Chapter 4

Time Varying Fields, Simple Waves

4.1 Introduction

In this Chapter, the charge density ρ and current density $\mathbf{J}$ are generalized to be varying with time. The charge conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{4.1}$$

imposes a constraint between the two quantities ρ (charge density in C m^{-3}) and $\mathbf{J}$ (the current density in A m^{-2}) and various electrodynamic laws must be consistent with this basic law. A time varying magnetic flux induces an electric field through Faraday's law,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}. \tag{4.2}$$

Likewise, a time varying electric field induces a magnetic field through the displacement current,

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \varepsilon_0 \frac{d}{dt}\int_S \mathbf{E} \cdot d\mathbf{S}, \tag{4.3}$$

even in the absence of the current $\mathbf{J} = 0$. As is well known, the displacement current played a crucial role in Maxwell's prediction that electromagnetic waves can propagate through vacuum. The generalized set of Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{4.4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4.5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4.6}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \tag{4.7}$$

is consistent with the charge conservation law, since the divergence of the LHS of the last equation indeed vanishes (recall that $\nabla \cdot (\nabla \times \mathbf{B}) = 0$ identically) which requires that

$$\nabla \cdot \mathbf{J} + \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{4.8}$$

4.2 Faraday's Law

In 1831, Faraday discovered that an electric current was induced along a conductor loop when a magnetic flux enclosed by the loop changed with time. This important discovery gave an answer to the old question prior to Faraday's time whether a magnetic field could induce an electric field because it had been known that an electric field, via an electric current, could induce magnetic field. What Faraday found was that an electric field (or electromotive force, emf) was induced by a time varying magnetic flux. The integral form of the Faraday's law,

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}, \tag{4.9}$$

was later put into a differential form by Maxwell,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{4.10}$$

The negative sign is due to Lenz and indicates that the emf is so induced as to oppose the *change* in the magnetic flux. It should be noted that many of the experiments originally done by Faraday were actually due to motional emf in which motion of conductors across a magnetic field was responsible for generation of emf without apparent time variation of the magnetic field itself. An object moving in a magnetic field experiences an effective electric field given by

$$\mathbf{E}_{eff} = \mathbf{v} \times \mathbf{B}. \tag{4.11}$$

This may be seen by noting that change in the magnetic flux enclosed by a loop consists of two parts, one due to time variation of the magnetic field,

$$d\Psi_1 = \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} dt,$$

and the other due to the change in the shape of the loop,

$$\begin{aligned} d\Psi_2 &= \int \mathbf{B} \cdot \frac{\partial \mathbf{S}}{\partial t} dt \\ &= \oint \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}) dt \\ &= -\oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} dt. \end{aligned}$$

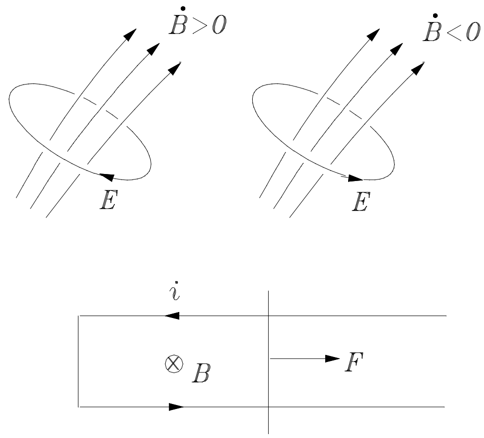


Figure 4-1: Induction of emf in time varying magnetic field (upper figures) and motional emf in static magnetic field (lower figure).

Then the total flux change is

$$\frac{d\Psi}{dt} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot dS - \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l},$$

and the emf induced along the loop is given by

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}. \tag{4.12}$$

An emf can be generated by letting a conductor move across a stationary magnetic field as done in most electric generators.

4.3 Displacement Current and Wave Equation

As shown in Introduction, the displacement current density

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

plays a crucial role for the Maxwell's equations to be consistent with the charge conservation law,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

The magnetic field induced by the displacement current may be best visualized in a capacitor being slowly charged as shown in Fig.4-2. The current flows on the surface of electrodes. The radially

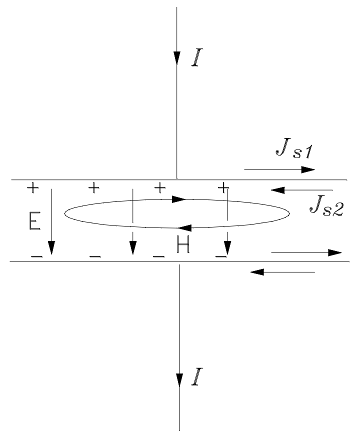


Figure 4-2: The surface currents on the capacitor electrode are consistent with the magnetic fields both outside and inside.

outward surface current on the outer surface of the upper plate,

$$J_{s1} = \frac{I}{2\pi\rho}, \tag{4.13}$$

is consistent with the boundary condition for the magnetic field,

$$\mathbf{n}_1 \times \mathbf{H} = \mathbf{J}_{s1}, \tag{4.14}$$

where

$$\mathbf{H} = -\frac{I}{2\pi\rho}\mathbf{e}_\phi, \tag{4.15}$$

is the magnetic field expected from the Ampere's law. In the space between the electrodes, no conduction current exists but there exists an azimuthal magnetic field,

$$B_\phi(\rho) = -\frac{1}{2}\mu_0\varepsilon_0\frac{\partial E_z}{\partial t}\rho, \tag{4.16}$$

as if there were a uniform conduction current equal to

$$J_z = \varepsilon_0\frac{\partial E_z}{\partial t}. \tag{4.17}$$

The magnetic field is required to exist to satisfy the boundary condition at the inner electrode surface,

$$\mathbf{n}_2 \times \mathbf{H} = \mathbf{J}_{s2}, \tag{4.18}$$

where

$$J_{s2} = -\frac{I}{2\pi a^2}\rho, \tag{4.19}$$

is the radially inward surface conduction current on the inner surface of the upper plate. Since

$$I = \frac{dq}{dt}, \tag{4.20}$$

and

$$E_z = -\frac{q}{\varepsilon_0 \pi a^2}, \tag{4.21}$$

we find

$$H_\phi = -\varepsilon_0 \frac{\partial E_z}{\partial t}\frac{1}{2}\rho. \tag{4.22}$$

This is consistent with the Maxwell's equation,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi) = \varepsilon_0 \frac{\partial E_z}{\partial t}, \tag{4.23}$$

or its integral form,

$$2\pi\rho H_\phi = \pi\rho^2 \varepsilon_0 \frac{\partial E_z}{\partial t}. \tag{4.24}$$

As is well known, the displacement current was instrumental for Maxwell to predict that electromagnetic fields obey a wave equation. To see what wave equations the electromagnetic fields should satisfy, let us take a curl of the Faraday's law,

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times (\nabla \times \mathbf{E}) &= -\frac{\partial}{\partial t}\nabla \times \mathbf{B}. \end{aligned} \tag{4.25}$$

The LHS can be expanded as

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

Since

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \ \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right),$$

Eq. (4.25) reduces to the following inhomogeneous wave equation,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = \frac{1}{\varepsilon_0}\nabla\rho + \mu_0 \frac{\partial \mathbf{J}}{\partial t}. \tag{4.26}$$

where c is the speed of light in vacuum,

$$c^2 = \frac{1}{\varepsilon_0 \mu_0}. \tag{4.27}$$

Likewise, the magnetic field obeys

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = -\mu_0 \nabla \times \mathbf{J}. \tag{4.28}$$

If the displacement current were absent (which, incidentally, is equivalent to the assumption that $c \to \infty$), both fields would merely satisfy vector Poisson's equation,

$$\nabla^2 \mathbf{E} = \frac{1}{\varepsilon_0}\nabla\rho + \mu_0 \frac{\partial \mathbf{J}}{\partial t},$$

$$\nabla^2 \mathbf{B} = -\mu_0 \nabla \times \mathbf{J},$$

which do not exhibit any propagation nature with a finite speed.

Equation (4.26) can be solved symbolically as

$$\mathbf{E} = \frac{1}{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}\left(\frac{1}{\varepsilon_0}\nabla\rho + \mu_0 \frac{\partial \mathbf{J}}{\partial t}\right), \tag{4.29}$$

where

$$\frac{1}{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}, \tag{4.30}$$

is the propagator integral operator which yields a retarded solution for the electric field,

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= -\frac{1}{4\pi\varepsilon_0}\int \frac{\nabla'\rho(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|}dV' - \frac{\mu_0}{4\pi}\int \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial}{\partial(t-\tau)}\mathbf{J}(\mathbf{r}', t-\tau)dV' \\
&= -\frac{1}{4\pi\varepsilon_0}\nabla\int \frac{\rho(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|}dV' - \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int \frac{\mathbf{J}(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|}dV',
\end{aligned} \tag{4.31}$$

where

$$\tau = \frac{|\mathbf{r}-\mathbf{r}'|}{c}, \tag{4.32}$$

is the time required for electromagnetic disturbances to propagate over a distance $|\mathbf{r}-\mathbf{r}'|$ between the source at $\mathbf{r}'$ and the observer at $\mathbf{r}$. Note that

$$\begin{aligned}
&\int \frac{\nabla'\rho(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|}dV' \\
&= \int \left[\nabla'\left(\frac{\rho(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}', t-\tau)\right]dV' \\
&= \nabla\int \frac{\rho(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|}dV'.
\end{aligned}$$

In Eq. (4.31),

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|}dV', \ \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|}dV',$$

are the retarded scalar and vector potential, respectively. For a single charged particle moving at a velocity $\mathbf{v}_p(t)$, the scalar potential becomes

$$\begin{aligned}\Phi(\mathbf{r},t) &= \frac{e}{4\pi\varepsilon_0}\int\frac{\delta\left[\mathbf{r}'-\mathbf{r}_p\left(t-\frac{1}{c}\left|\mathbf{r}-\mathbf{r}'\right|\right)\right]}{\left|\mathbf{r}-\mathbf{r}'\right|}dV' \\ &= \frac{e}{4\pi\varepsilon_0}\left.\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})\left|\mathbf{r}-\mathbf{r}_p\right|}\right|_{\text{ret}},\end{aligned} \tag{4.33}$$

where $\cdots|_{\text{ret}}$ means every time dependent quantity is to be evaluated at the retarded time t' to be determined implicitly from

$$t-t'-\frac{1}{c}\left|\mathbf{r}-\mathbf{r}_p(t')\right|=0.$$

Likewise, the vector potential due to a moving charge is

$$\mathbf{A}(\mathbf{r},t)=\frac{e\mu_0}{4\pi}\left.\frac{\mathbf{v}}{(1-\mathbf{n}\cdot\boldsymbol{\beta})\left|\mathbf{r}-\mathbf{r}_p\right|}\right|_{\text{ret}}. \tag{4.34}$$

These potentials were first formulated by Lienard and Wiechert. We will use the potentials in Chapter 8 in formulating the radiation electromagnetic fields due to moving charges.

Equation (4.26) can also be written as

$$\begin{aligned}\mathbf{E} &= \frac{1}{\nabla^2-\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}}\left(\frac{1}{\varepsilon_0}\nabla\rho+\mu_0\frac{\partial\mathbf{J}}{\partial t}\right) \\ &= \frac{1}{\nabla^2-\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}}\left(\frac{1}{\varepsilon_0}\nabla\rho+\mu_0\frac{\partial\mathbf{J}_l}{\partial t}+\mu_0\frac{\partial\mathbf{J}_t}{\partial t}\right),\end{aligned} \tag{4.35}$$

where $\mathbf{J}_l$ is the longitudinal component of the current density satisfying

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}_l=0, \tag{4.36}$$

and $\mathbf{J}_t=\mathbf{J}-\mathbf{J}_l$ is the transverse component which does not affect the charge density because of the identity (or definition)

$$\nabla\cdot\mathbf{J}_t=0. \tag{4.37}$$

Since $\nabla\times\mathbf{J}_l=0$, it follows

$$\nabla(\nabla\cdot\mathbf{J}_l)=\nabla^2\mathbf{J}_l,$$

and a symbolic solution for the longitudinal current density is

$$\mathbf{J}_l=-\frac{1}{\nabla^2}\nabla\frac{\partial\rho}{\partial t}. \tag{4.38}$$

Substituting into Eq. (4.35), we obtain

$$\mathbf{E} = \frac{1}{\varepsilon_0 \nabla^2}\nabla\rho + \frac{1}{\nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}}\mu_0\frac{\partial \mathbf{J}_t}{\partial t}. \tag{4.39}$$

This formulation is consistent with the choice of Coulomb gauge for the scalar and vector potentials as we will see in more detail in Chapter 6. Solution for the electric field is

$$\mathbf{E}(\mathbf{r}, t) = -\frac{1}{4\pi\varepsilon_0}\nabla\int\frac{\rho(\mathbf{r}', t)}{|\mathbf{r}-\mathbf{r}'|}dV' - \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int\frac{\mathbf{J}_t(\mathbf{r}', t-\tau)}{|\mathbf{r}-\mathbf{r}'|}dV'. \tag{4.40}$$

Note that in this formulation, the Coulomb electric field (the first term in the RHS) is non-retarded and instantaneous. This unphysical result is a consequence of the Coulomb gauge which singles out the transverse current $\mathbf{J}_t$. In fact, the non-retarded longitudinal electric filed is cancelled by a term contained in the last term and all physically observable electromagnetic fields are retarded because of the finite propagation speed c.

4.4 Fields and Potentials

By now, it is clear that there exists two kinds of electric field, one defined in terms of the scalar potential,

$$\mathbf{E}_1 = -\nabla\Phi, \tag{4.41}$$

and another originating from time varying magnetic field,

$$\nabla\times\mathbf{E}_2 = -\frac{\partial\mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla\times\mathbf{A}. \tag{4.42}$$

The field $\mathbf{E}_1$ is longitudinal because $\nabla\times\mathbf{E}_1 = 0$. In Eq. (4.42), the longitudinal components of $\mathbf{E}$ and $\mathbf{A}$ vanish. In fact Maxwell's equations do not impose any conditions on the longitudinal component of $\mathbf{A}$, and choice of $\nabla\cdot\mathbf{A}$ is arbitrary. Therefore, we are allowed to assume

$$\begin{aligned}\mathbf{E} &= -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A}\\ &= -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A}_l - \frac{\partial}{\partial t}\mathbf{A}_t,\end{aligned}$$

where $\mathbf{A}_l$ and $\mathbf{A}_t$ are the longitudinal and transverse component of the vector potential. Substitution in $\nabla\cdot\,\mathbf{E} = \rho/\varepsilon_0$ yields

$$\nabla\cdot\mathbf{E} = -\nabla^2\Phi - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A}_l = \frac{\rho}{\varepsilon_0}. \tag{4.43}$$

In Coulomb gauge, $\mathbf{A}_l$ is so chosen as to satisfy

$$\text{Coulomb gauge: } \nabla\cdot\mathbf{A}_l = 0. \tag{4.44}$$

(Note that $\mathbf{A}_l$ may not be zero identically.) Eq. (4.43) becomes

$$\nabla^2 \Phi_C(\mathbf{r},t) = -\frac{\rho(\mathbf{r},t)}{\varepsilon_0}, \tag{4.45}$$

where Φ_C is the scalar potential in the Coulomb gauge. It should be emphasized that both Φ_C and ρ are time dependent and in Coulomb gauge the scalar potential responds to a change in the charge density instantaneously. In Lorenz gauge, $\nabla \cdot \mathbf{A}_l$ is assigned as

$$\text{Lorenz gauge: } \nabla \cdot \mathbf{A}_l + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0. \tag{4.46}$$

In this case, the scalar potential obeys a wave equation,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi_L = -\frac{\rho}{\varepsilon_0}. \tag{4.47}$$

This yields a retarded solution,

$$\Phi_L(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho\left(\mathbf{r}', t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right) dV'. \tag{4.48}$$

Proof is straightforward if the following are noted:

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta\left(|\mathbf{r}-\mathbf{r}'|\right),$$

$$\nabla^2 \rho\left(\mathbf{r}', t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\rho\left(\mathbf{r}', t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right).$$

For the vector potential, we rewrite the Ampere-Maxwell's law

$$\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right),$$

in terms of the potentials as

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left((\nabla \cdot \mathbf{A}) + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \mathbf{J}. \tag{4.49}$$

In Coulomb gauge with $\nabla \cdot \mathbf{A} = 0$, this reduces to

$$\begin{aligned} \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\Phi \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= \mu_0 \mathbf{J}_t, \end{aligned} \tag{4.50}$$

since the longitudinal component of the current $\mathbf{J}_l$ vanishes through the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_l = 0.$$

Note that

$$\frac{\partial\rho}{\partial t} = \varepsilon_0\frac{\partial}{\partial t}\nabla\cdot\mathbf{E} = -\varepsilon_0\frac{\partial}{\partial t}\nabla^2\Phi_C,$$

and thus

$$\mathbf{J}_l - \varepsilon_0\frac{\partial}{\partial t}\nabla\Phi_C = 0.$$

In Coulomb gauge, the vector potential is transverse.

In Lorenz gauge, Eq. (4.49) becomes a wave equation for $\mathbf{A}$,

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}, \tag{4.51}$$

where $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$, $\mathbf{A} = \mathbf{A}_l + \mathbf{A}_t$. In Lorenz gauge, the potentials are symmetric in the sense that both Φ and $\mathbf{A}$ satisfy the same wave equation,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\Phi/c\\ \mathbf{A}\end{pmatrix} = -\mu_0\begin{pmatrix}c\rho\\ \mathbf{J}\end{pmatrix}, \tag{4.52}$$

and appropriately form a covariant four vector $(\Phi/c, \mathbf{A})$. In contrast, the formulation of electromagnetic fields in terms of Coulomb gauge is not invariant under the Lorentz transformation. All potentials and fields are retarded in Lorenz gauge while in the Coulomb gauge, the scalar potential is non-retarded. The appearance of non-retarded scalar potential is due to the choice $\nabla\cdot\mathbf{A} = 0$, or the assumption that the vector potential is purely transverse.

4.5 Poynting Vector: Energy and Momentum Conservation

The complete set of Maxwell's equations is

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0},$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t},$$

$$\nabla\cdot\mathbf{B} = 0,$$

$$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right).$$

It is noted that the charge density ρ contains all kinds of charges, free charges, bound charges, etc. Likewise, the current density $\mathbf{J}$ contains all kinds of current, conduction currents, magnetization currents, etc. If free charges are singled out, the first equation can be written in terms of the displacement vector $\mathbf{D}$,

$$\nabla\cdot\mathbf{D} = \rho_{\text{free}}, \tag{4.53}$$

where

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E}, \tag{4.54}$$

with $\mathbf{P}$ being the polarization vector, or the electric dipole moment density. The current density is likewise decomposed into the part due to the motion of free charges and the other due to time variation of the polarization vector,

$$-\frac{\partial}{\partial t}\nabla\cdot\mathbf{P}+\nabla\cdot\mathbf{J}_p=0,$$

from which

$$\mathbf{J}_p=\frac{\partial\mathbf{P}}{\partial t}. \tag{4.55}$$

Therefore, the magnetic induction equation can be rewritten as

$$\nabla\times\mathbf{B}=\mu_0\left(\mathbf{J}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}+\frac{\partial\mathbf{P}}{\partial t}\right)=\mu_0\left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\right), \tag{4.56}$$

where the current density $\mathbf{J}$ consists of conduction and magnetization currents,

$$\mathbf{J}=\mathbf{J}_c+\mathbf{J}_m=\mathbf{J}_c+\nabla\times\mathbf{M}.$$

If the conduction current is singled out, Eq. (4.56) can be rewritten in terms of the vector $\mathbf{H}$,

$$\nabla\times\mathbf{H}=\mathbf{J}_c+\frac{\partial\mathbf{D}}{\partial t}, \tag{4.57}$$

where

$$\mathbf{H}=\frac{\mathbf{B}}{\mu_0}-\mathbf{M}.$$

In macroscopic applications, free charges and conduction currents are the quantities that can be controlled by external means and Eqs. (4.53) and (4.57) are often more convenient than the original forms.

The Maxwell's equations are also consistent with energy conservation law. To show this, we define the Poynting vector by

$$\mathbf{E}\times\mathbf{H},\ (\mathrm{W\ m^{-2}}). \tag{4.58}$$

As its dimensions imply, the Poynting vector indicates the flow of electromagnetic energy per unit area per unit time, that is, power density. The divergence of the Poynting vector is

$$\begin{aligned}\nabla\cdot(\mathbf{E}\times\mathbf{H}) &= (\nabla\times\mathbf{E})\cdot\mathbf{H}-(\nabla\times\mathbf{H})\cdot\mathbf{E}\\ &= -\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H}-\left(\mathbf{J}_c+\frac{\partial\mathbf{D}}{\partial t}\right)\cdot\mathbf{E}\\ &= -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2+\frac{1}{2}\varepsilon E^2\right)-\mathbf{E}\cdot\mathbf{J},\end{aligned}$$

provided that the permittivity ε and permeability μ are independent of the frequency ω. If not, the energy densities should be modified as

$$\frac{1}{2}\frac{\partial\left[\omega\varepsilon\left(\omega\right)\right]}{\partial\omega}E^2,\ \frac{1}{2}\frac{\partial\left[\omega\mu\left(\omega\right)\right]}{\partial\omega}H^2, \tag{4.59}$$

respectively. We will return to this problem shortly in the following Section. Integrating over a volume, we find

$$\int_V \nabla\cdot(\mathbf{E}\times\mathbf{H})\,dV = \oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} = -\frac{d}{dt}(U_m+U_e) - \int \mathbf{E}\cdot\mathbf{J}dV, \tag{4.60}$$

where

$$U_e = \frac{1}{2}\int \varepsilon E^2 dV, \quad U_m = \frac{1}{2}\int \mu H^2 dV, \tag{4.61}$$

are the total electric and magnetic energies stored in the volume and

$$\int \mathbf{E}\cdot\mathbf{J}dV,\ (\text{J s}^{-1} = \text{W}) \tag{4.62}$$

is the rate of electromagnetic energy conversion into other forms of energy, e.g., creation of heat through Joule dissipation and acceleration of charged particles. Therefore,

$$-\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S}, \tag{4.63}$$

can be interpreted as the electromagnetic power flow *into* the volume.

For a system consisting of charged particles, the momentum conservation can be shown in a similar way. The mechanical momentum $\mathbf{P}_m$ follows the equation of motion,

$$\frac{d\mathbf{P}_m}{dt} = \int (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})\,dV,\ (\text{N}). \tag{4.64}$$

Substituting

$$\rho = \varepsilon_0\nabla\cdot\mathbf{E}$$

and

$$\mathbf{J} = \frac{1}{\mu_0}\nabla\times\mathbf{B} - \varepsilon_0\frac{\partial\mathbf{E}}{\partial t},$$

we find

$$\begin{aligned}\frac{d\mathbf{P}_m}{dt} &= \int \left[\varepsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla\times\mathbf{B} - \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B}\right]dV \\ &= \int \left(\nabla\cdot\overleftrightarrow{\mathbf{T}} - \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}\times\mathbf{H}\right)dV,\end{aligned} \tag{4.65}$$

where

$$\overleftrightarrow{\mathbf{T}} = T_{ij} = \varepsilon_0 E_i E_j - \frac{1}{2}\varepsilon_0 E^2\delta_{ij} + \frac{1}{\mu_0}B_iB_j - \frac{1}{2\mu_0}B^2\delta_{ij},\ (\text{N m}^{-2}) \tag{4.66}$$

is the Maxwell's stress tensor and use is made of the following vector identities,

$$\nabla E^2 = \nabla(\mathbf{E}\cdot\mathbf{E}) = 2\mathbf{E}\times(\nabla\times\mathbf{E}) + 2(\mathbf{E}\cdot\nabla)\mathbf{E},$$

$$\nabla\cdot(\mathbf{E}\mathbf{E}) = (\nabla\cdot\mathbf{E})\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{E}$$

$$\nabla B^2 = \nabla(\mathbf{B}\cdot\mathbf{B}) = 2\mathbf{B}\times(\nabla\times\mathbf{B}) + 2(\mathbf{B}\cdot\nabla)\mathbf{B}$$

$$\nabla\cdot(\mathbf{B}\mathbf{B}) = (\nabla\cdot\mathbf{B})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{B}.$$

Eq. (4.65) suggests that the vector

$$\frac{1}{c^2}\mathbf{E}\times\mathbf{H},\ (\mathrm{N\ s\ m^{-3}}) \tag{4.67}$$

can be regarded as the electromagnetic momentum density and

$$\frac{1}{c}\mathbf{E}\times\mathbf{H},\ (\mathrm{N\ m^{-2}}) \tag{4.68}$$

as the electromagnetic momentum flux density. For a system of charged particles, momentum conservation thus requires inclusion of the electromagnetic momentum as well as mechanical momentum.

From the momentum density in Eq. (4.67), the angular momentum density associated with electromagnetic fields is naturally defined by

$$\frac{1}{c^2}\mathbf{r}\times(\mathbf{E}\times\mathbf{H}),\ (\mathrm{J\ s\ m^{-3}}) \tag{4.69}$$

and the total angular momentum associated with electromagnetic fields by

$$\mathbf{L} = \frac{1}{c^2}\int \mathbf{r}\times(\mathbf{E}\times\mathbf{H})dV,\ (\mathrm{J\ s}). \tag{4.70}$$

Its flux density is

$$\mathcal{R} = \frac{1}{c}\mathbf{r}\times(\mathbf{E}\times\mathbf{H}),\ (\mathrm{J\ m^{-2}}). \tag{4.71}$$

Both the momentum and angular momentum densities are proportional to the Poynting vector, namely, the energy flow. If for example a system is losing energy through radiation of electromagnetic energy, the system is necessarily losing momentum and angular momentum as well. Consider a charged particle undergoing circular motion, e.g., electron in a magnetic field. Since the electron is continuously accelerated by the centripetal force, it radiates electromagnetic energy. At the same time it loses its angular momentum to radiation. Therefore, it is natural to expect that electromagnetic fields (with proper polarization) carry an angular momentum with them.

4.6 Plane Electromagnetic Waves

One special mode of electromagnetic waves in free space is a plane wave in which the amplitude of electric and magnetic field remains constant. Without loss of generality, we may assume wave propagation in the z direction and an electric field in the x direction,

$$\mathbf{E}(z,t) = E_0 e^{i(kz-\omega t)}\mathbf{e}_x.$$

In free space, the electric field satisfies the wave equation,

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(z,t) = 0, \tag{4.72}$$

provided

$$\frac{\omega}{k} = c. \tag{4.73}$$

From

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t},$$

we find

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}, \tag{4.74}$$

where $\mathbf{k} = k\mathbf{e}_z$. The magnetic field is thus in the y direction,

$$\mathbf{B} = \frac{k}{\omega}E_0 e^{i(kz-\omega t)}\mathbf{e}_y, \tag{4.75}$$

and its amplitude is

$$B_0 = \frac{E_0}{c}, \text{ or } H_0 = \frac{E_0}{c\mu_0} = \frac{E_0}{Z}, \tag{4.76}$$

where

$$Z = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.8, \ (\Omega) \tag{4.77}$$

is the impedance of free space.

The electric and magnetic energy densities associated with a plane wave are the same, for

$$\frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 Z^2 H^2 = \frac{1}{2}\mu_0 H^2. \tag{4.78}$$

This equipartition of wave energy is similar to that in mechanical waves in which kinetic and potential energies are equal. The total wave energy density is therefore given by

$$u = 2\times\frac{1}{2}\varepsilon_0 E^2 = \varepsilon_0 E^2, \ (\text{J m}^{-3}) \tag{4.79}$$

and the Poynting flux may be written in terms of either the electric or magnetic field as

$$S_z = E_x H_y^* = c\varepsilon_0 E_x^2 = \frac{E_x^2}{Z}, \text{ (W m}^{-2}\text{)} \tag{4.80}$$

or

$$S_z = c\mu_0 H_y^2 = Z H_y^2. \tag{4.81}$$

For a harmonic wave with an amplitude E_0, the average (rms) wave energy density is given by

$$u_{\text{ave}} = \frac{1}{2}\varepsilon_0 E_0^2, \text{ (W m}^{-2}\text{)} \tag{4.82}$$

and corresponding rms Poynting flux is

$$S_{z\text{ave}} = \frac{1}{2} c\varepsilon_0 E_0^2 = \frac{1}{2}\frac{E_0^2}{Z}, \text{ (W m}^{-2}\text{)}. \tag{4.83}$$

Electromagnetic waves radiated by a localized source approach plane waves at a sufficiently large distance but they can never be pure plane waves. Plane waves can be constructed from two circularly polarized waves with opposite helicities, one rotating with positive helicity and another with negative helicity. Helicity of electromagnetic waves is related to the angular momentum associated with the waves. Evidently, a plane polarized wave carry zero angular momentum. A more general theory of electromagnetic radiation will be developed in Chapter 5.

In a dielectric medium, the Maxwell's equations are modified as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{B} = \mu_0 \frac{\partial (\varepsilon \mathbf{E})}{\partial t}, \tag{4.84}$$

where ε is the permittivity which in general depends on the wave frequency and spatial position and also the electric field. The origin of the permittivity is in the current induced by the electric field in a material medium. In the magnetic induction equation

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right),$$

if the current density is proportional to the electric field through a conductivity σ, $\mathbf{J} = \sigma \mathbf{E}$, we have

$$\nabla \times \mathbf{B} = \mu_0 \left(\sigma \mathbf{E} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = \mu_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{4.85}$$

where the permittivity is given by

$$\varepsilon = \varepsilon_0 \left(1 + i \frac{\sigma}{\omega \varepsilon_0} \right). \tag{4.86}$$

It should be noted that the current density is due to deviation of electron orbit from bound harmonic motion in molecules. The equation of motion for an electron placed in an oscillating electric field is

$$m\left(\frac{d^2}{dt^2}+\omega_0^2\right)x=-eE_0e^{-i\omega t}, \tag{4.87}$$

where m is the electron mass and ω_0 is the frequency of bound harmonic motion. The current density is

$$J=-ne\frac{dx}{dt}=\frac{i\omega}{\omega^2-\omega_0^2}\frac{eE}{m},$$

and the conductivity becomes

$$\sigma=\frac{i\omega}{\omega^2-\omega_0^2}\frac{ne^2}{m}, \tag{4.88}$$

where n is the number density of electrons. Then the permittivity is given by

$$\varepsilon(\omega)=\varepsilon_0\left(1-\frac{\omega_p^2}{\omega^2-\omega_0^2}\right), \tag{4.89}$$

where ω_p is the "plasma" frequency defined by

$$\omega_p^2=\frac{ne^2}{\varepsilon_0 m}. \tag{4.90}$$

The phase velocity of electromagnetic waves in a dielectric is

$$\frac{\omega}{k}=\frac{1}{\sqrt{\varepsilon(\omega)\mu_0}}, \tag{4.91}$$

and the group velocity is

$$\frac{d\omega}{dk}=\frac{1}{1+\frac{1}{2}\frac{\omega}{\varepsilon(\omega)}\frac{d\varepsilon}{d\omega}}\frac{\omega}{k}. \tag{4.92}$$

(At frequencies remotely separated from the resonance frequency ω_0, the group velocity coincides with the energy propagation velocity. Near the resonance, however, the group velocity exceeds c and it loses the meaning of energy propagation velocity. The concept of signal velocity was introduced by Brillouin.) The impedance is accordingly modified as

$$Z(\omega)=\frac{E}{H}=\sqrt{\frac{\mu_0}{\varepsilon(\omega)}}. \tag{4.93}$$

By definition, the group velocity is equal to

$$\frac{d\omega}{dk}=\frac{\text{Poynting flux}}{\text{Energy density}},$$

where the Poynting flux is

$$S = \frac{E^2}{Z} = \sqrt{\frac{\varepsilon(\omega)}{\mu_0}} E^2, \ \ (\text{W m}^{-2}).$$

Therefore the wave energy density in a dielectric medium is

$$\begin{aligned} u &= \left(1 + \frac{1}{2}\frac{\omega}{\varepsilon(\omega)}\frac{d\varepsilon}{d\omega}\right)\varepsilon(\omega)E^2 \\ &= \frac{1}{2}\frac{d}{d\omega}[\omega\varepsilon(\omega)]E^2 + \frac{1}{2}\mu_0 H^2 \\ &= \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{\omega_p^2(\omega^2+\omega_0^2)}{(\omega^2-\omega_0)^2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0 H^2, \ \ (\text{J m}^{-3}) \end{aligned}$$

where the relationship

$$\frac{1}{2}\varepsilon(\omega)E^2 = \frac{1}{2}\mu_0 H^2, \tag{4.94}$$

is used. This result is valid only if the group velocity can be regarded as energy propagation velocity which may not hold near the resonance $\omega \simeq \omega_0$ if the dielectric is dissipative.

The origin of the additional factor

$$\frac{1}{2}\omega\frac{d\varepsilon}{d\omega}E^2,$$

in the electric energy density is due to electron kinetic and potential energies in an oscillating electric field. From the equation of motion of a bound electron,

$$\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right)x = -\frac{e}{m}E,$$

we readily find the electron kinetic energy density,

$$\frac{1}{2}nmv^2 = \frac{1}{2}\frac{ne^2}{m}\frac{\omega^2}{(\omega^2-\omega_0)^2}E^2 = \frac{1}{2}\frac{\omega^2\omega_p^2}{(\omega^2-\omega_0)^2}\varepsilon_0 E^2, \tag{4.95}$$

and potential energy density

$$\frac{1}{2}nm\omega_0^2 x^2 = \frac{1}{2}\frac{\omega_0^2\omega_p^2}{(\omega^2-\omega_0)^2}\varepsilon_0 E^2. \tag{4.96}$$

Therefore, the total energy density associated with the electric field is

$$\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{\omega^2\omega_p^2}{(\omega^2-\omega_0)^2}\varepsilon_0 E^2 + \frac{1}{2}\frac{\omega_0^2\omega_p^2}{(\omega^2-\omega_0)^2}\varepsilon_0 E^2, \ \ (\text{J m}^{-3}) \tag{4.97}$$

which is consistent with that conveniently calculated from

$$\frac{1}{2}\frac{d}{d\omega}[\omega\varepsilon(\omega)]E^2. \tag{4.98}$$

Since the wave under consideration is strongly dispersive, there is no simple energy equipartition as in the case of nondispersive waves. Note that the electron kinetic and potential energies are the result of forced oscillations.

The fact that

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=\mathbf{E}\cdot\frac{\partial\left(\varepsilon\mathbf{E}\right)}{\partial t}, \tag{4.99}$$

is not always equal to

$$\frac{1}{2}\varepsilon\frac{\partial E^2}{\partial t}, \tag{4.100}$$

can be seen if we realize that the permittivity ε is in general frequency dependent. Waves become dispersive and wave amplitude is bound to decrease although slowly. In this case,

$$\begin{aligned}\frac{\partial\left(\varepsilon\mathbf{E}\right)}{\partial t} &= -i\omega\varepsilon\left(\omega\right)\mathbf{E}_0e^{-i\omega t}\\ &\simeq \left(-i\omega_0+\gamma\right)\left[\varepsilon\left(\omega_0\right)+i\gamma\frac{\partial\varepsilon}{\partial\omega_0}\right]\mathbf{E}_0e^{-i\omega t}\\ &= -i\omega_0\varepsilon\left(\omega_0\right)\mathbf{E}_0e^{-i\omega t}+\left[\varepsilon\left(\omega_0\right)+\omega_0\frac{\partial\varepsilon}{\partial\omega_0}\right]\frac{d\mathbf{E}_0}{dt}e^{-i\omega t},\end{aligned} \tag{4.101}$$

where γ (<0) is the damping rate of the amplitude,

$$\frac{d\mathbf{E}_0}{dt}=\gamma\mathbf{E}_0. \tag{4.102}$$

Therefore, time variation of electric energy density should be calculated from

$$\mathbf{E}\cdot\frac{\partial\left(\varepsilon\mathbf{E}\right)}{\partial t}=\frac{1}{2}\left[\varepsilon\left(\omega_0\right)+\omega_0\frac{\partial\varepsilon}{\partial\omega_0}\right]\frac{dE^2}{dt}, \tag{4.103}$$

and the electric energy density becomes

$$u_e=\frac{1}{2}\left[\varepsilon\left(\omega_0\right)+\omega_0\frac{\partial\varepsilon}{\partial\omega_0}\right]E^2. \tag{4.104}$$

Likewise, the magnetic energy density should be generalized as

$$u_m=\frac{1}{2}\left[\mu\left(\omega_0\right)+\omega_0\frac{\partial\mu}{\partial\omega_0}\right]H^2. \tag{4.105}$$

These expressions were originally formulated by von Laue.

4.7 Wave Reflection and Transmission - Normal Incidence

Reflection and transmission of plane electromagnetic waves at a boundary of two dielectrics can be conveniently analyzed in terms of impedance mismatch. Let a plane wave of amplitude E_i in air be incident normally on a flat dielectric surface of impedance Z_2 as shown in Fig.4-3. The impedance of air is very close to the vacuum impedance, $Z_1=377\ \Omega$. The incident Poynting flux is split into those of reflected (E_r) and transmitted (E_t) waves,

$$\frac{E_i^2}{Z_1} = \frac{E_r^2}{Z_1} + \frac{E_t^2}{Z_2}, \text{ (W m}^{-2}\text{)}. \tag{4.106}$$

At the boundary, continuity of the electric fields, which are all tangential to the surface, yields

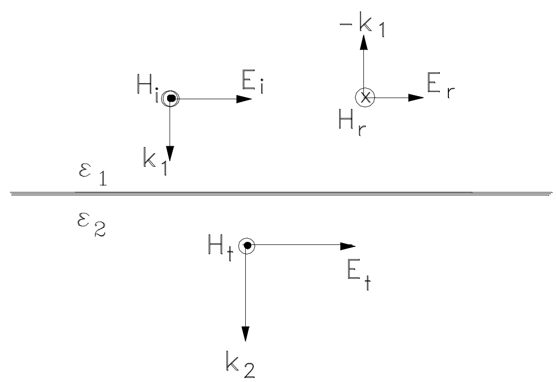


Figure 4-3: A plane wave incident normal to a dielectric boundary.

$$E_i + E_r = E_t. \tag{4.107}$$

Solving these equations for E_r and E_t, we find

$$E_r = \frac{Z_2 - Z_1}{Z_2 + Z_1} E_i, \tag{4.108}$$

$$E_t = \frac{2Z_2}{Z_2 + Z_1} E_i. \tag{4.109}$$

In terms of the magnetic fields,

$$Z_1 H_i^2 = Z_1 H_r^2 + Z_2 H_t^2, \tag{4.110}$$

$$H_i + H_r = H_t, \tag{4.111}$$

yield

$$H_r = \frac{Z_1 - Z_2}{Z_1 + Z_2} H_i, \quad H_t = \frac{2Z_1}{Z_1 + Z_2} H_i. \tag{4.112}$$

Note that the polarity of either electric or magnetic field of the reflected wave is reversed. This is understandable because the incident and reflected fields must satisfy the following vectorial relationships,

$$\mathbf{k} \times \mathbf{E}_i = \omega\mu_0 \mathbf{H}_i,$$

$$-\mathbf{k} \times \mathbf{E}_r = \omega\mu_0 \mathbf{H}_r.$$

Either E_r or H_r must change sign on reflection for the reflected Poynting flux to be in opposite direction relative to the incident flux.

It should be pointed out that the analysis presented above is solely based on energy conservation and no consideration was given to momentum conservation. In fact, if only the momenta associated with the three waves (incident, reflected, and transmitted) are considered, momentum is not conserved. Since the momentum flux densities associated with each wave are

$$\begin{aligned} \text{Incident wave} &: \frac{1}{c_1}\frac{E_i^2}{Z_1} = \varepsilon_1 E_i^2, \\ \text{Reflected wave} &: \varepsilon_1 E_r^2, \\ \text{Transmitted} &: \varepsilon_2 E_t^2, \end{aligned} \tag{4.113}$$

the following momentum unbalance emerges,

$$\varepsilon_1(E_i^2 + E_r^2) - \varepsilon_2 E_t^2 = \frac{2(Z_1^2 - Z_2^2)}{(Z_1 + Z_2)^2}\varepsilon_1 E_i^2 = \frac{2(Z_1 - Z_2)}{Z_1 + Z_2}\varepsilon_1 E_i^2. \tag{4.114}$$

This is actually taken up by the dielectric body as mechanical momentum. Recall that an infinitely massive body can absorb momentum without absorbing energy. If $Z_2 > Z_1$, the dielectric is pushed to the left. In fact, at the boundary, the electric energy densities are discontinuous, and the difference is

$$\begin{aligned} \frac{1}{2}\varepsilon_2 E_t^2 - \frac{1}{2}\varepsilon_1 E_t^2 &= (\varepsilon_2 - \varepsilon_1)\frac{2Z_2^2}{(Z_1 + Z_2)^2}E_i^2 \\ &= \frac{2(Z_1^2 - Z_2^2)}{(Z_1 + Z_2)^2}\varepsilon_1 E_i^2 \\ &= \frac{2(Z_1 - Z_2)}{Z_1 + Z_2}\varepsilon_1 E_i^2. \end{aligned}$$

This appears as a force per unit area on the boundary surface acting from the higher energy density side to the lower because electric force in the direction perpendicular to the field appears as pressure.

Reflection at a conductor surface can be analyzed in a similar manner by modifying the impedance appropriately. For a medium having a conductivity σ, the impedance is given by

$$Z = \sqrt{\frac{-i\omega\mu_0}{\sigma - i\omega\varepsilon_0}}. \tag{4.115}$$

This can be seen from the Maxwell's equation in the presence of conduction current,

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\ &= \sigma\mathbf{E} - i\omega\varepsilon_0\mathbf{E}, \end{aligned}$$

and the effective permittivity in a conductor is defined by

$$\varepsilon_{\text{eff}} = \varepsilon_0 - \frac{\sigma}{i\omega}.$$

Therefore, the impedance in a conductor is

$$Z = \sqrt{\frac{\mu_0}{\varepsilon_{\text{eff}}}} = \sqrt{\frac{-i\omega\mu_0}{\sigma - i\omega\varepsilon_0}}.$$

For ordinary conductors, the conduction current far dominates over the displacement current even in microwave frequency range. Then,

$$Z \simeq \sqrt{\frac{-i\omega\mu_0}{\sigma}} = (1-i)\sqrt{\frac{\omega\mu_0}{2\sigma}}. \tag{4.116}$$

A complex impedance indicates strong dissipation of electromagnetic energy. The magnitude of the impedance

$$|Z| = \sqrt{\frac{\omega\mu_0}{\sigma}}, \tag{4.117}$$

is much smaller than the free space impedance $Z_0 = \sqrt{\mu_0/\varepsilon_0} \simeq 377\ \Omega$ and electromagnetic waves incident on a conductor surface suffers strong reflection. However, reflection can never be complete. Bath room mirrors coated with aluminum has power reflection coefficient of about 90% at optical frequency $\omega \simeq 10^{14}$ rad/sec.

Example 1 ***Impedance Matching***

A dielectric film a quarter wavelength thick coated on a surface of another dielectric (say, optical glass) can eliminate reflection of electromagnetic waves normally incident if the impedance of the film is chosen to be a geometric mean,

$$Z_f = \sqrt{Z_0 Z_g}. \tag{4.118}$$

This condition follows from cancellation between two reflected waves, one at the air-film boundary and another at the film-glass boundary,

$$\frac{Z_f - Z_0}{Z_f + Z_0} + e^{i\pi}\frac{Z_g - Z_f}{Z_g + Z_f} = 0,$$

where the phase factor $e^{i\pi} = -1$ is due to additional propagation distance $\frac{\lambda_f}{4} \times 2 = \frac{\lambda_f}{2}$ of the wave reflected at the film-glass boundary. Here, λ_f is the wavelength in the dielectric film. Solving

$$\frac{Z_f - Z_0}{Z_f + Z_0} = \frac{Z_g - Z_f}{Z_g + Z_f},$$

for Z_f, we find

$$Z_f = \sqrt{Z_0 Z_g}.$$

Likewise, reflection from a conductor plate can be avoided by placing a thin conducting plate of thickness d at a quarter wavelength in front of the conductor surface if the conductance of the plate is chosen to be

$$\sigma = \frac{1}{Z_0 d}. \tag{4.119}$$

4.8 Reflection and Transmission at Arbitrary Incident Angle

For an arbitrary incident angle θ_1, reflection and transmission at a dielectric boundary can be analyzed by exploiting the boundary conditions for the electric and magnetic fields as follows. An incident wave continues to be assumed plane polarized. The refracted angle θ_2 is related to the incident angle θ_1 through the well known Snell's law,

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}, \tag{4.120}$$

where $n_1 = \sqrt{\varepsilon_1/\varepsilon_0}$ and $n_2 = \sqrt{\varepsilon_2/\varepsilon_0}$ are the indices of refraction of respective media. The Snell's law follows from the conservation of the wavenumber parallel to the boundary,

$$k_1 \sin\theta_1 = k_2 \sin\theta_2, \tag{4.121}$$

and the change in the wave propagation velocity,

$$\frac{\omega}{k_1} = \frac{1}{\sqrt{\varepsilon_1 \mu_0}}, \quad \frac{\omega}{k_2} = \frac{1}{\sqrt{\varepsilon_2 \mu_0}}, \tag{4.122}$$

or

$$\frac{k_1}{n_1} = \frac{k_2}{n_2}. \tag{4.123}$$

The normal wavenumber in the medium 2 is

$$k_z = k_2 \cos\theta_2 = k_2 \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1}.$$

This becomes pure imaginary when

$$\sin\theta_1 > \frac{n_2}{n_1}, \ n_2 < n_1,$$

This is the condition for total reflection which occurs when a wave is incident on a medium with a smaller index of refraction (e.g., from glass to air), $n_2 < n_1$. An imaginary wavenumber indicates exponential damping in the region n_2 from the surface,

$$E_0 e^{-k_z z},$$

where k_z is the damping factor in the axial (z) direction,

$$k_z = k_2 \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1 - 1}.$$

The wavenumber component along the surface is

$$k_\parallel = k_2 \frac{n_1}{n_2} \sin\theta.$$

If the second medium is vacuum (or air) $n_2 = 1$, $k_2 = \omega/c = k_0$, and

$$k_\parallel = k_0 n_1 \sin\theta.$$

In total reflection, electromagnetic fields in the region of smaller index of refraction exponentially decay from the surface. Such waves are called "evanescent." Important applications of evanescent waves are being found in high resolution microscopy even with visible light.

In analysis of wave reflection and transmission at a boundary between different media, use of the continuity of tangential electric and magnetic fields, $\mathbf{E}_t, \mathbf{H}_t$, is sufficient because continuity of normal components $\mathbf{D}_n, \mathbf{B}_n$ is redundant. From Maxwell's equation,

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \text{ or } \mathbf{k} \times \mathbf{H} = -\omega \mathbf{D},$$

it is evident that continuity of $\mathbf{D}_n$ demands continuity of $\mathbf{n} \cdot (\mathbf{k} \times \mathbf{H}) = -\mathbf{k} \cdot (\mathbf{n} \times \mathbf{H})$. However, the tangential component of $\mathbf{H}$, $\mathbf{n} \times \mathbf{H}$, is continuous and the wavenumber parallel to the boundary is also continuous. (Note that in $\mathbf{k} \cdot (\mathbf{n} \times \mathbf{H}) = (\mathbf{k} \times \mathbf{n}) \cdot \mathbf{H}$, only the parallel component of $\mathbf{k}$ appears.) The latter (continuity of $\mathbf{k} \times \mathbf{n}$) follows from the conservation of wave momentum parallel to the boundary which manifests itself in the form of well known Snell's law. Likewise, from

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \text{ or } \mathbf{k} \times \mathbf{E} = \omega \mathbf{B},$$

we see that continuity of $\mathbf{B}_n$ automatically follows because of continuity of tangential component of the electric field.

4.8.1 H in the Incident Plane (E tangential to the boundary)

We first consider the case in which the magnetic field of the incident wave is in the incident plane, or the electric field is parallel to the boundary surface. The reflected and refracted electric fields are also parallel to the surface and the continuity of the tangential component of the electric field gives

$$E_i + E_r = E_t. \tag{4.124}$$

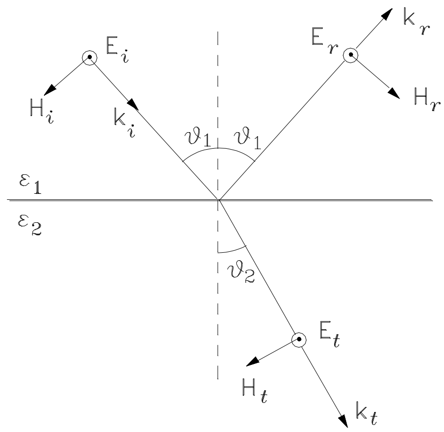


Figure 4-4: A plane wave with incident angle θ_1 when the magnetic field is in the incidence plane and the electric field is parallel to the boundary surface.

Since for each wave, the following vectorial relationship holds,

$$\mathbf{k} \times \mathbf{E} = \mu_0 \omega \mathbf{H}, \tag{4.125}$$

the continuity of tangential component of the $\mathbf{H}$ field yields

$$k_1 \cos\theta_1 (E_i - E_r) = k_2 \cos\theta_2 E_t. \tag{4.126}$$

Recalling the Snell's law, we thus find

$$E_r = \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_1 + \theta_2)} E_i, \tag{4.127}$$

$$E_t = \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)} E_i. \tag{4.128}$$

In the limit of normal incidence (small angles θ_1, θ_2), we recover

$$E_r = \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_1 + \theta_2)} E_i \simeq \frac{\sin\theta_2 - \sin\theta_1}{\sin\theta_1 + \sin\theta_2} E_i = \frac{n_1 - n_2}{n_2 + n_1} E_i = \frac{Z_2 - Z_1}{Z_2 + Z_1} E_i, \tag{4.129}$$

where

$$Z_i = \sqrt{\frac{\mu_0}{\varepsilon_i}}, \tag{4.130}$$

is the impedance of respective media.

When the conditions for total reflection are met, the impedance defined by the ratio between the tangential component of the electric and magnetic field,

$$Z_2^{\text{TE}} = \frac{E_2}{H_2 \cos\theta_2} = -i\sqrt{\frac{\mu_0}{\varepsilon_2}} \frac{1}{\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1 - 1}}, \tag{4.131}$$

is also pure imaginary which indeed ensures total reflection of electromagnetic waves,

$$\left| \frac{i\left|Z_2^{\text{TE}}\right| - Z_1}{Z_1 + i\left|Z_2^{\text{TE}}\right|} \right| = 1.$$

In total reflection, the phase difference between the incident and reflected waves is

$$\phi^{\text{TE}} = 2\tan^{-1}\left(\frac{Z_1}{\left|Z_2^{\text{TE}}\right|}\right) = 2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_1 - (n_2/n_1)^2}}{\cos\theta_1}\right). \tag{4.132}$$

4.8.2 E in the Incident Plane (H tangential to the boundary)

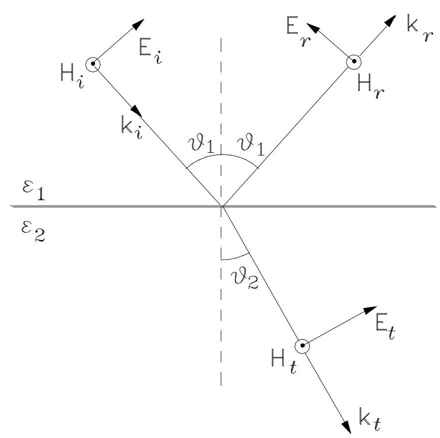


Figure 4-5: The electric field is in the incidence plane and the magnetioc field is paralllel to the boundary.

In this case, the magnetic field is parallel to the boundary plane and the continuity of tangential component of the magnetic field is simply

$$H_i + H_r = H_t. \tag{4.133}$$

For each wave, the electric field is related to the magnetic field through

$$\mathbf{E} = -\frac{1}{\omega\varepsilon}\mathbf{k} \times \mathbf{H}. \tag{4.134}$$

Therefore, continuity of the tangential component of the electric field yields

$$\frac{1}{n_1}\cos\theta_1(H_i - H_r) = \frac{1}{n_2}\cos\theta_2 H_t = \frac{1}{n_2}\cos\theta_2(H_i + H_r). \tag{4.135}$$

Solving for H_r, we find

$$H_r = \frac{\sin\theta_1\cos\theta_1 - \sin\theta_2\cos\theta_2}{\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2} H_i \tag{4.136}$$

$$= \frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)} H_i. \tag{4.137}$$

The transmitted magnetic field is

$$H_t = \left(1 + \frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}\right) H_i = \frac{2\sin\theta_1\cos\theta_1}{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2)} H_i.$$

The reflected electric field is

$$E_r = \frac{\tan(\theta_2 - \theta_1)}{\tan(\theta_1 + \theta_2)} E_i, \tag{4.138}$$

and transmitted electric field is

$$E_t = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2)} E_i. \tag{4.139}$$

If

$$\theta_1 + \theta_2 = \frac{\pi}{2}, \tag{4.140}$$

the reflected wave vanishes completely. Under this condition, Snell's law becomes

$$\frac{n_2}{n_1} = \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin\theta_1}{\cos\theta_1} = \tan\theta_1, \tag{4.141}$$

and this particular angle

$$\theta_{\text{B}} = \tan^{-1}\left(\frac{n_2}{n_1}\right), \tag{4.142}$$

is called the Brewster's angle. If incident wave is plane polarized with the magnetic field oriented parallel to, say, a surface of glass, reflection can be avoided at the Brewster's angle. This principle is often exploited in designing reflecting mirrors in lasers so that output laser beam has a high degree of planar polarization.

Because the reflection coefficient depends on wave polarization, unpolarized wave with random polarization becomes partially polarized on reflection and transmission at a dielectric boundary. The magnetic reflection coefficient derived in Eq. (4.136) will play an important role in analyzing the transition radiation discussed in Chapter 8.

When the conditions for total reflection are met, the impedance defined by the ratio between the tangential component of the electric and magnetic field is

$$Z_2^{\rm TM} = \frac{E_2 \cos\theta_2}{H_2} = -i\sqrt{\frac{\mu_0}{\varepsilon_2}}\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1 - 1}. \tag{4.143}$$

The phase difference between the incident and reflected waves is

$$\phi^{\rm TM} = 2\tan^{-1}\left(\frac{Z_1}{\left|Z_2^{\rm TM}\right|}\right) = 2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_1 - (n_2/n_1)^2}}{(n_2/n_1)^2\cos\theta_1}\right). \tag{4.144}$$

4.9 Circularly and Elliptically Polarized Plane Waves

Planar polarization discussed in the preceding section is a highly idealized mode of propagation of electromagnetic waves. A plane polarized wave can be decomposed into two circularly polarized plane waves of opposite helicity, one with positive helicity and another with negative helicity. Helicity of an electromagnetic wave is closely related with the angular momentum carried by the wave.

Circularly polarized waves propagating in the z-direction can be described by the electric field vectors,

$$\mathbf{E}_\pm(z,t) = E_0(\mathbf{e}_x \pm i\mathbf{e}_y)e^{i(kz-\omega t)}, \tag{4.145}$$

where the positive sign is for positive helicity and minus sign is for negative helicity. The sum of these two waves of opposite helicity trivially yields a plane polarized wave with the electric field in the x-direction. Corresponding magnetic fields are

$$\begin{aligned}\mathbf{B}_\pm &= \frac{1}{\omega}\mathbf{k}\times\mathbf{E}_\pm \\ &= \frac{k}{\omega}E_0(\mathbf{e}_y \mp i\mathbf{e}_x)e^{i(kz-\omega t)},\end{aligned} \tag{4.146}$$

and Poynting flux is

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}^* = 2\frac{E_0^2}{Z}\mathbf{e}_z, \tag{4.147}$$

where the factor 2 accounts for the two independent modes of equal amplitude.

A general form of mixed helicity may be written as

$$\mathbf{E}(z,t) = (E_1\mathbf{e}_x + E_2 i\mathbf{e}_y)e^{i(kz-\omega t)}, \tag{4.148}$$

where E_1 and E_2 are complex amplitude. Corresponding magnetic field,

$$\mathbf{B}(z,t) = \frac{k}{\omega}(E_1\mathbf{e}_y - E_2 i\mathbf{e}_x)e^{i(kz-\omega t)}, \tag{4.149}$$

is of course normal to the electric field,

$$\mathbf{E}\cdot\mathbf{B} = 0. \tag{4.150}$$

However, the scalar product $\mathbf{E} \cdot \mathbf{B}^*$ in general does not vanish.

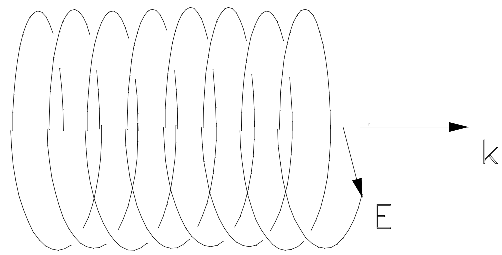


Figure 4-6: Trace of the head of the rotating electric field vector associated with a circularly polarized wave with positive helicity. In the case of negative helicity, the direction of rotation relative to the wavevector is reversed.

Example 2 ***Reflection and Transmission of Circularly Polarized Wave***

Consider a circularly polarized wave incident at an angle θ_i to a flat surface of a dielectric. The electric field of the incident wave may be decomposed into two components, one in the incident plane and another perpendicular to the incident plane,

$$\mathbf{E}_i = \mathbf{E}_{\parallel 0} + i\mathbf{E}_{\perp 0}.$$

The reflected wave of the parallel component is

$$E_{\parallel r} = \frac{\tan(\theta_2 - \theta_1)}{\tan(\theta_1 + \theta_2)} E_{\parallel 0},$$

while the reflected perpendicular component is

$$E_{\perp r} = \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_2 + \theta_1)} E_{\perp 0}.$$

The transmitted (refracted) components are

$$E_{\parallel t} = \frac{2\sin\theta_1 \cos\theta_2}{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2)} E_{\parallel 0},$$

$$E_{\perp t} = \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)} E_{\perp 0}.$$

Both reflected and transmitted waves are elliptically polarized. In particular, if the incident angle is at the Brewster's angle, $\theta_1 = \pi/2 - \theta_2$, $E_{\parallel r}$ vanishes and the reflected wave becomes plane polarized.

On total reflection of a circularly polarized electromagnetic wave, the reflected wave becomes elliptically polarized because the phases of TE and TM components differ as evident from Eqs. (4.132) and (4.144).

4.10 Stokes' Parameters

Consider an electromagnetic wave propagating in the z direction with electric field components

$$\mathbf{E}(\mathbf{r},t)=(E_x\mathbf{e}_x+E_y\mathbf{e}_{y.})e^{i(kz-\omega t)}. \tag{4.151}$$

The amplitudes E_x and E_y may be complex allowing for finite phase difference,

$$E_x=|E_x|\,e^{i\phi_x},\;\; E_y=|E_y|\,e^{i\phi_y}. \tag{4.152}$$

If $\phi=\phi_x-\phi_y=0$, the filed is a simple superposition of two linearly polarized waves. If ϕ is finite, the wave is in general elliptically polarized. In optics, direct measurement of the phase difference is not easy. What is normally measured is the intensity or the quadratic quantities of the electric field,

$$|E_x|^2,\;\; |E_y|^2,\;\; |E_x|\,|E_y|\cos(\phi_x-\phi_y),\;\text{and}\;\; |E_x|\,|E_y|\sin(\phi_x-\phi_y). \tag{4.153}$$

Stokes' parameters are defined, with $I=|E_x|^2+|E_y|^2$ the total intensity, by

$$\begin{aligned}
s_0&=\tfrac{1}{I}\left(|E_x|^2+|E_y|^2\right)=1,\\
s_1&=\tfrac{1}{I}\left(|E_x|^2-|E_y|^2\right),\\
s_2&=\tfrac{2}{I}\,|E_x|\,|E_y|\cos(\phi_x-\phi_y),\\
s_3&=\tfrac{2}{I}\,|E_x|\,|E_y|\sin(\phi_x-\phi_y),
\end{aligned} \tag{4.154}$$

and satisfy

$$s_1^2+s_2^2+s_3^2=1, \tag{4.155}$$

if the waves are purely coherent. If not, s_2 and s_3 are to be modified as

$$s_2=\frac{2}{I}\,\mathrm{Re}\,\overline{E_xE_y^*},\;\; s_3=\frac{2}{I}\,\mathrm{Im}\,\overline{E_xE_y^*}, \tag{4.156}$$

where the bar indicates time average. For incoherent waves,

$$s_1^2+s_2^2+s_3^2<1. \tag{4.157}$$

"Natural light" is characterized by a collection of many waves with random phases and complete depolarization, $s_1=s_2=s_3=0$ even though each wave may be highly monochromatic. For a plane wave polarized in the x direction,

$$s_1=1,\;\; s_2=s_3=0.$$

For a plane wave polarized in the direction $\theta = \pi/4$ (along the plane $x = y$),

$$s_1 = s_3 = 0, \;\; s_2 = 1.$$

For a circularly polarized wave with positive (negative) helicity,

$$s_1 = s_2 = 0, \;\; s_3 = \mp 1.$$

In experiments, Stokes' parameters can be determined by rotating a polarizer plate. $|E_x|^2$ and $|E_y|^2$ can be readily found by aligning the polarization direction along x and y direction, respectively. If the polarizer is at angle θ from the x axis, the intensity measured at that angle is

$$\begin{aligned} I(\theta) &= (|E_x|e^{i\phi_x}\cos\theta + |E_y|e^{i\phi_y}\sin\theta)\left(|E_x|e^{-i\phi_x}\cos\theta + |E_y|e^{-i\phi_y}\sin\theta\right) \\ &= |E_x|^2\cos^2\theta + |E_y|^2\sin^2\theta + 2\,|E_x|\,|E_y|\cos\theta\sin\theta\cos(\phi_x - \phi_y). \end{aligned}$$

By choosing $\theta = \pi/4$, for example, we have

$$I\left(\theta = \frac{\pi}{4}\right) = \frac{1}{2}\left[|E_x|^2 + |E_y|^2 + 2|E_x||E_y|\cos(\phi_x - \phi_y)\right]. \tag{4.158}$$

Finally, exploiting a uniaxial (or biaxial) crystal, a so-called quarter wavelength plate can be fabricated to induce $\pi/2$ *relative* phase delay between E_x and E_y due to the different propagation velocities of ordinary and extraordinary modes. The intensity in this case is

$$\begin{aligned} I'(\theta) &= (|E_x|e^{i\phi_x}\cos\theta + i|E_y|e^{i\phi_y}\sin\theta)\left(|E_x|e^{-i\phi_x}\cos\theta - i|E_y|e^{-i\phi_y}\sin\theta\right) \\ &= |E_x|^2\cos^2\theta + |E_y|^2\sin^2\theta - 2\,|E_x|\,|E_y|\cos\theta\sin\theta\sin(\phi_x - \phi_y) \end{aligned}$$

If $\theta = \pi/4$, this reduces to

$$I'\left(\theta = \frac{\pi}{4}\right) = \frac{1}{2}\left[|E_x|^2 + |E_y|^2 - 2|E_x||E_y|\sin(\phi_x - \phi_y)\right]. \tag{4.159}$$

Therefore, $|E_x|e^{i\phi_x}$ and $|E_y|e^{i\phi_y}$ and corresponding Stokes' parameters can be determined by measuring the four intensities,

$$I\,(\theta = 0)\,,\;\; I\left(\theta = \frac{\pi}{2}\right),\;\; I\left(\theta = \frac{\pi}{4}\right) \text{ and } I'\left(\theta = \frac{\pi}{4}\right).$$

The tensor defined by

$$I_{ij} = \overline{E_i E_j^*}, \;\; (i, j = x, y) \tag{4.160}$$

is called polarization tensor. Unless the field is purely coherent, the time averaged intensity $\overline{E_i E_j^*}$ may still vary slowly with time. I_{ik} is Hermitian and thus can be diagonalized through the eigenvalues λ_1 and λ_2 which are the roots of

$$\det\left(I_{ij} - \lambda\delta_{ij}\right) = 0. \tag{4.161}$$

Corresponding two polarization eigenvectors $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ can be determined from

$$I_{ij}n_j^{(1)} = \lambda_1 n_i^{(1)},\ I_{ij}n_j^{(2)} = \lambda_1 n_i^{(2)}. \tag{4.162}$$

In terms of the eigenvectors, the polarization tensor can be written in the form

$$\overleftrightarrow{\mathbf{I}} = \lambda_1\mathbf{n}^{(1)}\mathbf{n}^{(1)} + \lambda_2\mathbf{n}^{(2)}\mathbf{n}^{(2)}. \tag{4.163}$$

As an example, let us consider superposition of two plane polarized waves, one in the x direction and another in the direction $(\cos\theta, \sin\theta)$ in the $x-y$ plane. The relative phase between the two waves is assumed to be random and the intensities of the waves are I_1 and I_2. The electric field is

$$\mathbf{E} = (E_1 + E_2e^{i\phi}\cos\theta, E_2e^{i\phi}\sin\theta), \tag{4.164}$$

where $I_1 = E_1^2, I_2 = E_2^2$, and ϕ is random phase. Then

$$I_{ij} = \begin{pmatrix} I_1 + I_2\cos^2\theta & I_2\cos\theta\sin\theta \\ I_2\cos\theta\sin\theta & I_2\sin^2\theta \end{pmatrix}. \tag{4.165}$$

Eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(I_1 + I_2 \pm \sqrt{(I_1+I_2)^2 - 4I_1I_2\sin^2\theta}\right). \tag{4.166}$$

The ratio

$$\rho = \lambda_{\min}/\lambda_{\max} = \frac{I_1 + I_2 - \sqrt{(I_1+I_2)^2 - 4I_1I_2\sin^2\theta}}{I_1 + I_2 + \sqrt{(I_1+I_2)^2 - 4I_1I_2\sin^2\theta}}, \tag{4.167}$$

may be called the degree of depolarization. $\rho = 1$ corresponds to completely unpolarized state, while $\rho = 0$ corresponds to a plane polarized wave. When $I_1 = I_2 = I$, the polarization tensor reduces to

$$I_{ij} = I\begin{pmatrix} 1 + \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}. \tag{4.168}$$

Eigenvalues are

$$\lambda_{1,2} = 1 \pm \cos\theta, \tag{4.169}$$

and the degree of depolarization is

$$\rho = \frac{1-\cos\theta}{1+\cos\theta}. \tag{4.170}$$

The eigenvectors are

$$\mathbf{n}^{(1)} = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right), \mathbf{n}^{(2)} = \left(-\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right). \tag{4.171}$$

Derivation of the eigenvectors is left for exercise.

The Stokes' parameters s_i and the polarization tensor are related through

$$I_{ij} = \frac{1}{2} I \left(\delta_{ij} + \sum_{m=1}^{3} s_m \sigma_{ij}^{(m)} \right) = \frac{1}{2} I \left(\begin{array}{cc} 1 + s_1 & s_2 - i s_3 \\ s_2 + i s_3 & 1 - s_1 \end{array} \right), \tag{4.172}$$

where

$$\sigma_{ij}^{(1)} = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \sigma_{ij}^{(2)} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \sigma_{ij}^{(3)} = \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \tag{4.173}$$

are Pauli's spin matrices. When $s_1 = 1, s_2 = s_3 = 0$, I_{ij} reduces to

$$I_{ij} = I \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right),$$

which describes a plane polarized wave in the x direction. When $s_2 = 1, s_1 = s_3 = 0$,

$$I_{ij} = \frac{1}{2} I \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right).$$

This describes the case of plane polarization in the direction $\theta = \pi/4$. When $s_3 = 1, s_1 = s_2 = 0$,

$$I_{ij} = \frac{1}{2} I \left(\begin{array}{cc} 1 & -i \\ i & 1 \end{array} \right),$$

which describes circular polarization with positive helicity. Finally, $s_3 = -1, s_1 = s_2 = 0$ and corresponding

$$I_{ij} = \frac{1}{2} I \left(\begin{array}{cc} 1 & i \\ -i & 1 \end{array} \right),$$

describes circular polarization with negative helicity.

4.11 Propagation along a Conductor Rod

In this section, we analyze propagation of electromagnetic waves along a conductor rod of radius a and conductivity σ. A simple transverse Magnetic (TM) mode is considered with the following field components, E_ρ, E_z, and H_ϕ. The axial electric field E_z outside the conductor rod satisfies the scalar wave equation,

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) E_z(\rho, z, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E_z(\rho, z, t). \tag{4.174}$$

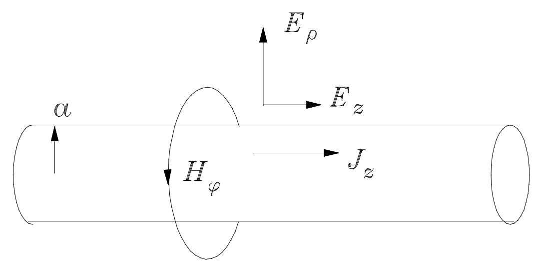


Figure 4-7: Field profiles of electromagentic wave propagating along a conductor rod.

For a harmonic wave with time dependence $e^{-i\omega t}$, this reduces to

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right) E_z(\rho, z) = 0. \tag{4.175}$$

Furthermore, since the wave is propagating along the rod, we may single out the z dependence in the form e^{ikz} and reduce the wave equation to

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \kappa^2\right) E_z(\rho) = 0, \tag{4.176}$$

where

$$\kappa^2 = \left(\frac{\omega}{c}\right)^2 - k^2. \tag{4.177}$$

Elementary solutions to Eq. (4.176) are the Bessel functions $J_0(\kappa\rho), N_0(\kappa\rho)$ and their linear combinations,

$$H_0^{(1)}(\kappa\rho) = J_0(\kappa\rho) + iN_0(\kappa\rho), \tag{4.178}$$

$$H_0^{(2)}(\kappa\rho) = J_0(\kappa\rho) - iN_0(\kappa\rho), \tag{4.179}$$

known as Hankel functions of the first and second kind, respectively. Their asymptotic behavior at large argument is

$$H_0^{(1)}(x) \to \sqrt{\frac{2}{\pi x}} \exp\left[i\left(x - \frac{\pi}{4}\right)\right],$$

$$H_0^{(2)}(x) \to \sqrt{\frac{2}{\pi x}} \exp\left[-i\left(x - \frac{\pi}{4}\right)\right],$$

indicating radially outward and inward propagation, respectively. In the present case, both k and κ are complex because of dissipation in the rod. It is reasonable to assume radially outward propagation (radiation from the rod) and we choose

$$E_z(\rho) = E_0 H_0^{(1)}(\kappa\rho). \tag{4.180}$$

From $\nabla \cdot \mathbf{E} = 0$, we find the radial component of the electric field,

$$E_\rho(\rho) = -\frac{ik}{\kappa} E_0 H_1^{(1)}(\kappa\rho), \tag{4.181}$$

and from $\nabla \times \mathbf{E} = i\omega\mu_0 \mathbf{H}$, the azimuthal component of the magnetic field,

$$H_\phi(\rho) = -i\omega\varepsilon_0 \frac{E_0}{\kappa} H_1^{(1)}(\kappa\rho), \tag{4.182}$$

where use is made of

$$\frac{d}{dx}\left(xH_1^{(1)}(x)\right) = xH_0^{(1)}(x),$$

$$\frac{d}{dx} H_0^{(1)}(x) = -H_1^{(1)}(x).$$

The boundary condition at the rod surface is

$$\frac{E_z(\rho = a)}{H_\phi(\rho = a)} = -Z = -\sqrt{\frac{-i\omega\mu_0}{\sigma}} \text{ (surface impedance)}, \tag{4.183}$$

or

$$\frac{\kappa H_0^{(1)}(\kappa a)}{H_1^{(1)}(\kappa a)} = i\omega\varepsilon_0 Z. \tag{4.184}$$

For a given wave frequency ω, this equation determines κ and thus k, the axial wavenumber and in this respect, it is a dispersion relation.

For a small impedance Z, κ approaches 0, and the axial wavenumber k becomes

$$k = \frac{\omega}{c},$$

as expected. Noting

$$\lim_{x\to 0} H_0^{(1)}(x) \to 1 + i\frac{2}{\pi}\left(\ln(x/2) + \gamma_E\right), \ \lim_{x\to 0} H_1^{(1)}(x) \to \frac{2}{\pi}\frac{1}{x}, \tag{4.185}$$

we see that the axial electric field becomes negligible and the transverse fields approach those of TEM mode,

$$E_\rho(\rho) = E_0 \frac{a}{\rho}, \ H_\phi(\rho) = \frac{E_\rho(\rho)}{Z_0}, \tag{4.186}$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the impedance of free space. The fields in coaxial cables used for transmission of electromagnetic waves can be approximated by those given above,

$$E_\rho = \frac{V}{\ln(b/a)}\frac{1}{\rho}, \quad H_\phi = \frac{E_\rho}{Z} = \frac{I}{2\pi\rho}, \tag{4.187}$$

where V is the potential difference between the inner and outer conductor with radii a and b,respectively, I is the current and $Z = \sqrt{\mu_0/\varepsilon}$ with ε the permittivity of the insulating material filling the cable. The characteristic impedance of the cable is

$$Z_{\text{cable}} = \frac{V}{I} = \frac{\sqrt{\mu_0/\varepsilon}}{2\pi} \ln\left(\frac{b}{a}\right). \tag{4.188}$$

Similarly, the impedance of a parallel wire transmission line with conductor radius a and separation distance D is approximately given by

$$Z \simeq \frac{\sqrt{\mu_0/\varepsilon_0}}{\pi} \ln\left(\frac{D}{a}\right), \quad D \gg a. \tag{4.189}$$

4.12 Skin Effects in Conductors

In a conductor, the conduction current dominates over the displacement current and a simple Ohm's law,

$$\mathbf{J} = \sigma \mathbf{E}, \tag{4.190}$$

may be assumed where σ (S/m) is the conductivity. Eliminating the electric field between the Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \ \nabla \times \mathbf{B} \simeq \mu_0 \sigma \mathbf{E},$$

yields the following diffusion equation for the magnetic field,

$$\nabla^2 \mathbf{B} = \mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t}.$$

If the field is oscillating at a frequency ω, we obtain the following ordinary differential equation,

$$\nabla^2 \mathbf{B} + i\omega\mu_0\sigma\mathbf{B} = 0.$$

Penetration of the magnetic field into a conductor slab can be described by

$$\frac{d^2 B_y}{dz^2} + i\omega\mu_0\sigma B_y = 0. \tag{4.191}$$

This has a bounded solution

$$B(z) = B_0 e^{ikz},$$

where

$$k = \sqrt{i\omega\mu_0\sigma} = \frac{1+i}{\sqrt{2}}\sqrt{\omega\mu_0\sigma}. \tag{4.192}$$

The magnetic field decays exponentially from the conductor surface and the quantity

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}, \ \text{(m)} \tag{4.193}$$

is called the skin depth. Damping of the electromagnetic fields is evidently due to Ohmic dissipation in the conductor.

For a cylindrical conductor rod with radius $a(\ll c/\omega)$ in which an axial current $J_z(\rho)e^{-i\omega t}$ is excited, J_z obeys

$$\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+i\omega\mu_0\sigma\right)J_z(\rho)=0, \tag{4.194}$$

This has a bounded solution

$$J_z(\rho)=\sigma E_0\frac{J_0(k\rho)}{J_0(ka)}, \tag{4.195}$$

where E_0 is the electric field at the rod surface. The impedance per unit length of the rod can thus be defined by

$$\frac{Z}{l}=\frac{E_0}{I}=\frac{k}{2\pi a\sigma}\frac{J_0(ka)}{J_1(ka)},\ \text{(Ohms/m)}. \tag{4.196}$$

where use is made of

$$I=2\pi\int_0^a J_z(\rho)\rho d\rho=2\pi\frac{a}{k}\sigma E_0\frac{J_1(ka)}{J_0(ka)}. \tag{4.197}$$

In the low frequency limit $|k|\,a\ll 1$, series expansion of the Bessel functions can be exploited,

$$J_0(x)\simeq 1-\frac{x^2}{4},\ J_1(x)\simeq\frac{x}{2}\left(1-\frac{x^2}{8}\right).$$

The impedance reduces to

$$\frac{Z}{l}=\frac{1}{\pi a^2\sigma}-i\omega\frac{\mu_0}{8\pi}, \tag{4.198}$$

where

$$\frac{L_i}{l}=\frac{\mu_0}{8\pi}\ \text{(H/m)} \tag{4.199}$$

is the well known internal inductance of a cylindrical rod carrying a uniform current (no skin effect). In the high frequency limit (strong skin effect), the ratio $J_0(ka)/J_1(ka)$ approaches $-i$, and the impedance is

$$\frac{Z}{l}=\frac{1}{2\pi a\sigma\delta}(1-i). \tag{4.200}$$

This is also reasonable since with strong skin effect, the current flow is limited in a thin layer with an area $2\pi a\delta$. Note that the inductive reactance is identical to the resistance in this limit.

If a conductor cylinder is placed in an oscillating axial magnetic field $H_{z0}e^{-i\omega t}$ as in inductive rf heating, the current flows in the azimuthal direction. $J_\phi(\rho)$ satisfies

$$\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{1}{\rho^2}+i\omega\mu_0\sigma\right)J_\phi(\rho)=0,\ \rho<a.$$

An appropriate solution is

$$J_\phi(\rho)=k\frac{J_1(k\rho)}{J_0(ka)}H_{z0},\ \rho<a, \tag{4.201}$$

where k is still given by

$$k = \sqrt{i\omega\mu_0\sigma}.$$

4.13 Skin Effect in a Plasma

A charge neutral plasma contains equal amount of positive and negative charge density. When placed in an electric field oscillating at a high frequency, a plasma current due to electron motion is induced. The conductivity can be found by letting $\omega_0 = 0$ in Eq. (4.88) (because in a plasma, electrons are free),

$$\sigma = \frac{i}{\omega}\frac{ne^2}{m}, \tag{4.202}$$

and corresponding permittivity

$$\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right). \tag{4.203}$$

In a collisional plasma with electron collision frequency ν_c, this is modified as

$$\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega\left(\omega + i\nu_c\right)}\right), \tag{4.204}$$

as can be easily worked out by introducing a finite collision frequency in the equation of motion.

For a frequency much smaller than the plasma frequency, the permittivity becomes negative,

$$\varepsilon \simeq -\varepsilon_0 \frac{\omega_p^2}{\omega^2}, \tag{4.205}$$

and wave propagation is forbidden. The wavenumber is complex in this case

$$k = i\frac{\omega_p}{c}, \tag{4.206}$$

and the wave amplitude decays in a manner

$$E_0 e^{i(kz-\omega t)} = E_0 \exp\left(-\frac{\omega_p}{c}z\right) e^{-i\omega t}, \tag{4.207}$$

where z is the distance in the plasma from the vacuum-plasma boundary. This means that an electromagnetic wave incident on a plasma cannot penetrate into plasma except for a distance of the order of the skin depth defined by

$$\delta = \frac{c}{\omega_p}. \tag{4.208}$$

The wave is completely reflected if the plasma is collisionless. Reflection of low frequency radio waves by the ionoshperic plasma and is a well known example.

In a collisional plasma with $\omega \ll \omega_p, \nu_c$, the skin depth is modified as

$$\delta \simeq \frac{c}{\omega_p}\sqrt{\frac{\omega}{2\nu_c}}. \tag{4.209}$$

Derivation of this formula is left for exercise.

Strictly speaking, the collisionless skin depth $\delta = c/\omega_p$ is valid for a cold plasma with negligible electron temperature. To implement the effects of finite electron temperature, it is necessary to employ a kinetic theory to find a conductivity σ. The electron velocity distribution function f $(\mathbf{r}, \mathbf{v}, t)$ obeys the kinetic equation,

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f - \frac{e}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0. \tag{4.210}$$

As it is, it is a nonlinear equation because the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ associated with a wave affect the distribution function f. Let us assume a wave propagating along the z direction with electric field polarized in the x direction $E_x(z,t) = E_0 e^{i(kz-\omega t)}$. The distribution function may be linearized as $f = f_1 + F_M$, where f_1 is the perturbation and F_M is unperturbed Maxwellian distribution. Noting

$$\frac{\partial F_M(v^2)}{\partial \mathbf{v}} = -\frac{m\mathbf{v}}{T_e}F_M(v^2), \tag{4.211}$$

and thus

$$(\mathbf{v}\times\mathbf{B})\cdot\frac{\partial F_M(v^2)}{\partial \mathbf{v}} = 0,$$

we find

$$f_1 = -\frac{\frac{e}{T_e}E_x}{i(kv_z - \omega)}v_x F_M. \tag{4.212}$$

The current density can be found from the first order moment,

$$\mathbf{J}_e = -ne\int \mathbf{v} f_1 d\mathbf{v}. \tag{4.213}$$

Only J_x is nonvanishing, and given by

$$\begin{aligned} J_x &= -i\frac{ne^2}{T_e}E_x\int \frac{v_x^2}{kv_z - \omega}F_M d\mathbf{v} \\ &= -i\frac{ne^2}{mkv_{Te}}E_x\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{t-\zeta}dt \\ &= -i\frac{ne^2}{mkv_{Te}}E_x Z(\zeta), \end{aligned} \tag{4.214}$$

where $\zeta = \omega/kv_{Te}$ with $v_{Te} = \sqrt{2T_e/m}$ being the thermal velocity of electrons and $Z(\zeta)$ is known as the plasma dispersion function. The permittivity is therefore given by

$$\varepsilon = \varepsilon_0 \left(1 + \frac{\omega_p^2}{\omega k v_{Te}} Z(\zeta) \right). \tag{4.215}$$

If the electron temperature is negligible $\zeta \gg 1$, the function $Z(\zeta)$ approaches

$$Z(\zeta) \simeq -\frac{1}{\zeta},$$

and we recover the case of cold plasma,

$$\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right).$$

In the opposite limit $\zeta \ll 1$,

$$Z(\zeta) \simeq i\sqrt{\pi},$$

and

$$\varepsilon \simeq i\sqrt{\pi}\varepsilon_0 \frac{\omega_p^2}{\omega k v_{Te}}.$$

The damping factor (inverse skin depth) in this limit is given by

$$\mathrm{Im}\, k = \frac{1}{\delta} \simeq \frac{\pi^{1/6}}{2} \left(\frac{\omega_p}{c} \right)^{2/3} \left(\frac{\omega}{v_{Te}} \right)^{1/3}. \tag{4.216}$$

This is often called anomalous skin effect. (This is probably misnomer because there is nothing anomalous in the derivation.)

4.14 Waves in Anisotropic Dielectrics

Some crystals exhibit anisotropy in polarizability. The permittivity in such media becomes a tensor, and the displacement vector **D** and electric field **E** are related through a dielectric tensor $\boldsymbol{\varepsilon}$,

$$\mathbf{D} = \boldsymbol{\varepsilon} \cdot \mathbf{E}, \text{ or } D_i = \varepsilon_{ij} E_j, \tag{4.217}$$

where $\boldsymbol{\varepsilon}$ is a diagonal tensor consisting of three permittivities in each axial direction, x, y, and z,

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}. \tag{4.218}$$

In uniaxial crystals, polarization occurs preferentially along one axis, say, $\varepsilon_x = \varepsilon_y \neq \varepsilon_z$. In anisotropic media, the phase and group velocities are in general oriented in different directions and the well known double refraction phenomenon (already known in the 17th century) can be explained in terms of the dielectric anisotropy.

The Maxwell's equations to describe wave propagation in an anisotropic medium in which there are no sources ($\rho_{\text{free}} = 0,\ \mathbf{J} = 0$) are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},\ \nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t} = \mu_0 \boldsymbol{\varepsilon} \cdot \frac{\partial \mathbf{E}}{\partial t}. \tag{4.219}$$

Then

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \boldsymbol{\varepsilon} \cdot \frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{4.220}$$

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} + \mu_0 \boldsymbol{\varepsilon} \cdot \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{4.221}$$

Note that in an anisotropic medium, the divergence of the electric field, $\nabla \cdot \mathbf{E}$, does not necessarily vanish, although

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}} = 0, \tag{4.222}$$

must hold. After Fourier decomposition, Eq. (4.221) reduces to

$$\mathbf{k}(\mathbf{k} \cdot \mathbf{E}) - k^2 \mathbf{E} + \omega^2 \mu_0 \boldsymbol{\varepsilon} \cdot \mathbf{E} = 0, \tag{4.223}$$

or

$$(k^2 \delta_{ij} - k_i k_j - \omega^2 \mu_0 \varepsilon_{ij}) E_j = 0. \tag{4.224}$$

The dispersion relation of electromagnetic waves in a uniaxial crystal is therefore given by the determinant,

$$\det(k^2 \delta_{ij} - k_i k_j - \omega^2 \mu_0 \varepsilon_{ij}) = 0. \tag{4.225}$$

Introducing $\hat{\varepsilon}_{ij} = \varepsilon_{ij}/\varepsilon_0$ and the index of refraction,

$$\mathbf{n} = \frac{c}{\omega} \mathbf{k}, \tag{4.226}$$

we rewrite the dispersion relation as

$$\det(n^2 \delta_{ij} - n_i n_j - \hat{\varepsilon}_{ij}) = 0. \tag{4.227}$$

In uniaxial crystals, polarizability along one axis (say, z-axis) differs from those along other axes. We assume

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon_\perp, \quad \varepsilon_z \equiv \varepsilon_\parallel. \tag{4.228}$$

Since the crystal is symmetric about the z-axis, the wave vector $\mathbf{k}$ can be assumed to be in the $x - z$ plane without loss of generality,

$$k_x = k_\perp, \quad k_z = k_\parallel, \tag{4.229}$$

or

$$n_\perp = \frac{ck_\perp}{\omega}, \quad n_\parallel = \frac{ck_\parallel}{\omega}. \tag{4.230}$$

The dispersion relation then becomes

$$\begin{vmatrix} n_\parallel^2 - \hat{\varepsilon}_\perp & 0 & -n_\perp n_\parallel \\ 0 & n^2 - \hat{\varepsilon}_\perp & 0 \\ -n_\perp n_\parallel & 0 & n_\perp^2 - \hat{\varepsilon}_\parallel \end{vmatrix} = 0. \tag{4.231}$$

Expanding the determinant, we find

$$n^2 = \hat{\varepsilon}_\perp, \tag{4.232}$$

and

$$\hat{\varepsilon}_\parallel n_\parallel^2 + \hat{\varepsilon}_\perp n_\perp^2 - \hat{\varepsilon}_\parallel \hat{\varepsilon}_\perp = 0. \tag{4.233}$$

The first mode of propagation is independent of the propagation angle as if in an isotropic medium. It is called the ordinary mode because it maintains the properties of electromagnetic waves in isotropic media. The electric field is in the y-direction and the dispersion relation in Eq. (4.232) is equivalent to the following differential equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \mu_0 \varepsilon_\perp \omega^2\right) E_y(x, y) = 0. \tag{4.234}$$

In this mode, the phase and group velocities are in the same direction (in the direction of $\mathbf{k}$) although in magnitude they differ because of the frequency dependence of the permittivity $\varepsilon_\perp(\omega)$.

The second mode has a peculiar property and for this reason is called extraordinary mode. Let the angle between $\mathbf{k}$ and the z-axis be θ. The dispersion relation in Eq. (4.233) becomes

$$\left(\hat{\varepsilon}_\parallel \cos^2\theta + \hat{\varepsilon}_\perp \sin^2\theta\right)\left(\frac{ck}{\omega}\right)^2 - \hat{\varepsilon}_\parallel \hat{\varepsilon}_\perp = 0. \tag{4.235}$$

The phase velocity given by

$$\frac{\omega}{\mathbf{k}} = c\sqrt{\frac{\cos^2\theta}{\hat{\varepsilon}_\perp} + \frac{\sin^2\theta}{\hat{\varepsilon}_\parallel}}\mathbf{e}_k, \tag{4.236}$$

is directed along $\mathbf{k}$. The group velocity is

$$\begin{aligned} \frac{d\omega}{d\mathbf{k}} &= \frac{\partial\omega}{\partial k}\mathbf{e}_k + \frac{1}{k}\frac{\partial\omega}{\partial\theta}\mathbf{e}_\theta \\ &= \frac{\omega}{k}\mathbf{e}_k + \frac{c^2}{\omega/k}\left(\frac{1}{\hat{\varepsilon}_\parallel} - \frac{1}{\hat{\varepsilon}_\perp}\right)\sin\theta\cos\theta\,\mathbf{e}_\theta. \end{aligned} \tag{4.237}$$

This is evidently not parallel to the phase velocity unless $\hat{\varepsilon}_\parallel = \hat{\varepsilon}_\perp$ (isotropic dielectric). The magnitude of the group velocity is always larger than that of the phase velocity.

Electromagnetic waves in a plasma confined by a magnetic field is another example of anisotropic medium which accommodates variety of waves. A simple case of cold plasma will be considered in

Problem 4.11.

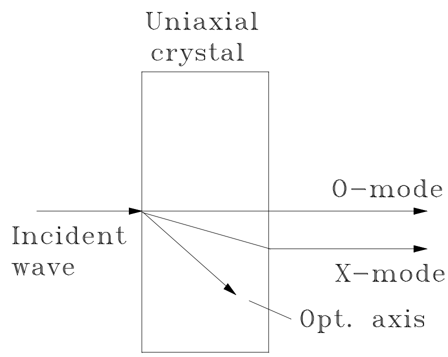


Figure 4-8: Double refraction of randomly polarized incident wave by a uniaxial crystal. Propagation of the O-mode (ordinary mode) is unaffected but that of the X-mode (extraordinary mode) is. The group velocity of the X-mode in the crystal deviates from the incident direction.

An important consequence of the presence of the ordinary and extraordinary modes is the well known double refraction caused by some crystals. Consider a light beam with random polarizations incident normal to a surface of a uniaxial crystal as shown in Fig. 4-8. Unless the optical axis (z-axis in the geometry assumed) coincides with the direction of the incident beam, the beam is split into two beams at the surface. In the crystal, one beam propagates along the direction of the incident beam and another at an angle. The phase velocities of the ordinary and extraordinary modes are in the same direction as the incident beam but the group velocity of the extraordinary mode deviates from the incident direction. Wave energy propagates at the group velocity and thus beam splitting occurs.

Some isotropic dielectrics can become uniaxial media if placed in an electric field. This is because the permittivity is in general nonlinear and the component in the direction of the field becomes field-dependent,

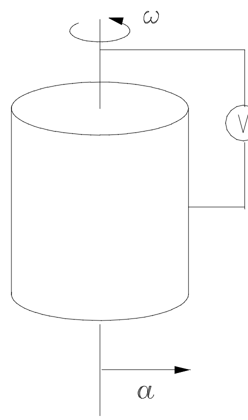
$$\varepsilon = \varepsilon(0) + \alpha E^n, \tag{4.238}$$

where α is a constant, $n = 1$ is for Pockel's effect and $n = 2$ is for Kerr effect. Pockel's effect in some liquids is widely used for laser switching and optical modulation.

Problems

4.1 A sphere of radius a carries a charge q which is decreasing due to emission of charge in every radial direction. Show that there is no magnetic field even though there exists a radial conduction current density.

4.2 A cylindrical permanent magnet with radius a and a uniform axial magnetization M_z is rotating at an angular frequency ω about its axis. What is the emf induced in the closed circuit shown?



Problem 4.2. Unipolar generator.

4.3 The permittivity of an unmagnetized plasma is given by

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right),$$

where ω_{pe} is the electron plasma frequency,

$$\omega_{pe} = \sqrt{\frac{ne^2}{\varepsilon_0 m_e}}.$$

Show that the energy density of a plane electromagnetic wave in a plasma is

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\varepsilon_0 \frac{\omega_{pe}^2}{\omega^2} E^2 + \frac{1}{2}\mu_0 H^2 = \varepsilon_0 E^2,$$

and interpret the term

$$\frac{1}{2}\varepsilon_0 \frac{\omega_{pe}^2}{\omega^2} E^2.$$

4.4 An isotropic dielectric has a permittivity in the form

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2} \right).$$

Show that the group velocity does not exceed c. What if a finite dissipation is allowed,

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 + 2i\gamma\omega - \omega_0^2} \right),$$

where γ is a damping constant? For simplicity, assume $\omega_p = \omega_0$ and plot Re $(d\omega/dk)$ as a function of ω for various γ/ω_0.

4.5 Show that if a dielectric medium is loss free, its dielectric tensor should be Hermitian,

$$\varepsilon_{ij} = \varepsilon_{ji}^*.$$

Note: In the absence of external magnetic field, the tensor is symmetric $\varepsilon_{ij} = \varepsilon_{ji}$. In this case, the loss-less condition is Im $(\varepsilon_{ij}) = 0$.

4.6 Find the permittivity and thickness of a dielectric film to be coated on a glass surface to eliminate reflection of light wave of wavelength $\lambda = 550$ nm. Assume that the glass has an index of refraction of $n = 1.5$.

4.7 A light beam linearly polarized is incident on a glass ($n_{\text{glass}} = 1.5$) surface at an angle $\theta_i = 45°$. Find the amplitude and polarization of the reflected and refracted waves. Consider both cases of polarization, electric field in the incident plane and magnetic field in the incident plane.

4.8 A light beam of circular polarization in glass is incident on a flat glass-air boundary at an angle $\theta_i = 60°$. Find the polarization of totally reflected beam.

4.9 A uniaxial crystal has $\varepsilon_{xx} = \varepsilon_{yy} = 1.3\varepsilon_0$ and $\varepsilon_{zz} = 1.5\varepsilon_0$. The optical axis is at angle 70° to a flat surface. A light beam of random polarization is incident normal to the surface. Find the propagation direction of the extraordinary mode in the crystal.

4.10 A uniaxial crystal has the following permittivity tensor,

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} 3\varepsilon_0 & 0 & 0 \\ 0 & 3\varepsilon_0 & 0 \\ 0 & 0 & 2\varepsilon_0 \end{pmatrix}.$$

A light beam in air is incident on a flat surface($x - y$ plane)of the crystal at an angle θ_i from the normal which coincides with the optical axis z. Show that the refraction law for the extraordinary mode is

$$\tan\theta_r = \frac{\sqrt{\hat{\varepsilon}_\perp}\sin\theta_i}{\sqrt{\hat{\varepsilon}_\parallel\left(\hat{\varepsilon}_\parallel - \sin^2\theta_i\right)}},$$

where $\hat{\varepsilon}_\perp = 3$, $\hat{\varepsilon}_\parallel = 2$. What is the refraction law for the ordinary mode?

4.11 Find the impedance per unit length of a copper wire of radius 3 mm at $f = 60$, 10^3 and 10^6 Hz. Copper conductivity is $\sigma = 5.8 \times 10^7$ S/m at room temperature.

4.12 Using the equation of motion for electrons in a cold magnetized plasma,

$$m_e \frac{\partial \mathbf{v}}{\partial t} = -e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}_0\right),$$

where $\mathbf{B}_0$ (external magnetic field) is in z direction, show that the dielectric tensor is given by

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{bmatrix} 1 - \dfrac{\omega_p^2}{\omega^2 - \Omega^2} & -i\dfrac{\Omega\omega_p^2}{\omega(\omega^2 - \Omega^2)} & 0 \\ i\dfrac{\Omega\omega_p^2}{\omega(\omega^2 - \Omega^2)} & 1 - \dfrac{\omega_{ps}^2}{\omega^2 - \Omega^2} & 0 \\ 0 & 0 & 1 - \dfrac{\omega_p^2}{\omega^2} \end{bmatrix}$$

where $\omega_p = \sqrt{ne^2/m_e\varepsilon_0}$ is the plasma frequency and $\Omega_e = eB_0/m_e$ is the cyclotron frequency. The wavevector $\mathbf{k}$ may be assumed to be $\mathbf{k} = k_\perp \mathbf{e}_x + k_\parallel \mathbf{e}_z$ without loss of generality because of axial symmetry. Note that the tensor is Hermitian. (In this analysis, the electron temperature is ignored.)

4.13 A laser beam passes through a glass window of refractive index 1.50 into water with refractive index 1.33. The beam is E-polarized (electric field in the incident plane). Design the glass window to avoid reflection at both surfaces.