielding length.

### 8.13 Bremsstrahlung

Whenever a charged particle collides with another charged particle, electromagnetic radiation occurs due to acceleration by Coulomb force. Collisions between like particles (e.g., electron-electron) emit quadrupole radiation while collisions between unlike particles (e.g., electron-ion) emit dipole radiation. Bremsstrahlung is due to collisions between electrons and ions and provides a basic mechanism for x-ray production.

Let an electron approach an ion having a charge $Ze$ with a velocity $v$ ($\ll c$) and impact parameter $b$. The acceleration due to Coulomb force is of the order of

$$a \approx -\frac{1}{4\pi\varepsilon_0} \frac{Ze^2}{mb^2}, \quad (8.210)$$

and consequent radiation power can be estimated from the Larmor’s formula,

$$P \approx \frac{1}{4\pi\varepsilon_0} \frac{2e^2a^2}{3c^3}. \quad (8.211)$$

The total energy radiated can in principle be found by integrating the power over time along the electron trajectory. However, in experiments, one is seldom interested in measuring radiation power or energy associated with a single electron. What is more relevant is the radiation associated with a beam of electrons impinging on an ion. In this case, some electrons have impact parameters vanishingly small. However, the impact parameter has a lower bound imposed by the uncertainty principle,

$$b_{\text{min}} \approx \frac{\hbar}{p}, \quad (8.212)$$

where $p$ is the electron momentum. For an impact parameter $b$, the time duration in which the acceleration is significant is

$$\Delta t \approx \frac{b}{v}. \quad (8.213)$$
Therefore, energy radiated by a single electron is

\[
\mathcal{E} = P \Delta t \\
\simeq \frac{1}{(4\pi\varepsilon_0)^3} \frac{2}{3c^3} \left( \frac{Ze^3}{m} \right)^2 \frac{1}{b^3 v}.
\] (8.214)

For an electron beam having a density \( n \), the radiation power can thus be estimated from

\[
P = n_e v \int_{b_{\text{min}}}^{\infty} \mathcal{E} 2\pi b \, db \\
= \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \left( \frac{Ze^3}{m} \right)^2 n_e mnv \frac{v}{c^3 h}.
\] (8.215)

If the ion density is \( n_i \), the quantity \( Pn_i \) defines the power density,

\[
\frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \left( \frac{Ze^3}{m} \right)^2 n_e n_imv \frac{v}{c^3 h} = \text{W/m}^3.
\] (8.216)

The frequency spectrum of radiation energy may qualitatively be found as follows. Since the characteristic time of acceleration

\[
\tau = \frac{b}{v},
\] (8.217)

is short, radiation occurs as an impulse and the spectrum is flat in the region \( 0 < \omega < v/b \), and vanishes for \( \omega > v/b \). Since the minimum impact parameter is

\[
b_{\text{min}} \simeq \frac{h}{mv},
\] (8.218)

the upper limit of the frequency spectrum extends to

\[
\omega_{\text{max}} \simeq \frac{mv^2}{h}, \quad \text{or } h\omega \lesssim mv^2.
\] (8.219)

This is essentially a statement of energy conservation, that is, the maximum photon energy emitted during bremsstrahlung is limited by the incident electron kinetic energy, which is reasonable. Therefore, the frequency spectrum of bremsstrahlung is

\[
I(\omega, b) = \begin{cases} 
\frac{1}{(4\pi\varepsilon_0)^3} \frac{2}{3c^3} \left( \frac{Ze^3}{m} \right)^2 \frac{1}{m^2 v^2 b^2}, & 0 < \omega < \frac{v}{b}, \\
0, & \omega > \frac{v}{b}.
\end{cases}
\] (8.220)

\( I(\omega, b) \) has dimensions of J/frequency. It is convenient to introduce a radiation cross-section \( \chi(\omega) \)
defined by

\[ \chi(\omega) = \int_{b_{\text{min}}}^{b_{\text{max}}} I(\omega, b) 2\pi b db \]

\[ = \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \frac{(Ze^3)^2}{mc^2mv^2} \ln \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right) \]

\[ = \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \frac{(Ze^3)^2}{mc^2mv^2} \ln \left( \frac{mv^2}{\hbar\omega} \right). \quad (8.221) \]

The integral over the frequency,

\[ \int \chi(\omega) d\omega, \quad (8.222) \]

evidently diverges at the lower end \( \omega \to 0 \). To remedy this difficulty, Bethe and Heitler recognized that if the velocity \( v \) is understood as the mean value of the initial and final velocities, i.e., before and after emission of a photon,

\[ v = \frac{1}{2} (v_{\text{initial}} + v_{\text{final}}) \]

\[ = \frac{1}{\sqrt{2m}} \left( \sqrt{\mathcal{E}} + \sqrt{\mathcal{E} - \hbar\omega} \right), \quad (8.223) \]

the integral remains finite,

\[ \int \chi(\omega) d\omega \quad = \quad \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \frac{(Ze^3)^2}{mc^2mv^2} \int_0^{\omega_{\text{max}}} \ln \left( \frac{\left( \sqrt{\mathcal{E}} + \sqrt{\mathcal{E} - \hbar\omega} \right)^2}{\hbar\omega} \right) d\omega \]

\[ = \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \frac{(Ze^3)^2}{mc^2\hbar} \int_0^1 \ln \left( \frac{1 + \sqrt{1 - x}}{\sqrt{x}} \right) dx, \quad x = \frac{\hbar\omega}{\mathcal{E}}, \]

\[ = \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \frac{(Ze^3)^2}{mc^3\hbar}. \quad (8.224) \]

The integral in the intermediate step is unity. Multiplying by the ion density \( n_i \), we thus obtain the bremsstrahlung rate per unit length,

\[ \frac{dE}{dz} = n_i \int \chi(\omega) d\omega \]

\[ = \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \frac{(Ze^3)^2 n_i}{mc^3\hbar}, \quad (8.225) \]

and radiation power density,

\[ P/V = \frac{1}{(4\pi\varepsilon_0)^3} \frac{4\pi}{3} \frac{(Ze^3)^2 n_e n_i v}{mc^3\hbar}, \quad (W \text{ m}^{-3}). \quad (8.226) \]

This agrees with the earlier qualitative estimate in Eq. (8.216).

Relativistic correction to the classical bremsstrahlung formulae can be readily found if we move
to the electron frame wherein the electron velocity is nonrelativistic. Since the energy and frequency
are Lorentz transformed in the same manner, and the transverse dimensions are Lorentz invariant,
it follows that the radiation cross-section $\chi(\omega)$ is Lorentz invariant,

$$\chi_{\text{lab}}(\omega_{\text{lab}}) = \chi'(\omega'),$$

where the primed quantities are those in the electron frame. The frequencies $\omega_{\text{lab}}$ and $\omega'$ are related
through the relativistic Doppler shift,

$$\omega' = \gamma \omega_{\text{lab}}(1 - \beta \cos \theta), \quad \omega_{\text{lab}} = \gamma \omega'(1 + \beta \cos \theta'),$$

(8.227)

where $\theta$ and $\theta'$ are the angles with respect to the electron velocity in each frame. Since in the
electron frame, the radiation is confined in a small angle about $\theta' = \pi/2$, we have

$$\omega' \simeq \frac{\omega_{\text{lab}}}{\gamma}.$$ (8.228)

The collision time is shortened by the factor $\gamma$ since the transverse field is intensified by the same
factor through Lorentz transformation. Therefore, the maximum impact parameter is modified as

$$b_{\text{max}} = \frac{\gamma v}{\omega'} = \frac{\gamma^2 v}{\omega_{\text{lab}}},$$

(8.229)

and the radiation cross-section in the laboratory frame becomes

$$\chi(\omega) = \chi'(\omega') = \frac{1}{(4\pi \varepsilon_0)^3} \frac{4\pi e^4 (Ze)^2}{3} \frac{\ln \left( \frac{\gamma^2 m v^2}{\hbar \omega} \right)}{m^2 c^4 v^2}.$$ (8.230)

It is noted that the minimum impact parameter remains unchanged through the transformation
because it is essentially the Compton length based on the uncertainty principle.

### 8.14 Radiation due to Electron-Electron Collision

In this case the dipole radiation is absent because in the center of mass frame, two electrons stay
at opposite positions, $r_1 = -r_2$, and the dipole moment identically vanishes. The lowest order
radiation process is that due to electric quadrupole. (The magnetic dipole moment also vanishes.)
The quadrupole moment tensor is

$$Q_{ij} = \frac{1}{2} e x_i x_j,$$

(8.231)

where $x_i$ is the $i$-th component of the relative distance $2r$. In Chapter 5, a general formula for the
quadrupole radiation power has been derived. Noting

$$\frac{d^3}{dt^3}(x_i x_j) = \ddot{x}_i x_j + 3\dot{x}_i \dot{x}_j + 3\ddot{x}_i \ddot{x}_j + x_i \dddot{x}_j,$$

(8.232)
Figure 8-11: Colliding electrons in the center of mass frame. The impact parameter is $b$.

\[ \ddot{x}_i = \frac{1}{4\pi\varepsilon_0} \frac{2e^2}{mr^3} x_i, \]  
\[ \dddot{x}_i = \frac{1}{4\pi\varepsilon_0} \frac{2e^2}{m} \left( \frac{x_i}{r^3} - \frac{3x_i v_r}{r^4} \right), \]  

we find

\[ \frac{d^3}{dt^3}(x_ix_j) = \frac{1}{4\pi\varepsilon_0} \frac{2e^2}{mr^3} \left( 4(v_ix_j + v_jx_i) - \frac{6v_r x_ix_j}{r} \right), \]

\[ \dddot{Q} = \frac{1}{4\pi\varepsilon_0} \frac{e^3}{mr^3} \left( 4(v_ix_j + v_jx_i) - \frac{6v_r x_ix_j}{r} \right) \]

Substituting this into the quadrupole radiation power,

\[ P = \frac{1}{4\pi\varepsilon_0} \frac{1}{60e^5} \left[ 3 \sum_{ij} \dddot{Q}_{ij}^2 - \left( \sum_i \dddot{Q}_{ii} \right)^2 \right], \]

we obtain, after somewhat lengthy calculations,

\[ P = \frac{1}{4\pi\varepsilon_0} \frac{2}{15e^5} \left( \frac{e^3}{m} \right)^2 \frac{v^2 + 11v_\phi^2}{r^4}, \]

where $v_\phi$ is the $\phi$ component of the velocity related to the initial angular momentum $bv_0 = r(t)v_\phi(t)$ with $v_0$ the velocity at $r \to \infty$. Energy conservation reads

\[ \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{4\pi\varepsilon_0} \frac{e^2}{2r}. \]
Then
\[ v^2 + 11v_0^2 = v_0^2 - \frac{4e^2}{4\pi \varepsilon_0 mr} + 11 \left( \frac{(bv_0)^2}{r^2} \right), \] (8.240)
and the total radiation energy can be found by integrating the radiation power over time along the trajectory,
\[ E = \frac{1}{4\pi \varepsilon_0} \frac{2}{15e^3} \left( \frac{e^2}{m} \right)^2 \int_{-\infty}^{\infty} \frac{1}{r^4} \left( v_0^2 - \frac{4e^2}{4\pi \varepsilon_0 mr} + 11 \left( \frac{(bv_0)^2}{r^2} \right) \right) dt. \] (8.241)
This can be converted to an integral over the distance \( r \) by noting
\[ dt = \frac{dr}{v_r} = \frac{dr}{\sqrt{v_0^2 - \frac{(bv_0)^2}{r^2} - \frac{4e^2}{4\pi \varepsilon_0 mr}}}, \] (8.242)
\[ E = \frac{1}{4\pi \varepsilon_0} \frac{2}{15e^3} \left( \frac{e^2}{m} \right)^2 \times 2 \int_{r_{\text{min}}}^{\infty} \frac{1}{r^4} \frac{v_0^2 - \frac{4e^2}{4\pi \varepsilon_0 mr} + 11 \left( \frac{(bv_0)^2}{r^2} \right)}{\sqrt{v_0^2 - \frac{(bv_0)^2}{r^2} - \frac{4e^2}{4\pi \varepsilon_0 mr}}} dr, \] (J) (8.243)
where \( r_{\text{min}} \) is the distance of the closest approach,
\[ r_{\text{min}} = \frac{1}{2v_0^2} \left( \frac{4e^2}{m} + \sqrt{\left( \frac{4e^2}{m} \right)^2 + 4(bv_0)^2} \right). \] (8.244)
However, the radiation energy by a single electron pair is of no practical interest. What is more relevant is the radiation power emitted by an electron beam impinging on a single electron which can be evaluated from
\[ P = 2\pi n v_0 \int_{0}^{\infty} \mathcal{E}(b) b db \]
\[ = \frac{1}{4\pi \varepsilon_0} \frac{4}{15e^3} \left( \frac{e^2}{m} \right)^2 \times 2\pi n v_0 \int_{0}^{\infty} b db \int_{r_{\text{min}}}^{\infty} \frac{1}{r^4} \frac{v_0^2 - \frac{4e^2}{4\pi \varepsilon_0 mr} + 11 \left( \frac{(bv_0)^2}{r^2} \right)}{\sqrt{v_0^2 - \frac{(bv_0)^2}{r^2} - \frac{4e^2}{4\pi \varepsilon_0 mr}}} dr, \] (W) (8.245)
The double integral reduces to
\[ \int_{0}^{\infty} b db \int_{r_{\text{min}}}^{\infty} dr \frac{1}{r^4} \frac{v_0^2 - \frac{4e^2}{4\pi \varepsilon_0 mr} + 11 \left( \frac{(bv_0)^2}{r^2} \right)}{\sqrt{v_0^2 - \frac{(bv_0)^2}{r^2} - \frac{4e^2}{4\pi \varepsilon_0 mr}}} \]
\[ = \frac{25}{3} v_0 \int_{r_{\text{min}}}^{\infty} \frac{1}{r^5} \left( r^2 - \frac{4re^2}{mv_0^3} \right)^{3/2} dr = \frac{5mv_0^3}{6e^2}. \] (8.246)
Therefore,

\[ P = \frac{1}{4\pi \varepsilon_0} \frac{4\pi ne^4 v_0^4}{9 \ mc^2}, \quad \text{(W).} \quad (8.247) \]

This defines a radiation cross-section,

\[ \sigma = \frac{P}{\frac{1}{2} nmv_0^3} = \frac{8\pi v}{9} \frac{r_e^2}{c}, \quad \text{(m}^2) \quad (8.248) \]

where \( \frac{1}{2} nmv_0^3 \) is the energy flux density of the beam and

\[ r_e = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{mc^2} = 2.85 \times 10^{-15} \text{ m}, \]

is the classical radius of electron.
Problems

8.1 A charged particle with charge $e$ and mass $m$ undergoes motion in external electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Show that the radiation power is given by

$$P(t') = \frac{1}{4\pi\varepsilon_0} \frac{2e^2}{3mc^3} \gamma^6 \left[ a^2 - (\beta \times a)^2 \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2e^2}{3m^2c^3} \gamma^2 \left[ (\mathbf{E} + \mathbf{v} \times \mathbf{B})^2 - (\beta \cdot \mathbf{E})^2 \right],$$

where $a$ is the acceleration. Since $a^2 - (\beta \times a)^2$ is positive definite, so is $(\mathbf{E} + \mathbf{v} \times \mathbf{B})^2 - (\beta \cdot \mathbf{E})^2$.

Note: In an instantaneous rest frame of the particle, $v = \beta = 0$ and $\gamma = 1$. The radiation power in that frame is thus with respect to the proper time,

$$\frac{d\mathcal{E}'}{d\tau} = P(\tau) = \frac{1}{4\pi\varepsilon_0} \frac{2e^2}{3m^2c^3} \mathbf{E}'^2,$$

where $\mathbf{E}'$ is the electric field also in that frame. Since the electric field is Lorentz transformed as

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel, \quad \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}),$$

we readily see that

$$P(\tau) = P(t'),$$

that is, the proper radiation power is Lorentz invariant. This is not surprising because both time and energy are Lorentz transformed in a similar manner.

8.2 Carry out the integration in the radiation power due to acceleration perpendicular to the velocity

$$P(t') = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{4\pi c} \left| \beta' \right|^2 \int \frac{1}{(1 - \beta \cos \theta)^5} \left[ (1 - \beta \cos \theta)^2 - \frac{1}{\gamma^2} \sin^2 \theta \cos^2 \phi \right] d\Omega$$

to obtain

$$P(t') = \frac{1}{4\pi\varepsilon_0} \frac{2e^2}{3c} \left| \beta' \right|^2 \gamma^4.$$ 

Plot the angular distribution of the radiation intensity

$$f(\theta) = \frac{1}{(1 - \beta \cos \theta)^6} \left[ (1 - \beta \cos \theta)^2 - \frac{1}{\gamma^2} \sin^2 \theta \cos^2 \phi \right],$$

for $\phi = 0$ and $\pi/2$ in a highly relativistic case.

8.3 A nonrelativistic electron with velocity $v$ collides head-on with a heavy negative ion having a charge $-Ze$. Find the total energy radiated. What is the dominant frequency component?
8.4 A highly relativistic electron passes by a proton at an impact parameter $b$. Assuming that the electron trajectory is unaffected by the proton and radiation, show that the total energy radiated is
\[
E = \frac{1}{4\pi\varepsilon_0} \frac{\pi e^6}{12 m_e^2 c^3 b^3} \left( \frac{4 - \beta^2}{1 - \beta^2} \right) \approx \frac{1}{4\pi\varepsilon_0} \frac{\pi e^6 \gamma^2}{4 m_e^2 c^4 b^3},
\]
What is the dominant frequency component contained in the radiation field?

8.5 An electron having an initial energy $E(t = 0) = \gamma_0 mc^2$ undergoes cyclotron motion in a magnetic field $B$. Show that the rate of energy loss to radiation is given by
\[
-\frac{dE(t)}{dt} = \frac{1}{4\pi\varepsilon_0} \frac{2e^4 B^2}{m^4 c^5} (E^2 - m^2 c^4).
\]
Integrate this and find an expression for $E(t)$.

8.6 An electron is placed in a plane wave $E_0 e^{i(kx - \omega t)}$. If $\hbar \omega \ll mc^2$, electron recoil may be ignored. Show that the radiation power is
\[
P = I_0 \frac{8\pi}{3} r_e^2,
\]
where $I_0 = c\varepsilon_0 E_0^2$ (W/m$^2$) is the intensity of the incident wave and
\[
r_e = \frac{e^2}{4\pi\varepsilon_0 mc^2} = 2.8 \times 10^{-15} \text{ (m)},
\]
is the classical radius of electron.
\[
\sigma = \frac{8\pi}{3} r_e^2,
\]
is known as Thomson scattering cross section.

Note: For $\hbar \omega \gg mc^2$, Klein and Nishina, solving the Dirac equation to take into account electron recoil, obtained
\[
\sigma \approx \pi r_e^2 \frac{mc^2}{\hbar \omega} \left( \frac{1}{2} + \ln \frac{2\hbar \omega}{mc^2} \right).
\]

8.7 The light from Crab nebula is believed to be synchrotron radiation emitted by highly energetic electrons trapped in a weak intergalactic magnetic field. If the electron energy is $2 \times 10^{11}$ eV and $B = 10^{-7}$ T, what is the dominant wavelength component?

8.8 Perform numerically the integration in the radiation power of $l$-th harmonic of synchrotron radiation,
\[
P_l = \frac{1}{4\pi\varepsilon_0} \frac{(\varepsilon_0 \omega_0)^2 l}{c} \int_0^\pi \frac{\sin(2\beta l \sin x)}{\beta \sin x} \left( \beta^2 \cos 2x - 1 \right) \cos(2lx) dx,
\]
for the case $\beta = 0.99$ ($\gamma \approx 7$) and plot the result as a function of $l$.

8.9 Estimate the bremsstrahlung cooling rate (in W/m$^3$) of a hydrogen plasma for an electron temperature of 10 keV and plasma density of $n = 10^{20}$ m$^{-3}$. If the plasma is confined by a magnetic field of 5 T, what is the cooling rate due to cyclotron radiation if reabsorption by
the plasma is ignored? (Cyclotron radiation can be reabsorbed by a plasma if its density is high enough and its contribution to energy loss may be ignored.)

8.10 Assuming a permittivity in the form

\[ \varepsilon(\omega) = \left( 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2} \right) \varepsilon_0, \]

carry out the integration in the rate of Cherenkov radiation,

\[ -\frac{dE}{dz} = \frac{1}{4\pi\varepsilon_0^2} \int_0^\infty \omega \left( 1 - \frac{1}{\beta^2} \right) d\omega. \]

8.11 Two identical charges are moving along a circular orbit of radius \( a \) at an angular velocity \( \omega \).

Assume nonrelativistic motion, \( \omega a \ll c \). Angular separation of the charges is \( \phi_0 \). If \( \phi_0 = 0 \), the problem reduces to a charge \( 2e \) rotating at \( \omega \) which radiates as a dipole. If \( \phi_0 = \pi \), the dipole moment vanishes and higher order quadrupole radiation should be considered. At what angular separation \( \phi_0 \) does the dipole radiation power become equal to quadrupole radiation power? Is it necessary to consider magnetic dipole radiation?

8.12 Four identical charges are placed along a circle of radius \( a \) with equal angular spacing \( \pi/2 \).

If the system rotates about the axis at angular velocity \( \omega \), what is the lowest order radiation power? What would happen if the number of charges \( N \) increases?

8.13 Generalize the quadrupole radiation in electron-electron collision to relativistic velocity.