Chapter 6

Harmonic Expansion of Electromagnetic Fields

6.1 Introduction

For a given current source \( J(r, t) \), the vector potential can in principle be found by solving the inhomogeneous vector wave equation,

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A(r, t) = -\mu_0 J(r, t),
\]

provided the Lorenz gauge is chosen. In source free region, the electromagnetic fields \( E \) and \( H \) can then be deduced from the vector potential through

\[
B = \nabla \times A,
\]

\[
\varepsilon_0 \mu_0 \frac{\partial E}{\partial t} = \nabla \times B, \text{ (in source-free region } J = 0).\]

In this Chapter, we develop spherical harmonic expansion of the electromagnetic fields. In experiments, the electromagnetic fields, not the potentials, are measured. Radiation electromagnetic fields can be decomposed into two basic vector components, Transverse Magnetic (TM) and Transverse Electric (TE), where “transverse” is with respect to the direction of wave propagation \( r \). These fundamental modes are normal to each other and provide convenient base vectors in analysis of radiation fields.

6.2 Wave Equations and Green’s Function

The set of Maxwell’s equations,

\[
\nabla \cdot E = \frac{\rho}{\varepsilon_0}, \quad (6.1)
\]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (6.2) \]
\[ \nabla \cdot \mathbf{B} = 0, \quad (6.3) \]
\[ \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \quad (6.4) \]

can be reduced to two inhomogeneous wave equations for the two potentials \( \Phi \) and \( \mathbf{A} \) as follows. Substitution of
\[ \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}, \quad (6.5) \]
into Eq. (6.1) yields
\[ \nabla^2 \Phi + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\rho}{\varepsilon_0}. \quad (6.6) \]

Also, substitution of \( \mathbf{B} = \nabla \times \mathbf{A} \) and Eq. (6.5) into Eq. (6.4) yields
\[ \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \mathbf{J}. \quad (6.7) \]

If the Lorenz gauge
\[ \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0, \quad (6.8) \]
is chosen for \( \nabla \cdot \mathbf{A} \) (i.e., the longitudinal component of the vector potential), then Eqs. (6.6) and (6.7) are completely decoupled, and reduce, respectively, to
\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Phi = -\frac{\rho}{\varepsilon_0}, \quad (6.9) \]
\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\mu_0 \mathbf{J}. \quad (6.10) \]

If the Coulomb gauge
\[ \nabla \cdot \mathbf{A} = 0, \]
is chosen instead, that is, if the longitudinal component of the vector potential is chosen to be zero, such decoupling cannot be achieved,
\[ \nabla^2 \Phi_C = -\frac{\rho}{\varepsilon_0}, \quad (6.11) \]
\[ \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \nabla \Phi = -\mu_0 \mathbf{J}. \quad (6.12) \]

Since the longitudinal current \( \mathbf{J}_l \) satisfies the charge conservation law,
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_l = 0, \quad (6.13) \]
it follows that
\[ \frac{1}{c^2} \frac{\partial}{\partial t} \nabla^2 \Phi_C = \mu_0 \nabla \cdot \mathbf{J}_l, \quad (6.14) \]
and the wave equation for the vector potential in Coulomb gauge involves only the transverse component of the vector potential,

\[ \nabla^2 A_t - \frac{1}{c^2} \frac{\partial^2 A_t}{\partial t^2} = -\mu_0 J_t. \] (6.15)

As briefly pointed out in Chapter 3, the scalar potential \( \Phi \) in Eq. (6.11) is not subject to retardation due to finite propagation speed of electromagnetic disturbance. The part of the electric field generated from the gradient of the scalar potential

\[ E = -\nabla \Phi_C - \frac{\partial A}{\partial t}, \]

is also nonretarded because of instantaneous propagation. Physical (retarded) electric field is contained in the vector potential. In the Lorenz gauge, both potentials are appropriately retarded and we will therefore employ Lorenz gauge. (As we will see in the following section, the instantaneous Coulomb field is in fact cancelled by a term in the retarded transverse vector potential.)

The scalar potential \( \Phi \) and three cartesian components of the vector potential \( A_i \) all satisfy the wave equation in the form

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A_i = -f_i(r, t), \] (6.16)

where \( f_i(r, t) \) is a source function, either the charge density or the current density. The Green’s function for the scalar wave equation \( G(r, t) \) can be found as a solution for the following singular wave equation,

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(r - r', t - t') = -\delta(r - r')\delta(t - t'), \] (6.17)

with the boundary condition that

\[ G = 0 \text{ at } r = \infty \text{ and } t = \pm \infty. \]

Once the Green’s function is found, the solution to the original wave equation can be written down in the form of convolution,

\[ A_i(r, t) = \int G(r - r', t - t') f_i(r', t') dV' dt'. \] (6.18)

We seek a solution for the Green’s function by the method of Fourier transform. Let the Fourier transform of \( G(r, t) \) be \( g(k, \omega) \),

\[ g(k, \omega) = \int dV \int d\tau e^{-i(k \cdot R - \omega \tau)} G(R, \tau), \] (6.19)

\[ G(R, \tau) = \frac{1}{(2\pi)^4} \int d^3k \int d\omega g(k, \omega) e^{i(k \cdot R - \omega \tau)}, \] (6.20)
where \( \mathbf{R} = \mathbf{r} - \mathbf{r}', \tau = t - t' \). Then, Eq. (6.17) in the Fourier-Laplace space \((\mathbf{k}, \omega)\) becomes

\[
\left( -k^2 + \frac{\omega^2}{c^2} \right) g(\mathbf{k}, \omega) = -1,
\]
or

\[
g(\mathbf{k}, \omega) = \frac{1}{k^2 - (\omega/c)^2}.
\] (6.21)

Substitution of \( g(\mathbf{k}, \omega) \) into Eq. (6.20) yields

\[
G(\mathbf{R}, \tau) = \frac{1}{(2\pi)^4} \int d\mathbf{k} \int d\omega \frac{e^{i(k\mathbf{R} - \omega\tau)}}{k^2 - (\omega/c)^2}.
\]

Integration over \( \omega \) can be done easily,

\[
\int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{k^2 - (\omega/c)^2} d\omega = 2\pi \frac{c}{k} \sin(kc\tau).
\] (6.22)

(Note that with a new variable \( s = -i\omega \), the integral reduces to

\[
-ic^2 \int_{-i\infty}^{i\infty} \frac{e^{st}}{s^2 + (ck)^2} ds = \frac{2\pi}{ck} \int_{-i\infty}^{i\infty} \frac{ckes^t}{s^2 + (ck)^2} ds = \frac{2\pi}{ck} \sin(kc\tau),
\] (6.23)

which is the standard inverse Laplace transform. Causality is appropriately handled in inverse Laplace transform.) Then,

\[
G(\mathbf{R}, \tau) = \frac{c}{(2\pi)^3} \int_0^{\infty} dk \int_0^{\pi} d\theta \int_0^{2\pi} d\phi k \sin(ck\tau) e^{ikR\cos \theta} \sin \theta,
\] (6.24)

where \( \theta \) is the polar angle in the \( \mathbf{k} \) space measured from the direction of \( \mathbf{R} = \mathbf{r} - \mathbf{r}' \), and \( R = |\mathbf{r} - \mathbf{r}'| \).

The integrations can be carried out as follows:

\[
G(\mathbf{R}, \tau) = \frac{c}{(2\pi)^2} \int_0^{\infty} dk \sin(ck\tau) \int_0^{\pi} e^{ikR\cos \theta} \sin \theta d\theta
\]
\[
= \frac{c}{4\pi^2 R} \int_0^{\infty} 2 \sin(ck\tau) \sin(kR) dk
\]
\[
= \frac{c}{4\pi R} \left[ \delta(c\tau - R) - \delta(c\tau + R) \right],
\] (6.25)

where in the final step, use is made of

\[
\int_0^{\infty} \cos(ak) dk = \pi \delta(a).
\]

The function

\[
\frac{c}{R} \delta(c\tau - R) = \frac{1}{R} \delta \left( \tau - \frac{R}{c} \right),
\] (6.26)
describes an impulse propagating radially outward at a speed \( c \), and the function

\[
\frac{c}{R} \delta(c \tau + t) = \frac{1}{R} \delta \left( \tau + \frac{R}{c} \right),
\]

(6.27)

describes an impulse propagating radially inward. For radiation fields due to a source spatially limited, only the outgoing solution is physically meaningful and we adopt it as the Green’s function,

\[
G(R, \tau) = \frac{1}{4 \pi R} \delta \left( \tau - \frac{R}{c} \right)
= \frac{1}{4 \pi \left| \mathbf{r} - \mathbf{r}' \right|} \delta \left( t - t' - \frac{\left| \mathbf{r} - \mathbf{r}' \right|}{c} \right), \quad (\text{m}^{-1}\text{s}^{-1}).
\]

(6.28)

It is straightforward to check that \( G(R, t) \) satisfies the inhomogeneous wave equation in Eq. (6.17) if the following identity is recalled,

\[
\nabla^2 \frac{1}{\left| \mathbf{r} - \mathbf{r}' \right|} = -4 \pi \delta \left( \mathbf{r} - \mathbf{r}' \right).
\]

Having found the Green’s function for the wave equation, the general solution to the inhomogeneous wave equation,

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A_i = -f_i(\mathbf{r}, t),
\]

(6.29)

can be written down as

\[
A_i(\mathbf{r}, t) = \frac{1}{4 \pi} \int dV' \int dt' \frac{f_i(\mathbf{r}', t')}{ \left| \mathbf{r} - \mathbf{r}' \right|} \delta \left( t - t' - \frac{\left| \mathbf{r} - \mathbf{r}' \right|}{c} \right).
\]

(6.30)

If the source function \( f(\mathbf{r}, t) \) is separable in the form,

\[
f_i(\mathbf{r}, t) = f_i(\mathbf{r}) e^{-i \omega t},
\]

(6.31)

Eq. (6.30) reduces to

\[
A_i(\mathbf{r}, t) = \frac{1}{4 \pi} e^{-i \omega t} \int \frac{e^{i k \left| \mathbf{r} - \mathbf{r}' \right|}}{ \left| \mathbf{r} - \mathbf{r}' \right|} f_i(\mathbf{r}') dV',
\]

(6.32)

where \( k = \omega / c \). The spatial part of this solution,

\[
A_i(\mathbf{r}) = \frac{1}{4 \pi} \int \frac{e^{i k \left| \mathbf{r} - \mathbf{r}' \right|}}{ \left| \mathbf{r} - \mathbf{r}' \right|} f_i(\mathbf{r}') dV',
\]

(6.33)

satisfies the Helmholtz equation,

\[
(\nabla^2 + k^2) A_i = -f_i(\mathbf{r}).
\]

(6.34)

Corresponding Green’s function is

\[
G(\mathbf{r} - \mathbf{r}') = \frac{e^{i k \left| \mathbf{r} - \mathbf{r}' \right|}}{4 \pi \left| \mathbf{r} - \mathbf{r}' \right|},
\]

(6.35)
which satisfies the Helmholtz equation,

\[
\left( \nabla^2 + k^2 \right) G \left( \mathbf{r} - \mathbf{r}' \right) = -\delta \left( \mathbf{r} - \mathbf{r}' \right).
\] (6.36)

The formulation developed here agrees with that in the preceding Chapter which has been derived through a qualitative argument based on the retarded nature of electromagnetic disturbance. In static cases \( \omega = 0, k = 0 \), we trivially recover the static potentials,

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV',
\]

\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'.
\]

In the Green’s function

\[
G(\mathbf{r} - \mathbf{r}', t - t') = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \delta \left( t - t' - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right),
\] (6.37)

the retarded nature of electromagnetic disturbance can be clearly seen. For a source located at \( \mathbf{r}' \) and observer at \( \mathbf{r} \), electromagnetic phenomena observed at time \( t \) is due to source disturbance created at the time \( |\mathbf{r} - \mathbf{r}'|/c \) seconds earlier than \( t \).

### 6.3 Coulomb and Lorenz Gauges

The divergence of the vector potential \( \nabla \cdot \mathbf{A} \) can be assigned an arbitrary scalar function without affecting the electromagnetic fields \( \mathbf{E} \) and \( \mathbf{B} \). The Maxwell’s equations do not specify \( \nabla \cdot \mathbf{A} \). In fact, potential transformation involving a scalar function \( \lambda \),

\[
\mathbf{A}' = \mathbf{A} + \nabla \lambda, \quad \Phi' = \Phi - \frac{\partial \lambda}{\partial t},
\] (6.38)

does not affect the electromagnetic fields,

\[
\mathbf{E}' = -\nabla \Phi' - \frac{\partial \mathbf{A}'}{\partial t} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E},
\]

\[
\mathbf{B}' = \nabla \times \mathbf{A}' = \nabla \times \mathbf{A} = \mathbf{B}.
\]

In Lorenz gauge characterized by

\[
\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0,
\] (6.39)

the function \( \lambda \) must satisfy the homogeneous wave equation,

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \lambda_L = 0,
\] (6.40)
and in the Coulomb gauge choice $\nabla \cdot \mathbf{A} = 0$, $\lambda$ must satisfy Laplace equation

$$\nabla^2 \lambda_C = 0. \quad (6.41)$$

In classical electrodynamics, the scalar function $\lambda$ can be chosen to be zero because a reasonable boundary condition $\lambda_{L,C} \to 0$ at $r \to \infty$ for both equations

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \lambda_L = 0, \quad \nabla^2 \lambda_C = 0,$$

allows only $\lambda_{L,C} = 0$.

In macroscopic electromagnetic analysis, Lorenz gauge defined by

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0,$$

is more convenient because all potentials and electromagnetic fields are explicitly retarded. In contrast, the scalar potential in the Coulomb gauge is not retarded and consequent Coulomb electric field is also nonretarded. Such instantaneous fields are clearly unphysical, and should be cancelled somehow by a counterpart. In this section, it is shown that the electromagnetic fields formulated in Lorenz gauge, which are all retarded appropriately, are indeed consistent with those in the Coulomb gauge. The instantaneous Coulomb electric field emerging in the Coulomb gauge is cancelled.

In Coulomb gauge, the scalar potential satisfies time dependent Poisson equation,

$$\nabla^2 \Phi(\mathbf{r}, t) = -\frac{\rho(\mathbf{r}, t)}{\varepsilon_0}. \quad (6.42)$$

Its solution is nonretarded,

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} dV', \quad (6.43)$$

and resultant Coulomb electric field is also nonretarded,

$$\mathbf{E}(\mathbf{r}, t) = -\nabla \Phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \quad (6.44)$$

As we have seen, the vector potential in Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) is transverse and obeys the wave equation,

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A}_t = -\mu_0 \mathbf{J}_t = -\mu_0 (\mathbf{J} - \mathbf{J}_l), \quad (6.45)$$

where $\mathbf{J}_l$ is the transverse component of the current density. Note that the longitudinal current $\mathbf{J}_l$ and the charge density $\rho$ are related through the charge conservation law,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_l = 0. \quad (6.46)$$
Solution for the transverse vector potential is retarded,

\[
A_t(r, t) = \frac{\mu_0}{4\pi} \int \frac{1}{|r - r'|} \mathbf{J}_t \left( r', t - \frac{|r - r'|}{c} \right) dV'.
\]  

(6.47)

The electric field in the Coulomb gauge is thus give by

\[
E_C(r, t) = -\nabla \Phi - \frac{\partial A_t}{\partial t} = \frac{1}{4\pi \varepsilon_0} \int \frac{(r - r') \rho(r', t)}{|r - r'|^3} dV' - \frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \int \frac{1}{|r - r'|} \mathbf{J}_t \left( r', t - \frac{|r - r'|}{c} \right) dV'.
\]  

(6.48)

The instantaneous Coulomb field (the first term in the RHS) must be somehow cancelled by a term stemming from the time derivative of the transverse vector potential for the electric field to be consistent with that from Lorenz gauge which guarantees that all fields are retarded.

It is convenient to work with spatial Fourier transforms of the potentials and fields. Since the inverse Laplace transform of the Fourier-Laplace Green’s function

\[
g(\mathbf{k}, \omega) = \frac{1}{k^2 - (\omega/c)^2},
\]  

(6.49)

is

\[
-\frac{c^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega \tau}}{\omega^2 - (ck)^2} d\omega = c^2 \frac{\sin(ck\tau)}{ck},
\]  

(6.50)

the particular solution for an inhomogeneous wave equation

\[
\left( \frac{1}{c^2} \frac{d^2}{dt^2} + k^2 \right) A(\mathbf{k}, t) = f(\mathbf{k}, t),
\]  

can be given in the form of convolution,

\[
A(\mathbf{k}, t) = \frac{c}{k} \int_0^{\infty} \sin(ck\tau) f(\mathbf{k}, t - \tau) d\tau.
\]  

(6.51)

Applying this to the transverse vector potential, we find

\[
A_t(\mathbf{k}, t) = \mu_0 \frac{c}{k} \int_0^{\infty} \sin(ck\tau) \left[ \mathbf{J}(\mathbf{k}, t - \tau) - \mathbf{J}(\mathbf{k}, t - \tau) \right] d\tau.
\]  

(6.52)

Since in the Coulomb gauge, the scalar potential is given by

\[
\Phi(\mathbf{k}, t) = \frac{\rho(\mathbf{k}, t)}{\varepsilon_0 k^2},
\]  

(6.53)
the electric field becomes

\[ E_C(k, t) = -\frac{i k}{\varepsilon_0 k^2} \rho(k, t) - \frac{\partial}{\partial t} A_t(k, t) \]

\[ = -\frac{i k}{\varepsilon_0 k^2} \rho(k, t) - \mu_0 \frac{c}{k} \frac{\partial}{\partial t} \int_0^\infty \sin(ck\tau) \left[ J(k, t - \tau) - J_t(k, t - \tau) \right] d\tau. \]  \quad (6.54)

Noting

\[ \frac{\partial}{\partial t} \int_0^\infty \sin(ck\tau) J_t(k, t - \tau) d\tau \]

\[ = - \int_0^\infty \sin(ck\tau) \frac{\partial}{\partial \tau} J_t(k, t - \tau) d\tau \]

\[ = ck \int_0^\infty \cos(ck\tau) J_t(k, t - \tau) d\tau, \]  \quad (6.55)

and

\[ J_t(k, t) = \frac{i k}{k^2} \rho(k, t), \]  \quad (6.56)

we find

\[ E_C(k, t) = -\frac{i k}{\varepsilon_0 k^2} \rho(k, t) - \mu_0 \frac{c}{k} \frac{\partial}{\partial t} \int_0^\infty \sin(ck\tau) J(k, t - \tau) d\tau \]

\[ + \frac{i k}{\varepsilon_0 k^2} \rho(k, t) - \frac{i k}{\varepsilon_0 k} \int_0^\infty \sin(ck\tau) \rho(k, t - \tau) d\tau \]

\[ = -\frac{i k}{\varepsilon_0 k} \int_0^\infty \sin(ck\tau) \rho(k, t - \tau) d\tau - \mu_0 \frac{c}{k} \frac{\partial}{\partial t} \int_0^\infty \sin(ck\tau) J(k, t - \tau) d\tau. \]  \quad (6.57)

Note that the instantaneous fields (the first and third terms in the RHS) exactly cancel each other and the electric field is appropriately retarded and consistent with the result in the Lorenz gauge,

\[ E_L(r, t) = -\frac{1}{4\pi \varepsilon_0} \nabla \int \frac{\rho(r', \tau)}{|r - r'|} dV' - \mu_0 \frac{\partial}{\partial t} \int \frac{J(r', \tau)}{|r - r'|} dV', \]  \quad (6.58)

where

\[ \tau = t - \frac{|r - r'|}{c}. \]

Fourier transform of Eq. (6.58) indeed recovers

\[ E_L(k, t) = -\frac{i k}{\varepsilon_0 k} \int_0^\infty \sin(ck\tau) \rho(k, t - \tau) d\tau - \mu_0 \frac{c}{k} \frac{\partial}{\partial t} \int_0^\infty \sin(ck\tau) J(k, t - \tau) d\tau, \]  \quad (6.59)

which is identical to the field formulated in the Coulomb gauge.

The magnetic field is generated by the transverse vector potential,

\[ B = \nabla \times A = \nabla \times A_t, \]  \quad (6.60)

(note that \( \nabla \times A_t = 0 \) by definition) and is thus explicitly retarded irrespective of the choice of the
gauge.

### 6.4 Elementary Spherical Waves

For a current source oscillating at a frequency $\omega$, the vector potential can be calculated from Eq. (6.33),

$$A(r,t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int \frac{e^{ik|r-r'|}}{|r-r'|} J(r')dV'.$$

Here, we wish to expand the spatial Green’s function,

$$G(r-r') = \frac{e^{ik|r-r'|}}{4\pi |r-r'|},$$

(6.61)

in terms of spherical harmonics as done in static cases,

**Static** : 

$$G(r-r') = \frac{1}{4\pi |r-r'|}$$

$$= \sum_{l,m} \frac{1}{2l+1} \frac{(r')^l}{r^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^{*}(\theta', \phi'), \ r > r'.$$

(6.62)

Evidently, the objective of the harmonic expansion is to identify electric and magnetic multipole components in the radiation electromagnetic fields. The Green’s function

$$G(r-r') = \frac{e^{ik|r-r'|}}{4\pi |r-r'|},$$

satisfies the singular Helmholtz equation,

$$(\nabla^2 + k^2) G = -\delta(r-r').$$

(6.63)

We wish to construct a series expansion of the Green’s function in terms of solutions to the homogeneous Helmholtz equation,

$$(\nabla^2 + k^2) f(r) = 0.$$  

(6.64)

Let the function $f(r)$ be separated in the form

$$f(r) = R(r)F(\theta, \phi).$$

Substitution into Eq. (6.64) yields

$$\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2} \right) R(r) = 0,$$

(6.65)

$$\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + l(l+1) \right] F(\theta, \phi) = 0,$$

(6.66)
where \( l(l + 1) \) is a separation constant. The angular function \( F(\theta, \phi) \) is unchanged from the static case,

\[
F(\theta, \phi) = Y_{lm}(\theta, \phi).
\]

To find solutions for the radial function \( R(r) \), let us assume

\[
R(r) = \frac{u(r)}{\sqrt{r}}.
\]

Substitution into Eq. (6.65) yields the following equation for \( u(r) \),

\[
\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + k^2 - \frac{(l + \frac{1}{2})^2}{r^2} u(r) = 0,
\]

whose solutions are Bessel functions of half-integer order,

\[
u(r) = J_{l+\frac{1}{2}}(kr), \quad N_{l+\frac{1}{2}}(kr).
\]

It is customary to introduce spherical Bessel functions defined by

\[
j_l(kr) = \sqrt{\frac{\pi}{2}} \frac{J_{l+\frac{1}{2}}(kr)}{\sqrt{kr}}, \quad n_l(kr) = \sqrt{\frac{\pi}{2}} \frac{N_{l+\frac{1}{2}}(kr)}{\sqrt{kr}},
\]

where the numerical factor \( \sqrt{\frac{\pi}{2}} \) is to let both functions approach the form of spherical wave,

\[
|j_l(kr)|, \quad |n_l(kr)| \to \frac{1}{kr} \text{ for } kr \gg 1.
\]

Noting that the asymptotic forms of the ordinary Bessel functions are

\[
J_n(x) \to \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{2n + 1}{4} \pi \right),
\]

\[
N_n(x) \to \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{2n + 1}{4} \pi \right),
\]

we observe that \( j_l(x) \) and \( n_l(x) \) asymptotically approach

\[
j_l(x) \to \frac{1}{x} \cos \left( x - \frac{l+1}{2} \pi \right), \quad x \gg 1
\]

\[
n_l(x) \to \frac{1}{x} \sin \left( x - \frac{l+1}{2} \pi \right), \quad x \gg 1.
\]

In radiation analyses, it is convenient to introduce spherical Hankel functions defined by

\[
\text{First kind: } h_l^{(1)}(x) = j_l(x) + i n_l(x),
\]
Second kind: \( h_2^2(x) = j_l(x) - i n_l(x) \). \hfill (6.78)

The asymptotic forms of these functions are:

\[
\begin{align*}
  h_1^1(x) &\to (-i)^{l+1} \frac{e^{ix}}{x}, \quad x \gg 1, \\
  h_1^2(x) &\to i^{l+1} \frac{e^{-ix}}{x}, \quad x \gg 1,
\end{align*}
\]

which describe outgoing and incoming spherical waves, respectively.

Having found elementary spherical harmonic solutions for the homogeneous Helmholtz equation, we are now ready to construct the Green’s function in terms of those elementary solutions. Since any linear combinations of the four radial functions \( j_l(kr) \), \( n_l(kr) \), \( h_1^1(kr) \) and \( h_1^2(kr) \) satisfy the Helmholtz equation, and the Green’s function should be bounded everywhere, we may assume

\[
G(r,r') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} j_l(kr') h_1^1(kr) Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi'), \quad \text{for } r > r',
\]

and

\[
G(r,r') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} j_l(kr) h_1^1(kr') Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi'), \quad \text{for } r < r',
\]

where \( A_{lm} \) is an expansion coefficient of the \((l,m)\) harmonic and continuity on the surface \( r = r' \) is imposed. Note that for a small argument, only \( j_l(x) \) remains bounded,

\[
j_l(x) \simeq \frac{x^l}{(2l+1)!!}, \quad x \ll 1.
\]

The spherical Bessel function of the second kind \( n_l(x) \) diverges at small \( x \) as

\[
n_l(x) \simeq -\frac{(2l-1)!!}{x^{l+1}}, \quad x \ll 1.
\]

The radially converging (incoming) solution \( h_1^2(kr) \) is discarded because we are interested in radiation of electromagnetic field from localized sources.

The coefficient \( A_{lm} \) can be determined through the familiar procedure exploiting the discontinuity in the radial derivative. Substituting Eqs. (6.81) and (6.82) into the original equation

\[
(\nabla^2 + k^2) G = -\delta(r - r') = -\frac{\delta(r - r')}{rr' \sin \theta} \delta(\theta - \theta') \delta(\phi - \phi'),
\]

multiplying both sides by \( Y_{lm}^*(\theta, \phi) \), and integrating the result over the entire solid angle by noting the orthogonality of the harmonic functions,

\[
\int Y_{lm}(\theta, \phi) Y_{lm}^*(\theta, \phi) d\Omega = \delta_{l'l'} \delta_{mm'},
\]

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we obtain
\[ A_{lm} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + k^2 - \frac{l(l + 1)}{r^2} \right) g_l(r, r') = -\frac{\delta(r - r')}{rr'}, \]  
(6.86)
where
\[ g_l(r, r') = \begin{cases} j_l(kr') h_l^{(1)}(kr), & r > r', \\ j_l(kr) h_l^{(1)}(kr'), & r < r' \end{cases} \]  
(6.87)
The derivative of the radial function \( g_l(r, r') \) is discontinuous at \( r = r' \) and thus second order derivative yields the singularity compatible with the RHS,
\[ \frac{d^2}{dr^2} g_l(r, r') = i k \frac{1}{(kr)^2} \delta(r - r'), \]  
(6.88)
where use is made of the Wronskian of the spherical Bessel functions,
\[ j_l(x)[h_l^{(1)}(x)]' - j_l'(x) h_l^{(1)}(x) = i \left[ j_l(x)n_l'(x) - j_l'(x)n_l(x) \right] = \frac{i}{x^2}. \]  
(6.89)
We thus find a rather simple result,
\[ A_{lm} = ik, \]  
(6.90)
and the desired spherical harmonic expansion of the Green’s function is
\[ e^{ik|r-r'|} = \begin{cases} ik \sum_{l=0}^{\infty} \sum_{m=-l}^{l} j_l(kr) h_l^{(1)}(kr') Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi'), & r < r', \\ ik \sum_{l=0}^{\infty} \sum_{m=-l}^{l} j_l(kr') h_l^{(1)}(kr) Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi'), & r > r'. \end{cases} \]  
(6.91)
In the limit of static fields \( k \to 0 \) (that is, \( \omega \to 0 \)), we indeed recover
\[ \frac{1}{4\pi |r-r'|} = \begin{cases} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l + 1} \frac{(r')^l}{r^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi'), & r > r', \\ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l + 1} \frac{r^l}{(r')^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi'), & r < r'. \end{cases} \]  
(6.92)
The expansion in Eq. (6.91) allows us to write the vector potential in the form
\[ A(r) = \frac{\mu_0}{4\pi} \int \frac{e^{ik|r-r'|}}{|r-r'|} J(r')dV' \]  
\[ = ik\mu_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} h_l^{(1)}(kr) Y_{lm}(\theta, \phi) \int J(r') j_l(kr') Y_{lm}^*(\theta', \phi')dV'. \]  
(6.93)
For cartesian components \( A_i(r), (i = x, y, z) \), Eq. (6.93) gives

\[
A_i(r) = ik\mu_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} h_l(kr)Y_{lm}(\theta, \phi) \int j_l(r') j_l(kr') Y_{lm}^*(\theta', \phi') dV'.
\]

However, for non-cartesian components (e.g., components in the spherical coordinates), it is necessary to single out the component of the current density \( J(r) \) in the same direction as the vector potential \( A \) at the observing position

\[
A(r) = ik\mu_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} h_l^{(1)}(kr)Y_{lm}(\theta, \phi)e_A \int e_A \cdot J(r') j_l(kr') Y_{lm}^*(\theta', \phi') dV',
\]

(6.94)

where \( e_A \) is the unit vector in the direction of \( A \), \( e_A = A / A \). The problem with this representation is that the direction of the vector potential \( e_A \) is not known a priori. It is more convenient if we can find basic electromagnetic field vectors which are normal to each other everywhere. The Transverse Electric (TE) and Transverse Magnetic (TM) eigenvectors will serve exactly for this purpose and form a convenient dyadic for radiation electromagnetic fields.

### 6.5 TE (Transverse Electric) and TM (Transverse Magnetic) Base Vectors

In electromagnetic wave problems, boundary conditions are usually imposed on the electric and magnetic fields \( E \) and \( B \), rather than the potentials \( \Phi \) and \( A \). (Note that the electromagnetic fields are gauge invariant, while the potentials are not.) Also, in experiments, what is usually measured is the electric field. For these reasons, it is desirable to formulate spherical harmonic expansion of the electric field.

In Chapter 3 on magnetostatics, we saw that \( \nabla \cdot A = 0 \) allows us to write the vector potential in the form

\[
A = \nabla \times F,
\]

where \( F \) is a vector field. In the Lorenz gauge, the divergence of \( A \) does not vanish,

\[
\nabla \cdot A = -\frac{1}{c^2} \frac{\partial \Phi}{\partial t},
\]

which suggests

\[
A = \nabla f + \nabla \times F,
\]

where \( f \) is a scalar field. However, the field \( f \) does not contribute to the magnetic field \( B = \nabla \times A \) since \( \nabla \times \nabla f \equiv 0 \) and we may continue to assume \( A = \nabla \times F \) even for time varying fields. (The scalar field \( f \) does affect the formulation of the electric field.) The vector field \( F \) may further be
decomposed into radial and transverse components as follows,

$$\mathbf{F} = r\phi + r \times \nabla \psi,$$

(6.95)

where $\phi$ and $\psi$ are scalar functions. If the vector potential satisfies the Helmholtz equation, so do $\phi$ and $\psi$,

$$(\nabla^2 + k^2)\phi = 0, \quad (\nabla^2 + k^2)\psi = 0,$$

(6.96)

and thus can be expanded in spherical harmonics. (Note that Laplacian $\nabla^2$ and operator $r \times \nabla$ commute.) The magnetic field in terms of the scalar functions is given by

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times (r \times \nabla \phi) - r \times \nabla \nabla^2 \psi$$

$$= \nabla \times (r \times \nabla \phi) + k^2 r \times \nabla \psi,$$

(6.97)

where use is made of

$$\nabla \cdot (r \times \nabla \psi) = 0.$$

(6.98)

The second term in the RHS is evidently transverse to $r$, that is, the scalar field $\psi$ is a generating function for transverse magnetic (TM) modes.

The scalar field $\phi$ generates transverse electric (TE) modes, since in the current free region, the electric field is given by

$$-\frac{i\omega}{c^2} \mathbf{E} = \nabla \times \mathbf{B}$$

$$= -r \times \nabla \nabla^2 \phi - \nabla \times (r \times \nabla \nabla^2 \psi)$$

$$= k^2 [r \times \nabla \phi + \nabla \times (r \times \nabla \psi)].$$

(6.99)

The TE and TM modes are the desired base vectors because vector functions $r \times \nabla \phi$ and $\nabla \times (r \times \nabla \psi)$ are normal to each other. To prove this, let us assume

$$\phi(r) = \sum A_{lm} g_l(r) Y_{lm}(\theta, \phi), \quad \psi(r) = \sum B_{lm} g_l(r) Y_{lm}(\theta, \phi).$$

(6.100)

Expanding

$$\nabla \times [g_l(r) r \times \nabla Y_{lm}] = \nabla g_l(r) \times [r \times \nabla Y_{lm}(\theta, \phi)] + g_l(r) \nabla \times (r \times \nabla Y_{lm})$$

$$= \nabla g_l(r) \times [r \times \nabla Y_{lm}(\theta, \phi)] - g_l(r) \frac{l(l+1)}{r} Y_{lm} \mathbf{e}_r - g_l(r) \nabla Y_{lm}$$

$$= -g_l(r) \frac{l(l+1)}{r} Y_{lm} \mathbf{e}_r - \frac{d}{dr} [rg_l(r)] \nabla Y_{lm},$$

(6.101)

we see that the two vectors $r \times \nabla Y_{lm}$ and $\nabla \times [g_l(r) r \times \nabla Y_{lm}]$ are indeed normal to each other, since $(r \times \nabla Y_{lm}) \cdot r = 0$ and $(r \times \nabla Y_{lm}) \cdot \nabla Y_{lm} = 0$. (The product

$$r (r \times \nabla Y_{lm}) \cdot \nabla Y_{lm}^* = -2im \frac{d}{d\theta} |Y_{lm}|^2,$$
evidently does not vanish. Interpret this.)

The operator

$$\mathbf{r} \times \nabla Y_{lm}(\theta, \phi) = i \mathbf{L} Y_{lm}(\theta, \phi), \quad \mathbf{L} = -i \mathbf{r} \times \nabla,$$

is the angular momentum operator and its explicit form is

$$\mathbf{r} \times \nabla Y_{lm}(\theta, \phi) = -e_\theta \frac{1}{\sin \theta} \frac{\partial Y_{lm}}{\partial \phi} + e_\phi \frac{\partial Y_{lm}}{\partial \theta}.$$  \hspace{1cm} (6.103)

Noting

$$\frac{\partial e_\phi}{\partial \phi} = -\sin \theta e_r - \cos \theta e_\theta,$$

we can readily show that

$$(\mathbf{r} \times \nabla) \cdot (\mathbf{r} \times \nabla Y_{lm}) = (\mathbf{r} \times \nabla)^2 Y_{lm} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y_{lm}}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y_{lm}}{\partial \phi^2} = -l(l+1)Y_{lm},$$  \hspace{1cm} (6.104)

or

$$L^2 Y_{lm} = l(l+1)Y_{lm}.  \hspace{1cm} (6.105)$$

Other useful properties are:

$$\nabla = e_r \frac{\partial}{\partial r} - \frac{i}{r^2} \mathbf{r} \times \mathbf{L},$$

$$\int (\mathbf{r} \times \nabla Y_{lm}) \cdot (\mathbf{r} \times \nabla Y_{lm}) d\Omega = l(l+1)\delta_{lm} \delta_{mm'},$$  \hspace{1cm} (6.107)

$$\nabla \cdot (\mathbf{r} \times \nabla Y_{lm}) = 0,$$  \hspace{1cm} (6.108)

$$L^2 Y_{lm} = \mathbf{L} L^2 Y_{lm}, \quad \nabla^2 L^2 Y_{lm} = \mathbf{L} \nabla^2 Y_{lm}, \quad \mathbf{L} \times \mathbf{L} Y_{lm} = i \mathbf{L} Y_{lm};$$

$$L_x = -i \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right), \quad L_y = -i \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right), \quad L_z = -i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -i \frac{\partial}{\partial \phi},$$  \hspace{1cm} (6.110)

$$L_{\pm} = L_x \pm i L_y = e^{\pm i \phi} \left( \pm \frac{\partial}{\partial \theta} \mp i \cot \theta \frac{\partial}{\partial \phi} \right),$$  \hspace{1cm} (6.111)

$$L_+ Y_{lm}(\theta, \phi) = \sqrt{l(l+1) - m(m+1)} Y_{l,m+1}(\theta, \phi),$$

$$L_- Y_{lm}(\theta, \phi) = \sqrt{l(l+1) - m(m-1)} Y_{l,m-1}(\theta, \phi),$$  \hspace{1cm} (6.112)

$$L_z Y_{lm}(\theta, \phi) = m Y_{lm}(\theta, \phi).$$  \hspace{1cm} (6.113)

We are now ready to expand the raw form of the vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') e^{i k|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} dV'$$

$$= \mu_0 i k \sum_{l=0}^{\infty} \sum_{m=-l}^{l} h_l^{(1)}(kr) Y_{lm}(\theta, \phi) \int j_l(kr') Y_{lm}^*(\theta', \phi') \mathbf{J}(\mathbf{r}') dV',$$  \hspace{1cm} (6.115)
in terms of the TE and TM base vectors. For TE modes, the vector potential should be in the form

$$\mathbf{A}^{\text{TE}} = \mathbf{r} \times \nabla \phi.$$  \hspace{1cm} (6.116)

Therefore, it is necessary to identify TE component of the source current $\mathbf{J}(\mathbf{r'})$, which can be effected by

$$J_{lm}^{\text{TE}}(\mathbf{r}) = j_l(kr)(\mathbf{r} \times \nabla Y_{lm}^*) \cdot \mathbf{J}(\mathbf{r}). \hspace{1cm} (6.117)$$

The TE component of the vector potential can thus be assumed in the form

$$\mathbf{A}^{\text{TE}}(\mathbf{r}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm} h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi) \int j_l(kr') [\mathbf{r}' \times \nabla Y_{lm}^*(\theta', \phi')] \cdot \mathbf{J}(\mathbf{r'}) dV', \hspace{1cm} (6.118)$$

where $a_{lm}$ is expansion coefficient. Note that in the summation over $l$, the monopole term $l = 0$ vanishes because $\nabla Y_{00} = 0$. It is convenient to choose $a_{lm}$ so that

$$a_{lm} \int (\mathbf{r} \times \nabla Y_{lm}) \cdot (\mathbf{r} \times \nabla Y_{lm}^*) d\Omega = 1,$$

or

$$a_{lm} = \frac{1}{l(l+1)}. \hspace{1cm} (6.119)$$

Corresponding TE base vector is

$$\mathbf{u}^{\text{TE}}_{lm}(\mathbf{r}) = \frac{1}{\sqrt{l(l+1)}} \mathbf{r} \times \nabla Y_{lm}(\theta, \phi), \hspace{1cm} (6.120)$$

which is normalized as

$$\int \mathbf{u}^{\text{TE}}_{lm}(\mathbf{r}) \cdot [\mathbf{u}^{\text{TE}}_{lm}(\mathbf{r})]^* d\Omega = 1. \hspace{1cm} (6.121)$$

The TE component of the vector potential is therefore given by

$$\mathbf{A}^{\text{TE}}(\mathbf{r}) = \mu_0 i k \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{1}{l(l+1)} h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi) \int j_l(kr') [\mathbf{r}' \times \nabla Y_{lm}^*(\theta', \phi')] \cdot \mathbf{J}(\mathbf{r'}) dV' \hspace{1cm} (6.122)$$

The current density $\mathbf{J}(\mathbf{r'})$ may be separated into the conduction current $\mathbf{J}_c$ and magnetization current $\mathbf{J}_m = \nabla \times \mathbf{M}(\mathbf{r})$,

$$\mathbf{J}(\mathbf{r}) = \mathbf{J}_c(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r}), \hspace{1cm} (6.123)$$

where $\mathbf{M}(\mathbf{r})$ is the magnetic dipole moment density.

For the TM modes, it is convenient to let the TM base vector have the same dimensions as TE
base vector. For this purpose, we rewrite the decomposition of the vector potential in the form

\[ A = r \times \nabla \phi + \frac{1}{k} \nabla \times (r \times \nabla \psi). \]  

(6.125)

Following the same procedure developed for the TE modes, we find

\[ A_{TM}^r(r) = i \mu_0 \frac{1}{r} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{1}{l(l+1)} \nabla \times [h_l^{(1)}(kr)r \times \nabla Y_{lm}(\theta, \phi)] \int \nabla' \times [j_l(kr')r' \times \nabla' Y_{lm}^*(\theta', \phi')] \cdot J(r')dV'. \]  

(6.126)

Corresponding TM base vector in the radiation zone is

\[ u_{lm}^{TM}(r) = \frac{1}{\sqrt{l(l+1)}} e_r \times [r \times \nabla Y_{lm}(\theta, \phi)], \]  

(6.127)

which is also normalized as

\[ \int u_{lm}^{TM}(r) \cdot [u_{lm}^{TM}(r)]^* d\Omega = 1. \]  

(6.128)

Note that in Eq. (6.126),

\[ \nabla' \times [j_l(kr')r' \times \nabla' Y_{lm}^*(\theta', \phi')] \cdot J(r') \]  

(6.129)

is the TM component of the current density.

For a small source \( kr' \ll 1 \), the spherical Bessel function \( j_l(kr') \) can be approximated by

\[ j_l(kr') \simeq \frac{(kr')^l}{(2l+1)!!}. \]  

(6.130)

Then, the moment for the TE modes becomes (the primes are omitted for brevity)

\[ \int [j_l(kr)r \times \nabla Y_{lm}(\theta, \phi)] \cdot J_c(r)dV \]

\[ = - \frac{k^l}{(2l+1)!!} \int r^l(r \times J_c) \cdot \nabla Y_{lm}^* dV \]

\[ = - \frac{k^l}{(2l+1)!!} \int \nabla(r^lY_{lm}^*) \cdot (r \times J_c)dV. \]  

(6.131)

This contains the current circulation \( r \times J_c \) which produces magnetic multipole moments. The magnetic multipole moment is in general defined by

\[ m_{lm} = \frac{1}{l+1} \int \nabla(r^lY_{lm}^*) \cdot (r \times J_c)dV. \]  

(6.132)

In terms of the magnetic moment \( m_{lm} \), the integral may be rewritten as

\[ \int [j_l(kr)r \times \nabla Y_{lm}(\theta, \phi)] \cdot J_c(r)dV = \frac{(l+1)k^l}{(2l+1)!!} m_{lm}. \]  

(6.133)
The contribution from the magnetization current $\nabla \times \mathbf{M}$ can be calculated in a similar manner,

$$m'_{lm} = -\int r^l Y_{lm}^* \nabla \cdot \mathbf{M}dV,$$

and the $(l,m)$ component of the TE vector potential is given by

$$A_{lm}^{TE}(r) = -\mu_0 i \frac{k^{l+1}}{l(2l+1)!!} (m_{lm} + m'_{lm}) h_{l}^{(1)}(kr) r \times \nabla Y_{lm}(\theta, \phi).$$

(6.134)

The radiation power associated with the vector potential can readily be calculated as follows:

$$P_{lm}^{TE} = r^2 \int Z |\mathbf{H}_{lm}^{TE}|^2 d\Omega,$$

(6.135)

where the magnetic field $\mathbf{H}_{lm}^{TE}(r)$ in the radiation region $kr \gg 1$ is approximately given by

$$\mathbf{H}_{lm}^{TE}(r) \simeq \frac{1}{\mu_0} i \mathbf{k} \times A_{lm}^{TE}$$

$$= k^{l+1}(-i)^l (m_{lm} + m'_{lm}) \frac{1}{l(2l+1)!!} \frac{1}{r} e^{ikr-\omega t} \mathbf{n} \times [r \times \nabla Y_{lm}(\theta, \phi)].$$

(6.136)

Then

$$|\mathbf{H}_{lm}^{TE}|^2 = \frac{k^{2(l+1)}}{l(2l+1)!!^2} |m_{lm} + m'_{lm}|^2 |r \times \nabla Y_{lm}(\theta, \phi)|^2.$$

(6.137)

Noting

$$\int |r \times \nabla Y_{lm}(\theta, \phi)|^2 d\Omega = l(l+1),$$

(6.138)

we find the radiation power in the form

$$P_{lm}^{TE} = \sqrt{\frac{\mu_0 l + 1}{\varepsilon_0}} \frac{k^{2(l+1)}}{l} \frac{1}{[(2l+1)!!]^2} |m_{lm} + m'_{lm}|^2$$

$$= \frac{1}{4\pi \varepsilon_0} \frac{4\pi}{4\pi c} \frac{l + 1}{l} \frac{k^{2(l+1)}}{[(2l+1)!!]^2} |m_{lm} + m'_{lm}|^2$$

(6.139)

$$= \frac{1}{4\pi \varepsilon_0} \frac{4\pi}{4\pi c} \frac{l + 1}{l} \frac{k^{2(l+1)}}{[(2l+1)!!]^2} |m_{lm}^{G} + m_{lm}^{G}|^2,$$

(6.140)

where the latter expression facilitates comparison with the formulation in the Gaussian unit system in which the magnetic multipole moment is defined by

$$m_{lm}^{G} = \frac{1}{c l + 1} \int \nabla (r^l Y_{lm}^*) \cdot (r \times \mathbf{J})dV,$$

(6.141)

Electric multipoles radiate TM modes. This may be seen from the TM component of the current
density,

\[
\int \nabla \times [j_i(kr) \mathbf{r} \times \nabla Y_{lm}^*(\theta, \phi)] \cdot \mathbf{J}(r) dV
\]

\[
= \int j_i(kr) [\mathbf{r} \times \nabla Y_{lm}^*(\theta, \phi)] \cdot \nabla \times \mathbf{J} dV
\]

\[
= - \int \nabla \times [j_i(kr) r Y_{lm}^*] \cdot \nabla \times \mathbf{J} dV
\]

\[
= - \int j_i(kr) r Y_{lm}^* \cdot [\nabla \times (\nabla \times \mathbf{J})] dV
\]

\[
= - \int j_i(kr) r Y_{lm}^* \cdot [\nabla (\nabla \cdot \mathbf{J}) - \nabla^2 \mathbf{J}] dV
\]

\[
= \int Y_{lm}^* \frac{d}{dr} [r j_i(kr)] \nabla \cdot \mathbf{J} dV - k^2 \int j_i(kr) Y_{lm}^* r \cdot \mathbf{J} dV,
\]  

(6.142)

where it is noted that the function \( j_i(kr) Y_{lm}^*(\theta, \phi) \) satisfies the Helmholtz equation,

\[
(\nabla^2 + k^2) j_i(kr) Y_{lm}^*(\theta, \phi) = 0.
\]

Again, if the current is separated into the condution current and magnetization currents, \( \mathbf{J} = \mathbf{J}_c + \nabla \times \mathbf{M} \), Eq. (6.142) can be rewritten as

\[
\int Y_{lm}^* \frac{d}{dr} [r j_i(kr)] \nabla \cdot \mathbf{J}_c dV - k^2 \int j_i(kr) Y_{lm}^* r \cdot \mathbf{J}_c dV
\]

\[
= \int Y_{lm}^* \frac{d}{dr} [r j_i(kr)] \nabla \cdot \mathbf{J}_c dV - k^2 \int j_i(kr) Y_{lm}^* r \cdot \mathbf{J}_c dV + k^2 \int j_i(kr) Y_{lm}^* \nabla \cdot (\mathbf{r} \times \mathbf{M}) dV.
\]  

(6.143)

For a small source \( kr' \ll 1 \), the first term in the RHS containing electric multipole moment is dominant. Recalling

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_c = 0,
\]

and

\[
j_i(kr) \simeq \frac{(kr)^l}{(2l + 1)!!} \quad \text{for} \; kr \ll 1,
\]

we may approximate Eq. (6.143) by

\[
\int Y_{lm}^* \frac{d}{dr} [r j_i(kr)] \nabla \cdot \mathbf{J}_c dV - k^2 \int j_i(kr) Y_{lm}^* r \cdot \mathbf{J}_c dV + k^2 \int j_i(kr) Y_{lm}^* \nabla \cdot (\mathbf{r} \times \mathbf{M}) dV
\]

\[
\simeq \frac{i c}{2l + 1} \frac{(kr)^{l+1}}{(2l + 1)!!} (q_{lm} + q_{lm}'),
\]  

(6.144)

where the second term is of order \( ka \ll 1 \) (\( a \) being the source size) and thus ignorable, \( q_{lm} \) is the
electric multipole moment defined earlier,

\[ q_{lm} = \int r^l Y_{lm}^*(\theta, \phi) \rho(r) dV; \quad (6.145) \]

and \( q'_{lm} \) is an effective electric quadrupole moment created by the magnetic dipole moment density,

\[ q'_{lm} = -\frac{ik}{(l+1)c} \int r^l Y_{lm}^*(\theta, \phi) \nabla \cdot (r \times \mathbf{M}) dV. \quad (6.146) \]

The radiation vector potential of TM mode is thus given by

\[ \mathbf{A}^{TM}_{lm}(r) = -\mu_0 c \frac{k^l}{l(2l+1)!!} q_{lm} \nabla \times \left[ h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi) \right], \quad (6.147) \]

and corresponding radiation power is

\[ P^{TM}_{lm} = \mu_0 c^3 \frac{k^{2(l+1)}}{[(2l+1)!!]^2} \frac{l+1}{l} |q_{lm} + q'_{lm}|^2 \]
\[ = \frac{1}{4\pi\varepsilon_0} \frac{4\pi c}{[(2l+1)!!]^2} \frac{l+1}{l} |q_{lm} + q'_{lm}|^2. \quad (6.148) \]

### 6.6 Radiation of Angular Momentum

In chapter 5, radiation of angular momentum from a circulating charge was discussed. In general, if a charge undergoes circular motion, it radiates (loses) angular momentum as well as energy, for the two quantities are intimately related. In this section, a general formulation for radiation of angular momentum is given. Since the momentum flux density associated with electromagnetic fields is given by

\[ \frac{1}{c} \mathbf{E} \times \mathbf{H}^*, \quad (6.149) \]

The angular momentum flux density associated with electromagnetic fields is

\[ \frac{1}{c} \mathbf{r} \times (\mathbf{E} \times \mathbf{H}^*). \quad (6.150) \]

In the radiation zone, this must be proportional to \( 1/r^2 \) if radiation of angular momentum accompanies radiation of energy. Therefore, terms proportional to \( 1/r^3 \) in the Poynting flux, which have been ignored in the calculation of radiation power, should be retained.

As a concrete example, let us consider TM modes due to electric multipoles. The vector potential of TM mode is

\[ \mathbf{A}^{TM}_{lm}(r) = -\mu_0 c \frac{k^l}{l(2l+1)!!} q_{lm} \nabla \times \left[ h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi) \right]. \quad (6.151) \]
Corresponding magnetic and electric fields are

\[ \mathbf{H}_{lm}^{\mathrm{TM}}(\mathbf{r}) = \frac{1}{\mu_0} \nabla \times \mathbf{A}_{lm}^{\mathrm{TM}} = -\frac{c}{\omega_0} \frac{k^{l+2}}{l(2l+1)!!} q_{lm} h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi), \]  

(6.152)

and

\[ \mathbf{E}_{lm}^{\mathrm{TM}}(\mathbf{r}) = -\frac{i}{\omega_0} \nabla \times \mathbf{H}_{lm}^{\mathrm{TM}} = -\frac{ic}{\omega_0} \frac{k^{l+2}}{l(2l+1)!!} q_{lm} \nabla \times [h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi)], \]  

(6.153)

where use is made of the identity

\[ \nabla \cdot [h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi)] = 0. \]

The expressions for the electromagnetic fields are exact. The radiation flux of angular momentum associated with an \((l, m)\) mode is

\[
\begin{align*}
\frac{1}{c} \mathbf{r} \times (\mathbf{E}_{lm}^{\mathrm{TM}} \times \mathbf{H}_{lm}^{*\mathrm{TM}}) &= \frac{ic}{\omega_0} \frac{k^{2(l+2)}}{l(2l+1)!!} q_{lm}^2 |\mathbf{r}| \left[ (\nabla \times (h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi)) \right] \times [h_l^{(1)*}(kr) \mathbf{r} \times \nabla Y_{lm}^*(\theta, \phi)] \\
&= -\frac{ic}{\omega_0} \frac{k^{2(l+2)}}{l(2l+1)!!} q_{lm}^2 \left( \mathbf{r} \cdot \nabla \times [h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi)] \right) h_l^{(1)*}(kr) \mathbf{r} \times \nabla Y_{lm}^*(\theta, \phi) \\
&= \frac{ic}{\omega_0} \frac{k^{2(l+2)}}{l(2l+1)!!} q_{lm}^2 l(l+1) |h_l^{(1)}(kr)|^2 Y_{lm}(\theta, \phi) \mathbf{r} \times \nabla Y_{lm}^*(\theta, \phi). \] 

(6.154)

In radiation zone \(kr \gg 1\), this reduces to

\[
\frac{ic}{\omega_0} \frac{k^{2(l+1)}}{l(2l+1)!!} \left( \frac{l+1}{l} \right) q_{lm}^2 Y_{lm}(\theta, \phi) \mathbf{r} \times \nabla Y_{lm}^*(\theta, \phi) \frac{1}{r^2},
\]

(6.155)

and the integration over the solid angle yields the rate of angular momentum radiation,

\[
\frac{d L_{lm}}{dt} = \frac{1}{4\pi \varepsilon_0} 4\pi c \frac{k^{2(l+1)}}{l(2l+1)!!} \left( \frac{l+1}{l} \right) q_{lm}^2 \frac{m}{\omega} \mathbf{e}_z,
\]

(6.156)

where use is made of

\[
\begin{align*}
\int Y_{lm}(\theta, \phi) \mathbf{r} \times \nabla Y_{lm}^*(\theta, \phi) d\Omega &= -im \int Y_{lm}(\theta, \phi) Y_{lm}^*(\theta, \phi) d\Omega \mathbf{e}_z \\
&= -im \mathbf{e}_z,
\end{align*}
\]

(6.157)
Comparing with the radiation power given in Eq. (6.148), we see that they are related through a simple ratio,

\[ \frac{\dot{L}_z}{\dot{P}} = \frac{m}{\omega}. \]  

(6.158)

Evidently, this is not consistent with the familiar quantum mechanical result,

\[ -\frac{\dot{L}_z}{\dot{P}} = \frac{h\sqrt{l(l+1)}}{h\omega} = \sqrt{l(l+1)}/\omega. \]  

(6.159)

The discrepancy is not surprising because the quantum mechanical formula is for a single photon while the result obtained in classical electrodynamics pertains to (infinitely) many photons. Quantum mechanical formula for \( N \) photons is

\[ -\frac{\dot{L}_z}{\dot{P}} = \frac{\sqrt{N^2m^2 + Nl(l+1) - m^2}}{N\omega}, \]  

(6.160)

and the classical result can be recovered in the limit of \( N \gg 1 \).

### 6.7 Some Examples

**Example 1 Spherical Dipole Antenna**

![Figure 6-1: Spherical antenna.](image)

Two ideally conducting hemispheres of radius \( a \) insulated from each other at the equator with a small gap are connected to an oscillating voltage source \( V_0 e^{-i\omega t} \). In this case, the surface current flows in the \( \theta \) direction and creates an azimuthal magnetic fields \( B_\phi \). The resultant radiation fields
are thus TM (Transverse Magnetic) and we may assume the magnetic field in the form

$$B = r \times \nabla \psi,$$  \hspace{1cm} (6.161)

where $\psi$ is a scalar function. Since the system is axially symmetric without dependence on the azimuthal angle $\phi$, the scalar function $\psi$ can be expanded in spherical harmonics as

$$\psi(r, \theta) = \sum_{l \geq 1} a_l h_l^{(1)}(kr) P_l(\cos \theta), \quad r > a,$$  \hspace{1cm} (6.162)

which yields the magnetic field

$$B = r \times \nabla \psi = - \sum_l a_l h_l^{(1)}(kr) P_l^1(\cos \theta) e_\phi, \quad r > a.$$  \hspace{1cm} (6.163)

Note that

$$\frac{d}{d \theta} P_l(\cos \theta) = - \sin \theta \frac{d}{d(\cos \theta)} P_l(\cos \theta) = - P_l^1(\cos \theta).$$  \hspace{1cm} (6.164)

The electric field associated with the magnetic field can be found from the Maxwell’s equation,

$$\varepsilon_0 \mu_0 \frac{\partial E}{\partial t} = \nabla \times B,$$  \hspace{1cm} (6.165)

which yields

$$E(r) = \frac{ic^2}{\omega} \left[ \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta B_\phi) e_r - \frac{1}{r} \frac{\partial}{\partial r} (r B_\phi) e_\theta \right].$$  \hspace{1cm} (6.166)

Substituting $B_\phi(r)$ in Eq.(6.163) yields the transverse component of the electric field,

$$E_\theta(r, \theta) = \frac{ic^2}{\omega} \sum_l a_l \left[ \frac{l}{r} h_l^{(1)}(kr) - k h_{l-1}^{(1)}(kr) \right] P_l^1(\cos \theta),$$  \hspace{1cm} (6.167)

where the following recurrence formula for the spherical Bessel functions has been used,

$$\frac{d}{dx} h_l^{(1)}(x) = \frac{l}{x} h_l^{(1)}(x) - h_{l+1}^{(1)}(x) = h_{l-1}^{(1)}(x) - \frac{l + 1}{x} h_l^{(1)}(x).$$  \hspace{1cm} (6.168)

Since the sphere is ideally conducting, the $\theta$ component of the electric field should vanish except at the gap. Therefore, at the sphere surface $r = a$, the electric field may be approximated by

$$E_\theta(r = a, \theta) = \frac{V_0}{a} \delta \left( \theta - \frac{\pi}{2} \right),$$  \hspace{1cm} (6.169)
which satisfies the obvious requirement that

$$a \int_0^\pi E_\theta(a, \theta) d\theta = V_0. \quad (6.170)$$

Then, for arbitrary angle $\theta$, the following relation must hold

$$\frac{V_0}{a} \delta \left( \theta - \frac{\pi}{2} \right) = \frac{ic^2}{\omega} \sum_l a_l \left[ \frac{l}{r} h_l^{(1)}(ka) - kh_l^{(1)}(ka) \right] P_l^1(\cos \theta). \quad (6.171)$$

Multiplying both sides by $P_l^1(\cos \theta) \sin \theta$ and integrating over $\theta$, we readily find the expansion coefficient $a_l$,

$$a_l = \frac{\omega}{ic^2} \frac{2l+1}{2l(l+1)} \frac{P_l^1(0)}{lh_l^{(1)}(ka) - kah_l^{(1)}(ka)} V_0, \quad (6.172)$$

where

$$P_l^1(0) = \begin{cases} (-1)^{l-1} \frac{l!!}{(l-1)!!}, & l \text{ odd} \\ 0, & l \text{ even} \end{cases} \quad (6.173)$$

and use has been made of the integral

$$\int_{-1}^1 P_l^1(x) P_l^1(x) dx = \frac{2}{2l+1} l(l+1) \delta_{l0}. \quad (6.174)$$

The disappearance of even harmonics is expected because of the up-down anti-symmetry of the problem. (Note that $P_l^1(x)$ is an odd function of $x$ if $l$ is odd.)

The surface current density $J_s$ on the sphere surface can be found from the boundary condition for the magnetic field,

$$J_s = n \times H_\phi(r = a, \theta)$$

$$= -H_\phi(a, \theta)e_\theta$$

$$= -\frac{1}{\mu_0} \sum_{l \geq 1} a_l h_l^{(1)}(ka) P_l^1(\cos \theta) e_\theta. \quad (6.175)$$

The current at the gap $\theta = \pi/2$ where the voltage $V_0$ appears is

$$I = 2\pi a J_s(\theta = \pi/2)$$

$$= \frac{2\pi \alpha}{\mu_0} \sum_{l \geq 1} a_l h_l^{(1)}(ka) P_l^1(0)$$

$$= \frac{i 2\pi \omega}{\mu_0 c^2} \sum_{l \geq 1} \frac{2l+1}{2l(l+1)} \frac{h_l^{(1)}(ka)}{lh_l^{(1)}(ka) - kah_l^{(1)}(ka)} \left[ P_l^1(0) \right]^2 V_0. \quad (6.176)$$
This defines the radiation admittance of the spherical dipole antenna,

\[
Y = \frac{I}{V_0} = \frac{i \pi \sqrt{\frac{\varepsilon_0}{\mu_0}}}{\sum_{l \geq 1} \frac{2l + 1}{l(l + 1)} \frac{kah_l^{(1)}(ka)}{lh_l^{(1)}(ka) - kah_{l-1}^{(1)}(ka)}} [P_l^1(0)]^2.
\]  

(6.177)

A spherical antenna has limited practical applications. However, the procedure developed in this example is applicable to more practical problems such as half wavelength dipole antennas. A thin rod antenna can be analyzed rigorously using the prolate spheroidal coordinates.

**Example 2 Circular Loop Antenna**

![Circular Loop Antenna Diagram](image)

*Figure 6-2: Circular loop antenna. ka is arbitrary.*

Let us consider a circular oscillating current \(I_0 e^{-i\omega t}\) carried by a thin conductor ring of radius \(a\). The current density may be written as

\[
\mathbf{J} = I_0 e^{-i\omega t} \frac{\delta(r - a)}{a} \delta \left(\theta - \frac{\pi}{2}\right) \mathbf{e}_\phi.
\]  

(6.178)

As shown in Chapter 5, if the ring radius is much smaller than the wavelength \(ka = \frac{\omega}{c} a \ll 1\), the problem reduces to radiation by a magnetic dipole with a dipole moment

\[
m_z(t) = \pi a^2 I_0 e^{-i\omega t}.
\]

For an arbitrary value of \(\omega a/c\), higher order multipole fields must be retained. Since the loop current radiates TE modes, the TE vector potential in Eq. (6.118) can be directly applied with a result

\[
\mathbf{A}^{TE}(r) = \mu_0 k \sum_l \frac{1}{l(l + 1)} h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{l0}(\theta, \phi) M_l,
\]  

(6.179)
where

\[ M_l = \int j_l(\kappa r') [r' \times \nabla Y_{l0}^*(\theta')] \cdot \mathbf{J}(r') dV', \quad (6.180) \]

is the TE component of the current density. Note that because of the axial symmetry, only \( m = 0 \) components are nonvanishing. Since

\[
r' \times \nabla Y_{l0}^*(\theta') = \frac{dY_{l0}}{d\theta'} e_{\phi'}
\]

\[
= -\sqrt{\frac{2l + 1}{4\pi}} P_l^1(\cos \theta') e_{\phi'},
\]

we find

\[
M_l = -2\pi a I_0 \sqrt{\frac{2l + 1}{4\pi}} j_l(ka) P_l^1(0).
\]  

The vector potential is therefore given by

\[
A_{\text{TE}}^\text{TE}(r) = -i\mu_0 I_0 ka \sum_l \frac{2l + 1}{2l(l + 1)} j_l(ka) h_l^{(1)}(\kappa r) P_l^1(0) P_l^1(\cos \theta) e_\phi.
\]  

(6.182)

and the magnetic field in the radiation zone \( kr \gg 1 \) is

\[
H(r) \simeq \frac{1}{\mu_0} \mathbf{\mathbf{k}} \times A_{\text{TE}}
\]

\[
= -ak I_0 \frac{e^{i\kappa r}}{r} \sum_{l \geq 1} (-i)^{l+1} \frac{2l + 1}{2l(l + 1)} j_l(ka) P_l^1(0) P_l^1(\cos \theta) e_\theta.
\]  

(6.183)

The radiation power associated with the \( l \)-th harmonic mode is

\[
P^l = r^2 Z_0 \int |H_l|^2 d\Omega
\]

\[
= Z_0 (ak I_0)^2 \left( \frac{2l + 1}{2l(l + 1)} \right)^2 \left[ j_l(ka) P_l^1(0) \right]^2 2\pi \int_0^\pi \left[ P_l^1(\cos \theta) \right]^2 \sin \theta d\theta
\]

\[
= \pi Z_0 (ak I_0)^2 \left[ j_l(ka) P_l^1(0) \right]^2 \frac{2l + 1}{l(l + 1)}, \quad l = 1, 3, 5, \ldots
\]  

(6.184)

In the long wavelength limit \( ka \ll 1 \), the dipole term \( l = 1 \) is dominant, and we recover

\[
P = \pi Z_0 (ak I_0)^2 \frac{(ka)^2}{6}
\]

\[
= \frac{1}{4\pi \varepsilon_0} \frac{2\omega^4 m^2}{3c^5},
\]  

(6.185)

where \( m = \pi a^2 I_0 \) is the magnetic dipole moment of the ring current.
6.8 Spherical Harmonic Expansion of a Plane Wave

A plane wave of planar polarization is characterized by constant amplitudes of electromagnetic fields and unidirectional propagation assumed here in the \( z \)-direction,

\[
E_0 e^{ikz} \mathbf{e}_x, \quad H_0 e^{ikz} \mathbf{e}_y.
\]  

The purpose of this section is to decompose the plane wave into spherical harmonics. Once achieved, such a representation will be very useful in analyzing scattering of electromagnetic wave by an object placed in the plane wave. Scattered waves are evidently no longer plane waves but they consist of many (often infinitely many) spherical harmonic waves. A key observation to be made is that there is one-to-one correspondence between spherical harmonic component in the incident plane wave and the one in the scattered wave. This is because a scattered harmonic mode characterized by mode numbers \((l, m)\) can only be produced by a mode in the incident wave having exactly the same angular dependence. For example, TM \((l, m)\) harmonic component in the scattered wave is generated by the same TM harmonic component contained in the incident wave.

![Figure 6-3: Scattered electromagnetic waves consist of spherical waves. The incident plane wave can be decomposed into spherical waves to facilitate analysis.](image)

The first step is to expand the propagator function \( e^{ikz} = e^{ikr \cos \theta} \) in terms of spherical harmonics. This can be effected by considering the limiting case of the scalar Green’s function,

\[
G(r, r') = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|} = \frac{i k}{\mathbf{r} \cdot \mathbf{r}'} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} j_l(kr) h_l^{(1)}(kr') Y_l^m(\theta, \phi) Y_l^m(\theta', \phi'), \quad r' > r. \]  

(6.187)

In the limit of \( r' \to \infty \), that is, when the source is far away, \( h_l^{(1)}(kr') \) approaches

\[
h_l^{(1)}(kr') \to (-i)^{l+1} \frac{e^{ikr'}}{kr'}. \]  

(6.188)
Noting also
\[ k|\mathbf{r}' - \mathbf{r}| \to kr' - \mathbf{k} \cdot \mathbf{r}, \quad r' \gg r, \]
we find
\[ e^{-ik\mathbf{r}} = 4\pi i \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^{l+1} j_l(kr)Y_{lm}^{*}(\theta, \phi)Y_{lm}(\theta', \phi'), \tag{6.189} \]
where the angles \((\theta', \phi')\) are those of the wavevector \(\mathbf{k} = (k, \theta', \phi').\) Since we have assumed a plane wave propagating in the \(z\)-direction, it follows that \(\theta' = 0\) and \(\phi'\) becomes irrelevant. Noting
\[ P_l^m(1) = \delta_{m0}, \]
and taking the complex conjugate of Eq. (6.189) yields the following identity,
\[ e^{ikz} = e^{ikr\cos \theta} = \sum_{l=0}^{\infty} i^l (2l + 1) j_l(kr)P_l(\cos \theta). \tag{6.190} \]

The electric field assumed to be in the \(x\)-direction can be converted into components in the spherical coordinates as
\[ E_0 \mathbf{e}_x = E_0 \nabla(r \sin \theta \cos \phi) = E_0(\sin \theta \cos \phi \mathbf{e}_r + \cos \theta \cos \phi \mathbf{e}_\theta - \sin \phi \mathbf{e}_\phi). \tag{6.191} \]

This field can be represented as a sum of TE and TM modes. To effect such a representation, we assume the following expansion,
\[ E_0 \mathbf{e}_x = \sum_{lm} a_{lm} j_l(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi) + \frac{1}{k} \sum_{lm} b_{lm} \mathbf{r} \times [j_l(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi)], \]
or substituting the expansion in Eq. (6.190) for \(e^{ikr\cos \theta}\) in the LHS,
\[ E_0 \sum_{l=0}^{\infty} i^l (2l + 1) j_l(kr)P_l(\cos \theta)(\sin \theta \cos \phi \mathbf{e}_r + \cos \theta \cos \phi \mathbf{e}_\theta - \sin \phi \mathbf{e}_\phi) \]
\[ = \sum_{l=1, m=1}^{\infty} \sum_{l=1, m=1}^{l} a_{lm} j_l(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi) + \frac{1}{k} \sum_{lm} b_{lm} \mathbf{r} \times [j_l(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi)], \tag{6.192} \]
where \(a_{lm}\) and \(b_{lm}\) are expansion coefficients for TE and TM modes, respectively. The TE coefficient \(a_{lm}\) can be determined by multiplying both sides by \(\mathbf{r} \times \nabla Y_{lm}^{*}(\theta, \phi)\) and integrating the result over the solid angle. Exploiting the following orthogonality relationships,
\[ \int [\mathbf{r} \times \nabla Y_{lm}(\theta, \phi)] \cdot [\mathbf{r} \times \nabla Y_{lm'}^{*}(\theta, \phi)] d\Omega = l(l + 1)\delta_{ll'}\delta_{mm'}, \]
\[ [\mathbf{r} \times \nabla Y_{lm}^{*}(\theta, \phi)] \cdot \nabla \times [j_l(kr) \mathbf{r} \times \nabla Y_{lm}(\theta, \phi)] = 0, \]
we find
\[ a_{lm} = E_0 \frac{i(2l+1)}{(l+1)} \int P_l(\cos \theta)(\cos \theta \cos \phi e_\theta - \sin \phi e_\phi) \cdot (r \times \nabla Y^*_{lm}) d\Omega. \] (6.193)

The presence of \( \cos \phi \) and \( \sin \phi \) functions in the integral makes only \( m = \pm 1 \) components nonvanishing. For \( m = +1 \), we find
\[ a_{l,1} = E_0 \frac{i(2l+1)}{(l+1)} \int P_l(\cos \theta)(\cos \theta \cos \phi e_\theta - \sin \phi e_\phi) \cdot (r \times \nabla Y^*_{l,1}) d\Omega, \] (6.194)

where
\[ Y^*_{l,1}(\theta, \phi) = -Y_{l,-1}(\theta, \phi) \]
\[ = -\sqrt{\frac{2l+1}{4\pi l(l+1)}} P^1_l e^{-i\phi}. \] (6.195)

Since
\[ \int P_l(\cos \theta)(\cos \theta \cos \phi e_\theta - \sin \phi e_\phi) \cdot (r \times \nabla Y^*_{l,1}) d\Omega \]
\[ = -i\pi \sqrt{\frac{2l+1}{4\pi l(l+1)}} \int_0^\pi P_l(\cos \theta) \left( \frac{\cos \theta}{\sin \theta} P^1_l(\cos \theta) + \frac{dP^1_l(\cos \theta)}{d\theta} \right) \sin \theta d\theta, \] (6.196)

and
\[ \int \frac{dP^1_l(\cos \theta)}{d\theta} P_l(\cos \theta) \sin \theta d\theta \]
\[ = -\int \frac{P^1_l(\cos \theta)}{d\theta} [P_l(\cos \theta) \sin \theta] d\theta \]
\[ = -\int P^1_l(\cos \theta) P_l(\cos \theta) \cos \theta d\theta + \int \left[ P^1_l(\cos \theta) \right]^2 \sin \theta d\theta, \] (6.197)

the integral reduces to
\[ \int \frac{P^1_l(\cos \theta)}{d\theta} P_l(\cos \theta) \left( \frac{\cos \theta}{\sin \theta} P^1_l(\cos \theta) + \frac{dP^1_l(\cos \theta)}{d\theta} \right) \sin \theta d\theta \]
\[ = \int \left[ P^1_l(\cos \theta) \right]^2 \sin \theta d\theta \]
\[ = \frac{2}{2l+1} l(l+1). \] (6.198)

Then, finally,
\[ a_{l,1} = -i^{l+1} \sqrt{\frac{\pi(2l+1)}{l(l+1)}} E_0. \] (6.199)
The coefficient \( a_{l,-1} \) can be found in a similar manner,
\[
a_{l,-1} = a_{l,1} = -i^{l+1} \sqrt{\frac{\pi(2l+1)}{l(l+1)}} E_0.
\]

The appearance of \( m = \pm 1 \) components in the spherical harmonic expansion of a plane wave is understandable, for a plane wave can be decomposed into two circularly polarized waves of opposite helicity.

The coefficient \( b_{lm} \) of the TM mode can be found in a similar manner. It is more convenient to work with the magnetic field,
\[
B = \frac{1}{i\omega} \nabla \times \mathbf{E}
\]
\[
= \frac{1}{i\omega} \sum_{l=1}^{\infty} a_{l,\pm 1} \nabla \times [j_l(kr) \nabla Y_{l,\pm 1}] + \frac{k}{i\omega} \sum_{l=1}^{\infty} b_{l,\pm 1} j_l(kr) \nabla Y_{l,\pm 1},
\]
where use is made of the identity,
\[
\nabla \times \nabla \times [j_l(kr) \nabla Y_{l,\pm 1}] = \nabla \nabla \cdot [j_l(kr) \nabla Y_{l,\pm 1}] - \nabla^2 [j_l(kr) \nabla Y_{l,\pm 1}] = 0 + k^2 j_l(kr) \nabla Y_{l,\pm 1}.
\]

The magnetic field associated with the plane wave is
\[
B_0 e^{ikr \cos \theta} e_y = \frac{E_0}{c} \sum_l i^l (2l+1) j_l(kr) P_l(\cos \theta) (\sin \theta \sin \phi e_r + \cos \theta \sin \phi e_\theta + \cos \phi e_\phi).
\]

Then,
\[
\frac{E_0}{c} \sum_{l=1}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos \theta) (\sin \theta \sin \phi e_r + \cos \theta \sin \phi e_\theta + \cos \phi e_\phi)
\]
\[
= \frac{1}{i\omega} \sum_{l=1}^{\infty} a_{l,\pm 1} \nabla \times [j_l(kr) \nabla Y_{l,\pm 1}] + \frac{k}{i\omega} \sum_{l=1}^{\infty} b_{l,\pm 1} j_l(kr) \nabla Y_{l,\pm 1}.
\]

Multiplying both sides by \( r \times \nabla Y_{l,\pm 1}^* \) and integrating the result over the solid angle, we find
\[
b_{l,\pm 1} = E_0 \frac{i^{l+1}(2l+1)}{l(l+1)} \int P_l(\cos \theta) \left( -\frac{\cos \theta}{\sin \theta} \frac{\partial Y_{l,\pm 1}^*}{\partial \phi} + \cos \phi \frac{\partial Y_{l,\pm 1}^*}{\partial \theta} \right) d\Omega
\]
\[
= \mp i^{l+1} \sqrt{\frac{\pi(2l+1)}{l(l+1)}} E_0
\]
\[
= \pm a_{l,\pm 1}.
\]

(Calculation steps are left for exercise.) Then the desired spherical harmonic expansion of a plane wave...
wave is
\[ E_0 e^{ikz} e_x = E_0 \sum_{l=1}^{l+1} \frac{i^l}{2^l} \sqrt{\frac{4\pi(2l+1)}{l(l+1)}} \left[ j_l(kr) \mathbf{r} \times \nabla Y_{l,\pm1}(\theta, \phi) \pm \frac{1}{k} \nabla \times \left[ j_l(kr) \mathbf{r} \times \nabla Y_{l,\pm1}(\theta, \phi) \right] \right]. \]  
(6.205)

The accompanying magnetic field is expanded as
\[ B_0 e^{ikz} e_y = -\frac{E_0}{c} \sum_{l=1}^{l+1} \frac{i^l}{2^l} \sqrt{\frac{\pi(2l+1)}{l(l+1)}} \left[ \frac{1}{k} \nabla \times \left[ j_l(kr) \mathbf{r} \times \nabla Y_{l,\pm1}(\theta, \phi) \right] \pm j_l(kr) \mathbf{r} \times \nabla Y_{l,\pm1}(\theta, \phi) \right]. \]  
(6.206)

These expansions of the plane wave greatly facilitate analysis of scattering of plane electromagnetic waves by an object. An important observation to be made is that the electromagnetic waves re-radiated (scattered) by an object also consist of TE and TM modes in the form

TE mode: \[ h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{l,\pm1}(\theta, \phi), \]  
(6.207)

TM mode: \[ \frac{1}{k} \nabla \times [h_l^{(1)}(kr) \mathbf{r} \times \nabla Y_{l,\pm1}(\theta, \phi)], \]  
(6.208)

since the boundary conditions for electric and magnetic fields at the object require that scattered wave should have exactly the same angular dependence as the incident plane wave. In examples to follow, we analyze scattering of a plane wave by ideally conducting sphere and by dielectric sphere.

### 6.9 Scattering by an Ideally Conducting Sphere

Let us consider a highly conducting sphere of radius \(a\) placed in a plane electromagnetic wave as shown in Fig.6-4. In two extreme cases, \(ka \ll 1\) (low frequency limit) and \(ka \gg 1\) (high frequency limit), the problem can readily be solved without detailed analysis based on the spherical harmonic expansion. If \(ka \ll 1\), the dipole approximation can be used. As we have seen, the electric dipole moment of a conducting sphere placed in a static electric field \(E_0\) is
\[ \mathbf{p} = 4\pi \varepsilon_0 a^3 \mathbf{E}_0, \]  
(6.209)

and the magnetic dipole moment of a superconducting sphere placed in a static magnetic field \(H_0\) is
\[ \mathbf{m} = -2\pi a^3 \mathbf{H}_0. \]  
(6.210)

These expressions remain valid in oscillating fields as long as the oscillation frequency is sufficiently small so that \(ka = \omega a/c \ll 1\). Each dipole radiates linearly independent modes. The radiation powers of the modes are thus additive. The power radiated by the electric dipole is
\[ P_E = \frac{1}{4\pi \varepsilon_0} \frac{2\omega A^2}{3c^3} = 4\pi \varepsilon_0 \frac{2\omega A^6}{3c^3} E_0^2, \]  
(6.211)
and that due to the magnetic dipole is

\[ P_M = \frac{1}{4\pi\varepsilon_0} \frac{2\omega^4 m^2}{3c^5} = \frac{1}{4} 4\pi\varepsilon_0 \frac{2\omega^4 a^6}{3c^3} E_0^2, \]  

(6.212)

where \( H_0 = E_0/Z, \) \( Z = \sqrt{\mu_0/\varepsilon_0} \) is substituted in the magnetic dipole moment. The total re-radiated (scattered) power is

\[ P = 4\pi\varepsilon_0 \frac{2\omega^4 a^6}{3c^3} E_0^2 \times \left( 1 + \frac{1}{4} \right), \]  

(6.213)

and corresponding scattering cross-section is given by

\[ \sigma = \frac{P}{S_1} = \frac{P}{c\varepsilon_0 E_0^2} = \left( \frac{8\pi}{3} + \frac{2\pi}{3} \right) a^2 (ak)^4. \]  

(6.214)

The dependence \( \sigma \propto k^4, \) which is common to all dipole radiation by objects much smaller than the wavelength of incident wave, is known as Rayleigh’s law.

The reader may wonder why the magnetic dipole radiates a power comparable with that by electric dipole in this case. As we have seen, the radiation fields due to a magnetic dipole is of higher order by a factor \( (ka)^2 \) compared with those due to electric dipole. The reason is as follows. The low order vector potential

\[ A(r) \approx \frac{\mu_0}{4\pi r} e^{ikr} \int J(r')(1 - i\mathbf{k} \cdot \mathbf{r'}) dV', \]  

(6.215)

must be applied separately for electric and magnetic dipoles because the surface currents on the conducting sphere induced by the incident electric and magnetic fields are entirely different. The current due to the electric field flows between the poles while the current induced by the magnetic field is azimuthal. The surface current in the polar direction is of order

\[ J_{s\theta} \approx a\omega \varepsilon_0 E_0 = akH_0, \]  

(6.216)

while the azimuthal current is of order

\[ J_{s\phi} \approx H_0. \]  

(6.217)

The polar current is smaller than the azimuthal current by a factor \( ak. \) (This is not surprising because current flows in the azimuthal direction without any hindrance while in the polar direction, current flow results in charge accumulation at the poles.) Therefore, the radiation fields from the electric and magnetic dipoles are comparable.

In the opposite limit of \( ka \gg 1 \) (high frequency or short wavelength limit), geometric and physical optics approximation can be applied. The problem reduces to reflection by the illuminated surface and diffraction by the other half surface in the shadow. Each surface contributes equally to the scattering cross-section,

\[ \sigma = 2\pi a^2, \]  

(6.218)
although the angular distribution of scattered Poynting fluxes are entirely different. We will revisit this problem in chapter 7.

![Geometry assumed in analysis of scattering by a conducting sphere.](image)

Figure 6-4: Geometry assumed in analysis of scattering by a conducting sphere.

For arbitrary value of $ka$, the spherical harmonic expansion of the plane wave can be exploited as follows. The scattered wave can be decomposed into TE and TM modes having the same angular dependence as those contained in the incident plane wave. We therefore assume the following expansion for the electric and magnetic fields of the scattered wave,

$$
E_{sc}(r) = -E_0 \sum_{l=1}^{l+1} \frac{\pi(2l+1)}{l(l+1)} (A_lM_{l,\pm1} \pm B_lN_{l,\pm1}), \ r > a \quad (6.219)
$$

$$
B_{sc}(r) = -B_0 \sum_{l=1}^{l+1} \frac{\pi(2l+1)}{l(l+1)} (A_lN_{l,\pm1} \pm B_lM_{l,\pm1}), \ r > a
$$

where $M_{l,\pm1}$ and $N_{l,\pm1}$ are the TE and TM base vectors,

$$
M_{l,\pm1}(r) = h_l^{(1)}(kr)r \times \nabla Y_{l,\pm1}, \quad (6.220)
$$

$$
N_{l,\pm1}(r) = \frac{1}{k} \nabla \times [h_l^{(1)}(kr)r \times \nabla Y_{l,\pm1}], \quad (6.221)
$$

and $A_l$ and $B_l$ are expansion coefficients to be determined. If the sphere is ideally conducting, the boundary conditions at the sphere surface are that the tangential component of the electric field and normal component of the magnetic field both vanish. Since the total field at the surface is the sum of incident and scattered waves, the explicit forms of the boundary conditions are:

$$
(E_{inc} + E_{sc})_t = 0, \quad \text{at } r = a, \quad (6.222)
$$

and

$$
(B_{inc} + B_{sc})_n = 0, \quad \text{at } r = a. \quad (6.223)
$$
In fact, the coefficients $A_l$ and $B_l$ can be determined from $(\mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{sc}})_t = 0$ alone,

$$A_l = -\frac{j_l(ka)}{h_l^{(1)}(ka)},$$

$$B_l = -\frac{\frac{d}{dr}[r j_l(kr)]}{\frac{d}{dr}[r h_l^{(1)}(kr)]} = \frac{[k a_j l(ka)]'}{[k a h_l^{(1)}(ka)]'},$$

where

$$[x j_l(x)]' = \frac{d}{dx}[x j_l(x)], \quad [x h_l^{(1)}(x)]' = \frac{d}{dx}[x h_l^{(1)}(x)].$$

(6.226)

Note that for $g_l(kr) = j_l(kr)$ or $h_l^{(1)}(kr)$,

$$\nabla \times [g_l(kr) \mathbf{r} \times \nabla Y_{l,\pm 1}] = -g_l(kr) \frac{l(l+1)}{r} Y_{l,\pm 1} \mathbf{e}_r - \frac{d}{dr}[r g_l(kr)] \nabla Y_{l,\pm 1},$$

(6.227)

and the coefficient $B_l$ has been determined from the tangential component of the electric field of the TM mode. The boundary condition for the magnetic field is automatically satisfied. (Verify this statement.)

![Figure 6-5: Normalized scattering cross-section of an ideally conducting sphere. $\sigma/(2\pi a^2)$ as a function of $ka$.](image)

The radiation power of the $l$-th harmonic mode can be readily found as

$$P_l = 2\varepsilon_0 E_0^2 \frac{\pi(2l+1)}{l(l+1)} r^2 \left( |A_l|^2 \int |M_{l1}|^2 d\Omega + |B_l|^2 \int |N_{l1}|^2 d\Omega \right)$$

$$= \varepsilon_0 E_0^2 \frac{2\pi(2l+1)}{k^2} \left( |A_l|^2 + |B_l|^2 \right),$$

(6.228)
where the factor 2 accounts for an equal contribution from the \((l, m = \pm 1)\) modes. Then the scattering cross-section is given by

\[
\sigma(ka) = \sum_l \frac{P_l}{c\varepsilon_0 E_0^2} = \frac{2\pi}{k^2} \sum_{l=1}^{\infty} (2l + 1) \left( |A_l|^2 + |B_l|^2 \right) = \frac{2\pi}{k^2} \sum_{l=1}^{\infty} (2l + 1) \left( \left| \frac{j_l(ka)}{h_l^{(1)}(ka)} \right|^2 + \left| \frac{ka j_l(ka)}{h_l^{(1)}(ka)'} \right|^2 \right), \quad (m^2). \tag{6.229}
\]

\(\sigma(ka)\) normalized by \(2\pi a^2\) (this is the scattering cross-section in the geometrical optics limit \(ka \gg 1\)) is plotted in Fig. ?? in the range \(0 < x(= ka) < 10\). In the long wavelength limit, \(ka \ll 1\), the spherical Bessel functions approach

\[
j_l(x) \rightarrow \frac{x^l}{(2l + 1)!!}, \quad [xj_l(x)]' \rightarrow \frac{l+1}{(2l + 1)!!} x^l \text{ for } x \ll 1, \tag{6.230}
\]

\[
h_l^{(1)}(x) \rightarrow -i \frac{(2l-1)!!}{x^{l+1}}, \quad [xh_l^{(1)}(x)]' \rightarrow i \frac{l(l+1)!!}{x^{l+1}} \text{ for } x \ll 1. \tag{6.231}
\]

The lowest order terms of \(l = 1\) remain finite in this limit,

\[
A_1 = -i \frac{1}{3} (ka)^3, \quad B_1 = i \frac{2}{3} (ka)^3, \tag{6.232}
\]

which yields a scattering cross-section

\[
\sigma \simeq \frac{10\pi}{3} k^4 a^6, \quad ka \ll 1. \tag{6.233}
\]

In short wavelength limit \(ka \gg 1\), the figure indicates that \(\sigma\) indeed approaches \(2\pi a^2\).

In radar engineering, the Poynting flux scattered back toward a radar is of main interest. In the geometry assumed in Fig. 6-4, the direction of back scattering corresponds to \(\theta = \pi\). Since

\[
Y_{l,1}(\theta, \phi) + Y_{l,-1}(\theta, \phi) = -2i \sqrt{\frac{2l+1}{4\pi l(l+1)}} P_l^1(\cos \theta) \sin \phi, \tag{6.234}
\]

\[
\frac{dP_l^1}{d\theta} + \frac{1}{\sin \theta} P_l^1 = 0, \quad \text{and} \quad \frac{dP_l^1}{d\theta} = (-1)^l \frac{l(l+1)}{2} \text{ at } \theta = \pi, \tag{6.235}
\]

the scattered field at \(\theta = \pi\) becomes

\[
E_{sc}(r) = -E_0 \sum_{l, \pm 1} i^{l+1} \sqrt{\frac{\pi(2l+1)}{l(l+1)}} (A_l M_{l, \pm 1} \pm B_l N_{l, \pm 1})
\]

\[
= -\frac{iE_0 e^{ikr}}{kr} \sum_l \frac{2l+1}{2} (-1)^l (A_l - B_l)E_x. \tag{6.236}
\]

At \(\theta = \pi\), the scattered field is plane polarized in the same direction as the incident electric field.
The Poynting flux at $\theta = \pi$ is

$$S_r(\theta = \pi) = \frac{c_0 E_0^2}{4r^2 k^2} \left| \sum_l (-1)^l (2l + 1) (A_l - B_l) \right|^2 .$$

The radar scattering cross section is defined by

$$\sigma_{radar} = \frac{4\pi}{c_0 E_0^2} r^2 S_r(\theta = \pi) = \frac{r^2}{k^2} \left| \sum_l (2l + 1) (-1)^l (A_l - B_l) \right|^2 ,$$

where $4\pi$ is the total solid angle.

**Example 3 Scattering by a Dielectric Sphere**

Scattering by a sphere of dielectric and magnetic properties can be analyzed in a similar manner. We assume a sphere of radius $a$ having relative permittivity $\varepsilon_r$ and relative permeability $\mu_r$ placed in a plane wave. In general, $\varepsilon_r$ and $\mu_r$ are both complex to account for dissipation (absorption) of electromagnetic energy. The incident wave is described by the fields

$$E_0 e^{ikz} e_x = -E_0 \sum_{l,\pm 1} i^{l+1} \sqrt{\frac{\pi (2l + 1)}{l(l + 1)}} \left[ j_l(kr) \times \nabla Y_{l,\pm 1}(\theta, \phi) \pm \frac{1}{k} \nabla \times [j_l(kr) \times \nabla Y_{l,\pm 1}(\theta, \phi)] \right] ,$$

and

$$H_0 e^{ikz} e_y = -\frac{E_0}{Z_0} \sum_{l,\pm 1} i^l \sqrt{\frac{\pi (2l + 1)}{l(l + 1)}} \left[ \frac{1}{k} \nabla \times [j_l(kr) \times \nabla Y_{l,\pm 1}(\theta, \phi)] \pm j_l(kr) \times \nabla Y_{l,\pm 1}(\theta, \phi) \right] .$$

where $Z_0 = \sqrt{\mu_0 / \varepsilon_0}$. The scattered wave continues to be assumed in the form

$$E_0(r) = E_0 \sum_{l,\pm 1} i^{l-1} \sqrt{\frac{\pi (2l + 1)}{l(l + 1)}} (A_l M_{l,\pm 1} \pm B_l N_{l,\pm 1}) , \ r > a$$

and

$$H_0(r) = -\frac{E_0}{Z_0} \sum_{l,\pm 1} i^l \sqrt{\frac{\pi (2l + 1)}{l(l + 1)}} (A_l N_{l,\pm 1} \pm B_l M_{l,\pm 1}) , \ r > a$$

where $Z_0 = \sqrt{\mu_0 / \varepsilon_0}$ is the impedance in the free space. The fields inside the sphere should be bounded at $r = 0$ and thus may be assumed as

$$E_i(r) = E_0 \sum_{l,\pm 1} i^{l-1} \sqrt{\frac{\pi (2l + 1)}{l(l + 1)}} \left( C_l j_l(kr) \times \nabla Y_{l,\pm 1} \pm D_l \frac{1}{kr} \nabla \times [j_l(kr) \times \nabla Y_{l,\pm 1}] \right) ,$$

and

$$H_i(r) = -\frac{E_0}{Z_0} \sum_{l,\pm 1} i^l \sqrt{\frac{\pi (2l + 1)}{l(l + 1)}} \left( C_l \frac{1}{kr} \nabla \times [j_l(kr) \times \nabla Y_{l,\pm 1}] \pm D_l j_l(kr) \times \nabla Y_{l,\pm 1} \right) ,$$

where $C_l$ and $D_l$ are constants determined by boundary conditions.
where

\[ Z_s = \sqrt{\frac{\mu_r}{\varepsilon_r}} Z_0, \quad k' = \sqrt{\varepsilon_r \mu_r} k, \quad (6.244) \]

to account for the impedance and index of refraction of the sphere. Continuity of the tangential components of the electric field \( \mathbf{E} \) and magnetic field \( \mathbf{H} \) yields the following conditions:

\[ j_l(ka) + A_l h_l^{(1)}(ka) = C_l j_l(k'a), \quad (6.245) \]

\[ [ka j_l(ka)]' + B_l [kah_l^{(1)}(ka)]' = \frac{1}{\sqrt{\varepsilon_r \mu_r}} D_l [k'a j_l(k'a)]', \quad (6.246) \]

\[ j_l(ka) + B_l h_l^{(1)}(ka) = \sqrt{\frac{\varepsilon_r}{\mu_r}} D_l j_l(k'a), \quad (6.247) \]

\[ [ka j_l(ka)]' + A_l [kah_l^{(1)}(ka)]' = \frac{1}{\mu_r} C_l [k'a j_l(k'a)]'. \quad (6.248) \]

where

\[ [x j_l(x)]' = \frac{d}{dx} [x j_l(x)], \quad [x h_l^{(1)}(x)]' = \frac{d}{dx} [x h_l^{(1)}(x)]. \quad (6.249) \]

Solving for the coefficients \( A_l \) and \( B_l \), we find

\[ A_l = -\frac{\mu_r j_l(k'a) [ka j_l(ka)]' - j_l(ka) [k'a j_l(k'a)]'}{h_l^{(1)}(ka) [k'a j_l(k'a)]' - \mu_r j_l(k'a) [kah_l^{(1)}(ka)]'}, \quad (6.250) \]

\[ B_l = \frac{\varepsilon_r j_l(k'a) [ka j_l(ka)]' - j_l(ka) [k'a j_l(k'a)]'}{h_l^{(1)}(ka) [k'a j_l(k'a)]' - \varepsilon_r j_l(k'a) [kah_l^{(1)}(ka)]'}. \quad (6.251) \]

The case of ideally conducting sphere can be recovered in the limit \( \varepsilon_r \to \infty \) and \( \mu_r \to 0 \). The scattering cross-section can readily be found,

\[ \sigma = \frac{2\pi}{k^2} \sum_{l=1}^{\infty} (2l + 1) \left( |A_l|^2 + |B_l|^2 \right). \quad (6.252) \]

In the long wavelength limit, \( ka, k'a \ll 1 \), the leading order terms are the dipoles, \( l = 1 \),

\[ A_1 = \frac{2}{3} \frac{\mu_r - 1}{\mu_r + 2} (ka)^3, \quad B_1 = \frac{2}{3} \frac{\varepsilon_r - 1}{\varepsilon_r + 2} (ka)^3. \quad (6.253) \]

For a dielectric sphere with \( \mu_r = 1 \), the scattering cross-section in the low frequency regime is thus given by

\[ \sigma \approx \frac{8\pi}{3} \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 k^4 a^6, \quad ka \ll 1. \quad (6.254) \]

In the limit of \( \varepsilon_r \gg 1 \), we recover the electric dipole portion of the cross-section of a conducting sphere,

\[ \sigma = \frac{8\pi}{3} k^4 a^6. \quad (6.255) \]

Note that the case of conducting sphere can be recovered, mathematically, in the limit \( \mu_r = 0 \) for
$A_1$ and $\varepsilon_r = \infty$ for $B_1$. The case of conducting sphere should be analyzed by incorporating the impedance

$$Z = \sqrt{\frac{-i\omega\mu}{-i\omega\varepsilon + \sigma_c}},$$

where $\sigma_c$ is the conductivity. Ideal conductor is characterized by $Z = 0$ which can be realized by letting $\sigma_c \to \infty$, or $\mu \to \infty$, or $\varepsilon \to \infty$.

### 6.10 Scattering by a Cylinder

In this section, we consider scattering of electromagnetic waves by a cylindrical object having a length $l$ sufficiently longer than the wavelength, $kl \gg 1$. If the incident wave propagates perpendicular to the cylinder axis, the problem becomes two dimensional. For general incident angle, the incident wave can be decomposed into normal and axial components. The axial component suffers little scattering and it is sufficient to consider only the case of normal incidence.

![Diagram of a cylinder with incident fields](image)

Figure 6-6: Two possible polarizations of the incident field relative to the cylinder axis, $E_i$ parallel to the axis (top) and perpendicular (bottom).

Scattering by a conducting cylinder can be analyzed in a manner similar to the case of conducting sphere. If the incident wavelength is much shorter than the radius of the cylinder, $ka \gg 1$, the geometrical optics approximation applies. Regardless of the polarization of the incident wave relative to the cylinder axis, the scattering cross-section per unit length of the cylinder is given by

$$\frac{\sigma}{l} = 2a + 2a = 4a, \quad ka \gg 1,$$

(6.256)

where $2a$ is the contribution from the illuminated surface and another $2a$ is the contribution from the shadow surface due to forward diffraction. Analysis for this case, which is quite parallel to the
case of a conducting sphere, is left for an exercise.

In long wavelength limit \(ka \ll 1\), the polarization direction of the incident wave becomes important. This is understandable because the current induced on the cylinder surface sensitively depends on the orientation of the incident electric field. As intuitively expected, the surface current flows much more easily along the axis than in azimuthal direction. Therefore, the scattering cross-section when \(E_i \parallel e_z\) should be much larger than the case \(E_i \perp e_z\).

The axial components of the electric and magnetic fields satisfy the following 2-dimensional scalar Helmholtz equation,

\[
\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + k^2 \right) \begin{pmatrix} E_z \\ H_z \end{pmatrix} = 0. \tag{6.257}
\]

General solutions can be found by assuming a form \(R(\rho)e^{im\phi}\), where the radial function satisfies the Bessel equation

\[
\left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2 \right) R(\rho) = 0, \tag{6.258}
\]

\[R(\rho) = J_m(k\rho), \ N_m(k\rho).\]

For wave analysis, it is convenient to define the Hankel functions of the first and second kinds by

\[H_m^{(1)}(k\rho) = J_m(k\rho) + iN_m(k\rho), \tag{6.259}\]
\[H_m^{(2)}(k\rho) = J_m(k\rho) - iN_m(k\rho). \tag{6.260}\]

The asymptotic form of the first kind is

\[H_m^{(1)}(k\rho) \to \sqrt{\frac{2}{\pi k\rho}} \exp \left( ik\rho - \frac{2m + 1}{4} \pi \right), \tag{6.261}\]

which has an amplitude dependence \(1/\sqrt{\rho}\) appropriate for cylindrical waves. \(H_m^{(1)}(k\rho)\) describes an outgoing wave and \(H_m^{(2)}(k\rho)\) an incoming wave.

We first analyze the case of incident electric field along the cylinder axis \(E_i \parallel e_z\). The incident wave is assumed to be propagating in the negative \(x\)-direction,

\[E_i(r) = E_0 e^{-ik\rho \cos \phi} \mathbf{e}_z. \tag{6.262}\]

The scattered electric field is also in the \(z\)-direction and we assume

\[E_z^{sc} = \sum_m a_m H_m^{(1)}(k\rho)e^{im\phi}. \tag{6.263}\]

If the cylinder is ideally conducting, the boundary condition is that the tangential component of
the electric field vanish at the cylinder surface,

\[ E_0 e^{-ika \cos \phi} + \sum_m a_m H_m^{(1)}(ka) e^{im\phi} = 0. \]  

(6.264)

Multiplying by \( e^{-im'\phi} \) and integrating over \( \phi \), we find the expansion coefficient \( a_m \),

\[
 a_m = -\frac{E_0}{2\pi H_m^{(1)}(ka)} \int_0^{2\pi} e^{-i(ka \cos \phi + m\phi)} d\phi
\]

\[
 = -(i)^m \frac{J_m(ka)}{H_m^{(1)}(ka)} E_0.
\]

(6.265)

and the scattered electric field is

\[
 E_z^{sc} = -E_0 \sum_m (-i)^m \frac{J_m(ka)}{H_m^{(1)}(ka)} H_m^{(1)}(k\rho) e^{im\phi}.
\]

(6.266)

The Poynting flux in the radiation zone \( k\rho \gg 1 \) is

\[
 S_\rho = \frac{E_0^2}{Z_0} \frac{2}{\pi k \rho} \sum_m \left| \frac{J_m(ka)}{H_m^{(1)}(ka)} \right|^2.
\]

(6.267)

and the total scattered power is given by

\[
 P = \rho \int_0^{2\pi} S_\rho d\phi
\]

\[
 = \frac{E_0^2}{Z_0} \frac{4}{k} \sum_m \left| \frac{J_m(ka)}{H_m^{(1)}(ka)} \right|^2.
\]

(6.268)

The scattering cross-section is

\[
 \sigma = \frac{4}{k} \sum_m \left| \frac{J_m(ka)}{H_m^{(1)}(ka)} \right|^2.
\]

The function

\[
 f(x) = \frac{1}{x} \sum_m \left| \frac{J_m(x)}{H_m^{(1)}(x)} \right|^2,
\]

(6.269)

is plotted below. In the short wavelength regime it approaches unity and the scattering cross section is \( \sigma/\ell = 4a \) as expected from the geometrical optics approximation. In the long wavelength regime \( ka \ll 1, m = 0 \) mode is dominant and the cross-section diverges in a manner

\[
 \frac{\sigma}{\ell} \sim \frac{\pi^2}{k} \frac{1}{\ln \left( \frac{ka}{2} \right) + \gamma_E^2}.
\]
where $\gamma_E = 0.5772 \ldots$ is the Euler’s constant. Note that in the limit $x \to 0$, $J_0(x) \to 1$ and

$$H_0^{(1)}(x) \to i \frac{2}{\pi} \left[ \ln \left( \frac{x}{2} \right) + \gamma_E \right], \quad x \to 0.$$  

Physically, the symmetric mode ($m = 0$) corresponds to radiation by a long cylindrical antenna having a length much larger than the wavelength. Fig.6-7 shows $\sigma/l$ normalized by $4a$ as a function of $x = ka$.

$$\frac{1}{x} \sum_{m=-10}^{10} \left| \frac{J_m(x)}{J_m(x) + jY_m(x)} \right|^2$$

![Figure 6-7: Scattering cross-section per unit length of a long conducting cylinder of radius a. $\sigma/4al$ as a function of $x = ka$. The electric field is polarized along the cylinder.](image_url)

If the incident electric field is perpendicular to the cylinder axis, it is more convenient to use the magnetic field which is axial. The incident magnet field is

$$B_0 e^{-ikp \cos \phi} \mathbf{e}_z,$$  

(6.270)

and we assume the scattered magnetic field in the form

$$B^sc_z(r) = \sum_m b_m H_m^{(1)}(kp)e^{im\phi}.  \quad (6.271)$$

Corresponding electric field can be found from the Maxwell’s equation,

$$\frac{1}{c^2} \frac{\partial \mathbf{E}^sc}{\partial t} = \nabla \times \mathbf{B}^sc.$$  

(6.272)
The $\phi$ component of the scattered electric field is thus

$$E_{\phi}^{sc}(\rho, \phi) = -ic \sum_{m} \frac{b_m}{k} \frac{\partial}{\partial \rho} [H_m^{(1)}(kp)] e^{im\phi},$$

while the $\phi$ component of the incident electric field is

$$E_{\phi}^i(\rho, \phi) = cB_0 \cos \phi e^{-ik\rho \cos \phi}.$$ 

The boundary condition of vanishing tangential component of the electric field at the cylinder surface yields

$$B_0 \cos \phi e^{-ika \cos \phi} - i \sum_{m} \frac{b_m}{k} \frac{\partial}{\partial a} [H_m^{(1)}(ka)] e^{im\phi} = 0.$$ 

Multiplying by $e^{im\phi}$ and integrating over $\phi$, we obtain

$$b_m = -B_0 (-i)^m \frac{[J_m(ka)]'}{[H_m^{(1)}(ka)]'}.$$ 

where the prime means differentiation with respect to the argument $ka$ and the following transformation is used,

$$\int_0^{2\pi} \cos \phi e^{-i(ka \cos \phi + m\phi)} d\phi = i(-i)^m 2\pi [J_m(ka)]'.$$

A resultant scattering cross-section is

$$\frac{\sigma}{l} = \frac{4}{k} \sum_{m} \left| \frac{[J_m(ka)]'}{[H_m^{(1)}(ka)]'} \right|^2.$$ 

Fig.6-8 shows $\sigma/l$ normalized by $4a$. In contrast to the preceding case $E_i = E_0 e_z$, the cross-section in the long wavelength regime $ka \ll 1$ is bounded and given

$$\frac{\sigma}{l} \approx \frac{3\pi^2}{4} k^3 a^4,$$

since if $ka \ll 1$, the dominant harmonics are

$$b_0 = \frac{i\pi B_0}{4} (ka)^2, \quad b_{\pm 1} = \pm \frac{\pi B_0}{4} (ka)^2.$$ 

In short wavelength regime $ka \gg 1$, the cross-section approaches $4a$ as in the case of axial polarization.
Figure 6-8: \((\sigma/4a)\) of a long conducting cylinder when the electric field is normal to the cylinder.

Problems

6.1 A conducting sphere is placed in a low frequency plane electromagnetic wave such that \(ka \ll 1\) where \(a\) is the sphere radius. Finding the effective electric and magnetic dipole moments, show that the scattering cross section of the sphere is given by

\[
\sigma = \frac{(8 + 2)\pi}{3} k^4 a^6,
\]

where 8 parts is due to electric polarization and 2 parts due to magnetic dipole. According to the radiation magnetic field due to a small source in Eq. (5.51),

\[
\mathbf{H}(r) \approx \frac{1}{4\pi c} \frac{e^{ikr}}{r} \left( \mathbf{p} \times \mathbf{n} + \frac{1}{c} \mathbf{n} \times (\mathbf{n} \times \mathbf{m}) - \frac{1}{2c} \mathbf{n} \times (\mathbf{n} \cdot \mathbf{Q}) \right),
\]

the radiation power due to magnetic dipole is supposed to be of higher order than that due to electric dipole by a factor \((ka)^2\). In scattering by a conducting sphere, they are comparable. Resolve this apparent paradox.

6.2 A conducting sphere of radius \(a\) has a complex index of refraction \(\sqrt{\varepsilon/\varepsilon_0} = n_r + in_i\). Determine the scattering cross-section as a function of \(ka\).

6.3 Show that the transverse component of the current density \(\mathbf{J}\) is given by

\[
\mathbf{J}_t = \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{\mathbf{J}(r')}{|\mathbf{r} - \mathbf{r}'|} dV',
\]
and consequently the longitudinal component by
\[
\mathbf{J}_l = -\frac{1}{4\pi} \nabla \nabla \cdot \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' = -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'.
\]

Hint:
\[
\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi \delta(\mathbf{r} - \mathbf{r}').
\]

6.4 The scattering cross section of small objects \((ka \ll 1 \text{ with } a \text{ being the size})\) obeys the Rayleigh’s law \(\sigma \propto k^4\) which is often used to explain blueness of sky and redness of sunset. Scattering of sunlight by air molecules requires fluctuation in the air density. Explain why. You may treat the atmosphere as a fluid.

6.5 A conducting sphere of radius \(a\) is coated with a material having a relative permittivity \(\varepsilon_r\) and permeability \(\mu_r\) (both may be complex). The thickness of coating is \(\delta\). Analyze low frequency scattering in the limit \(ka \ll 1\) and show that the radiation fields are characterized by the dipole terms
\[
A_1 = -i\frac{(kb)^3}{3} \left( \frac{2 + \rho - 2\mu_r(1 - \rho)}{2 + \rho + \mu_r(1 - \rho)} \right),
\]
\[
B_1 = -i\frac{(kb)^3}{3} \left( \frac{2(1 - \rho) - 2\varepsilon_r(1 + 2\rho)}{2(1 - \rho) + \varepsilon_r(1 + 2\rho)} \right),
\]
where \(b = a + \delta\), \(\rho = a^3/(a + \delta)^3\). Discuss possible effects of coating on the scattering cross section.

6.6 For a tangential electric field on the surface of a sphere of radius \(a\),
\[
\mathbf{E}_s(\theta, \phi) = E_\theta(\theta, \phi)\mathbf{e}_\theta + E_\phi(\theta, \phi)\mathbf{e}_\phi,
\]
find a general expression for the radiation electric field.

6.7 A conducting sphere of radius \(a\) has a narrow planar gap cut at polar angle \(\theta_0\). Show that the radiation admittance of the sphere is
\[
Y = \frac{i}{Z_0} 2\pi ka \sin^2 \theta_0 \sum_{l=1}^{\infty} \frac{2l + 1}{2l(l + 1)} [P_l^1(\cos \theta_0)]^2 \frac{h_l^{(2)}(ka)}{a h_l^{(2)}(ka)}, \quad \text{(Siemens)}
\]
where \(Z_0 = \sqrt{\mu_0/\varepsilon_0}\).

6.8 In the Lorenz gauge, the scalar and vector potentials are
\[
\Phi(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}') dV',
\]
\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \mathbf{J}(\mathbf{r}') dV',
\]
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where the charge density and current density are related through

\[-i\omega \rho + \nabla \cdot \mathbf{J} = 0.\]

Show that the electric field is given by

\[
\mathbf{E}(\mathbf{r}) = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} = \frac{i\omega \mu_0}{4\pi} \left( 1 + \frac{1}{k^2} \nabla \nabla \right) \cdot \int \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \mathbf{J}(\mathbf{r}') \, dV',
\]

where \( \mathbf{1} \) is the unit tensor. Furthermore, show that the concrete form of the tensor operator in the spherical coordinates is

\[
\left( 1 + \frac{1}{k^2} \nabla \nabla \right) \frac{e^{ikR}}{4\pi R} = \begin{bmatrix}
\frac{2}{(kR)^2} - \frac{2i}{kR} & 0 & 0 \\
0 & 1 + \frac{i}{kR} - \frac{1}{(kR)^2} & 0 \\
0 & 0 & 1 + \frac{i}{kR} - \frac{1}{(kR)^2}
\end{bmatrix} \frac{e^{ikR}}{4\pi R},
\]

where \( R = |\mathbf{r} - \mathbf{r}'| \). Note that the radial (longitudinal) component is proportional to

\[
\frac{1}{R} \left( \frac{2}{(kR)^2} - \frac{2i}{kR} \right).
\]

Hint:

\[
\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_r}{\partial \phi} = \sin \theta \mathbf{e}_\phi.
\]

6.9 The Lorenz gauge is characterized by

\[
\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0.
\]

(a) Show that

\[
\frac{\partial}{\partial t} \left( \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = 0,
\]

yields the Coulomb’s law,

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.
\]

(b) What does

\[
\nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = 0,
\]

yield?

(c) Show that

\[
\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0,
\]

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is consistent with the charge conservation law,

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \]

(d) Repeat a and b for the Coulomb gauge, namely, interpret

\[ \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = 0, \nabla \nabla \cdot \mathbf{A} = 0. \]