Chapter 2

Electrostatics II. Potential Boundary Value Problems

2.1 Introduction

In Chapter 1, a general formulation was developed to find the scalar potential $\Phi(\mathbf{r})$ and consequent electric field $\mathbf{E} = -\nabla\Phi$ for a given static charge distribution $\rho(\mathbf{r})$. In a system involving conductor electrodes, often the potential Φ is specified on electrode surfaces and one is asked to find the potential in the space off the electrodes. Such problems are called potential boundary value problems. In this case, the surface charge distribution on the electrodes is unknown and can only be found after the potential and electric field have been found in the vicinity of the electrode surfaces from

$$\sigma_s = \varepsilon_0 E_n, \quad (\mathrm{C/m^2})$$

where

$$E_n = -\frac{\partial\Phi}{\partial n},$$

is the electric field component normal to the conducting electrode surface with n the normal coordinate.

If the potential is specified on a closed surface, the potential off the surface is uniquely determined in terms of the surface potential. This is known as Dirichlet's boundary value problem and most problems we will consider belong to this category. Solving Dirichlet's problems is greatly facilitated by finding a suitable Green's function for a given boundary shape. However, except for simple geometries (*e.g.*, plane, sphere, cylinder, etc.), finding Green's functions analytically is not an easy task. For complicated electrode shapes, potential problems often have to be solved numerically.

Specifying the normal derivative $\partial\Phi/\partial n$ on a closed surface also uniquely determines the potential elsewhere. This category of boundary value problems is called Neumann problem. Physically, specifying the normal derivative of the potential on a closed surface corresponds to specifying the surface charge distribution on the surface through

$$\sigma = -\varepsilon_0 \frac{\partial\Phi}{\partial n}.$$

Then, the problem is reduced to finding the potential due to a prescribed charge distribution as worked out in Chapter 1.

Specifying both the potential itself and its normal derivative everywhere on a closed surface is in general overdetermining. However, in some problems, the potential is known in one part of a closed surface and its normal derivative in the remaining part. This constitutes the so-called mixed boundary value problem.

By introducing suitable coordinates transformation, some potential problems can be reduced to one dimensional, that is, the potential becomes a total function of a single coordinate variable. This happens if the Laplace equation and potential are completely separable, $\Phi(u_1, u_2, u_3) = F_1(u_1)F_2(u_2)F_3(u_3)$. There are some 30 known rectilinear coordinate systems developed in the past for specific purposes. As one example, we will study the oblate spheroidal coordinates because of its wide variety of applications in electrostatics and magnetostatics.

2.2 Dirichlet Problems and Green's Functions

If a charge is given to a conductor, the potential of the conductor becomes constant everywhere after a short transient time as shown in Chapter 1. Electrostatic state is thus quickly established. Since the volume charge density ρ should vanish *in* a conductor, all of the charge given to a conductor must reside entirely on the conductor surface in the form of singular surface charge density σ (C/m^2). The corresponding volume charge density involves a delta function

$$\rho = \sigma\delta(n - n_s),$$

where n is the coordinate normal to the surface and n_s indicates the location of the surface.

After static condition is established, the volume charge density and the electric field in a conductor both vanish. The potential of a conductor thus becomes constant $\Phi = \Phi_c = \text{const}$. If a charge q is given to an isolated conductor, the potential of the conductor relative to zero potential at infinity is uniquely determined and the proportional constant C defines the self-capacitance of the conductor,

$$C = \frac{q}{\Phi_c}, \text{ (F)}. \tag{2.1}$$

Let us consider a trivial case, a conducting sphere of radius a carrying a charge q. The potential outside the sphere is given by

$$\Phi(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}, \quad r \geq a. \tag{2.2}$$

The sphere potential is

$$\Phi_s = \frac{q}{4\pi\varepsilon_0 a}, \tag{2.3}$$

which determines the self-capacitance of the sphere,

$$C = 4\pi\varepsilon_0 a, \ \text{(F)}. \tag{2.4}$$

The outer potential $\Phi(r)$ can be written in the form

$$\Phi(r) = \frac{a}{r}\Phi_s, \quad r > a, \tag{2.5}$$

which indicates that the potential is uniquely determined if the sphere potential Φ_s is known. In general, if the potential is specified everywhere on a closed surface, the potential elsewhere off the surface is uniquely determined in terms of the surface potential $\Phi_s(\mathbf{r}_s)$ where $\mathbf{r}_s$ denotes the coordinates on the closed surface. This is known as Dirichlet's theorem and finding a potential for given boundary potential distribution on a closed surface is called Dirichlet's problem.

The same problem can also be solved in terms of the electric field on the sphere surface,

$$E_r = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}, \tag{2.6}$$

which can be replaced with a surface charge,

$$\sigma = \varepsilon_0 E_r = \frac{q}{4\pi a^2}, \ \text{(C/m}^2\text{)}. \tag{2.7}$$

The potential due to the uniform surface charge is

$$\begin{aligned}
\Phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\oint \frac{\sigma}{|\mathbf{r}-\mathbf{r}'|}dS \\
&= \frac{\sigma}{4\pi\varepsilon_0}2\pi a^2 \int_0^\pi \frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}\sin\theta d\theta \\
&= \frac{q}{4\pi\varepsilon_0 r}, \ (r > a)
\end{aligned} \tag{2.8}$$

where θ is measured from the direction of $\mathbf{r}$. (This is allowed because of symmetry. For $r < a$, the integral yields

$$\Phi(r) = \frac{q}{4\pi\varepsilon_0 a} = \ \text{const.}, \ (r < a, \ \text{interior})$$

which is also an expected result.) Since

$$E_r = -\frac{\partial \Phi}{\partial r} = \frac{\partial \Phi}{\partial n}, \tag{2.9}$$

where n is the normal coordinate on the surface *directed away from the volume of interest*, the potential can be rewritten as

$$\Phi(\mathbf{r}) = \frac{1}{4\pi}\oint_S \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial \Phi}{\partial n}dS'. \tag{2.10}$$

As this simple example indicates, potential boundary value problems can be solved in terms of either the surface potential Φ_s or its normal derivative, $\partial\Phi/\partial n$. The latter method may be regarded as a boundary value problem for the electric field.

Let us revisit the potential due to a prescribed charge distribution,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'. \tag{2.11}$$

The potential can be understood as a convolution between the charge density distribution $\rho(\mathbf{r})$ and the function

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|}, \tag{2.12}$$

which is the particular solution to the singular Poisson's equation

$$\nabla^2 G = -\delta(\mathbf{r}-\mathbf{r}'), \tag{2.13}$$

subject to the boundary condition that G vanish at infinity. The function G is called Green's function. Physically, the Green's function defined as a solution to the singular Poisson's equation is nothing but the potential due to a point charge placed at $\mathbf{r} = \mathbf{r}'$. In potential boundary value problems, the charge density $\rho(\mathbf{r})$ is unknown and one has to devise an alternative formulation in terms of boundary potential $\Phi_s(\mathbf{r})$. It is noted that the Green's function in Eq. (2.12) is the particular solution to the singular Poisson's equation and we still have freedom to add general solutions satisfying Laplace equation,

$$G = G_p + G_g, \tag{2.14}$$

where G_p is the particular solution and G_g is a collection of general solutions satisfying

$$\nabla^2 G_g = 0. \tag{2.15}$$

This freedom will play an important role in constructing a Green's function suitable for a given boundary shape as we will see shortly. In doing so, we exploit the following theorem:

Theorem 1 *Green's Theorem: For arbitrary scalar functions ϕ and ψ, the following identity holds,*

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \oint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}. \tag{2.16}$$

Proof of this theorem goes as follows. Gauss' theorem applied to the function $\phi\nabla\psi$ gives

$$\int_V \nabla\cdot(\phi\nabla\psi)dV = \oint_S (\phi\nabla\psi)\cdot d\mathbf{S}. \tag{2.17}$$

The LHS may be expanded as

$$\int_V \nabla\cdot(\phi\nabla\psi)dV = \int_V \left(\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi\right) dV. \tag{2.18}$$

Therefore,

$$\int_V \left(\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi\right) dV = \oint_S (\phi\nabla\psi)\cdot d\mathbf{S}. \tag{2.19}$$

Exchanging ϕ and ψ,

$$\int_V \left(\nabla\psi\cdot\nabla\phi + \psi\nabla^2\phi\right) dV = \oint_S (\psi\nabla\phi)\cdot d\mathbf{S}. \tag{2.20}$$

Subtracting Eq. (2.20) from Eq. (2.19) yields

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \oint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}, \tag{2.21}$$

which is the desired identity.

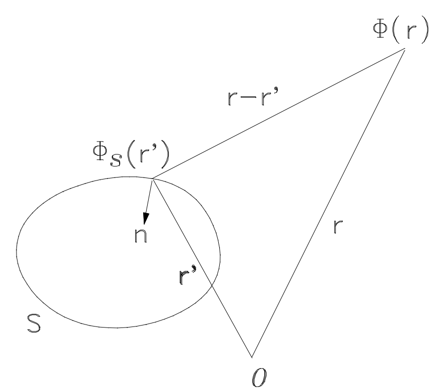


Figure 2-1: $\Phi_s(\mathbf{r'})$ is the potential specified on a closed surafce S, $\mathbf{n}$ is the coordinate normal to the surface directed away from the volume wherein the potential $\Phi(\mathbf{r})$ is to be evaluated.

We now apply the formula to electrostatic potential problems. Let ψ be the Green's function

$\psi = G$, satisfying

$$\nabla^2 G = -\delta(\mathbf{r} - \mathbf{r}'), \tag{2.22}$$

and $\phi = \Phi$ be the scalar potential satisfying Poisson's equation

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}. \tag{2.23}$$

Then, the terms in the LHS of Eq. (2.21) become

$$\begin{aligned} \int_V \Phi \nabla_{r'}^2 G dV' &= -\int_V \Phi\left(\mathbf{r}'\right) \delta(\mathbf{r} - \mathbf{r}') dV' \\ &= -\Phi(\mathbf{r}), \end{aligned}$$

provided the coordinates $\mathbf{r}$ resides in the volume V where we wish to find the potential, and

$$\int G \nabla^2 \Phi dV' = -\frac{1}{\varepsilon_0} \int G \rho(\mathbf{r}') dV'. \tag{2.24}$$

The RHS of Eq. (2.21) reduces to

$$\oint_S \left(\Phi \frac{\partial G}{\partial n} - G \frac{\partial \Phi}{\partial n} \right) dS, \tag{2.25}$$

where Φ is the potential on the closed surface and n is the coordinate normal to the surface directed *away* from the volume of interest as indicated in Fig.2-1. Therefore, the solution for the potential $\Phi(\mathbf{r})$ is given by

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \int_V G \rho(\mathbf{r}') dV' - \oint_S \left(\Phi_s \frac{\partial G}{\partial n} - G \frac{\partial \Phi_s}{\partial n} \right) dS. \tag{2.26}$$

At this stage, the Green's function is still arbitrary except it should satisfy the singular Poisson's equation in Eq. (2.22). The first term in the RHS allows evaluation of the potential for a given charge distribution as we saw earlier. The surface integral involves the potential on the closed surface Φ_s and its normal derivative, namely, the normal component of the electric field at the surface.

In usual boundary value problems, the potential on a closed surface is specified as a function of the surface coordinates. In this case, it is convenient to choose the Green's function so that it vanishes on the surface,

$$G = 0 \quad \text{on } S.$$

Then the last term in Eq. (2.26) vanishes, and the solution for the potential becomes

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' - \oint_S \Phi_s \frac{\partial G}{\partial n} dS', \quad G = 0 \text{ on } S. \tag{2.27}$$

In particular, if there are no charges in the region of concern $\rho = 0$, the potential is uniquely determined in terms of the surface potential alone,

$$\Phi(\mathbf{r}) = -\oint_S \Phi_s \frac{\partial G}{\partial n} dS', \quad \rho = 0 \text{ in } V, \ G = 0 \text{ on } S. \tag{2.28}$$

We have a freedom to make such a choice for the Green's function that it vanish on the closed surface S through adding general solutions to the particular solution of the singular Poisson's equation. Therefore, solving a potential boundary value problems for a given closed surface S boils down to finding a Green's function satisfying

$$\nabla^2 G = -\delta(\mathbf{r} - \mathbf{r}'), \qquad G = 0 \text{ on } S. \tag{2.29}$$

Once such an appropriate Green's function is found for a given surface shape S, the potential at arbitrary point can be found from Eq. (2.28) for a specified potential distribution $\Phi_s(\mathbf{r})$ on the surface.

In the following, Green's functions for some simple surface shapes will be found. It is noted that three dimensional Green's functions have dimensions of 1/length, two dimensional Green's functions are dimensionless, and one dimensional Greens functions have dimensions of length.

2.3 Examples of Green's Functions

2.3.1 Plane

Suppose that the potential is specified everywhere on an infinite (x, y) plane, $\Phi_s(x, y)$. The plane is closed at infinity and the method of Green's function is applicable. The Green's function is to be found as a solution to the equation

$$\nabla^2 G = -\delta(x - x')\delta(y - y')\delta(z - z'), \tag{2.30}$$

with the boundary condition $G = 0$, $z = 0$. Mathematically, the Green's function is equivalent to the potential due to a point charge placed near a *grounded* conducting plate that can be readily worked out using the method of image as shown in Fig.2-2,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}}\right), \tag{2.31}$$

where the second term in the RHS is the contribution from the image charge at the mirror point. Note that the Green's function is reciprocal and remains unchanged against the coordinates interchange,

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r}).$$

This is expected from the fact that the delta function in the original singular Poisson's equation is even,

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r}' - \mathbf{r}).$$

In the upper region $z > 0$,

$$\begin{aligned} \frac{\partial G}{\partial n} &= -\left.\frac{\partial G}{\partial z'}\right|_{z'=0} \\ &= -\frac{1}{2\pi}\frac{z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}. \end{aligned} \tag{2.32}$$

Therefore, for a surface potential $\Phi_s(x', y')$ specified as a function of (x', y'), the potential in the region $z > 0$ is given by

$$\Phi(\mathbf{r}) = \frac{z}{2\pi}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{\Phi_s(x', y')}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}. \tag{2.33}$$

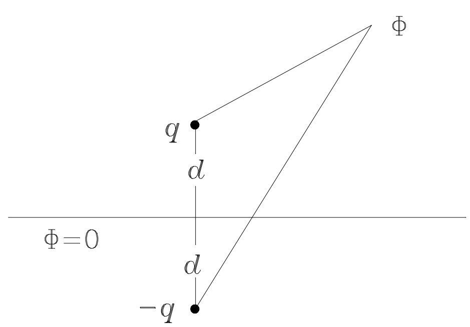


Figure 2-2: Image charge $-q$ for a large, grounded conducting plate. The potential due to q and $-q$ vanishes at the plate.

Let us apply this formula to the boundary condition on the (x, y) plane,

$$\Phi_s(\rho) = \begin{cases} V, & \rho < a \\ 0, & \rho > a \end{cases} \tag{2.34}$$

where $\rho = \sqrt{x^2 + y^2}$ is the radial distance on the plane as shown in Fig. 2-3. Physically, the boundary condition describes a large conducting plate which is grounded except for a circular region of radius a whose potential is maintained at V. The potential on the z-axis can be found

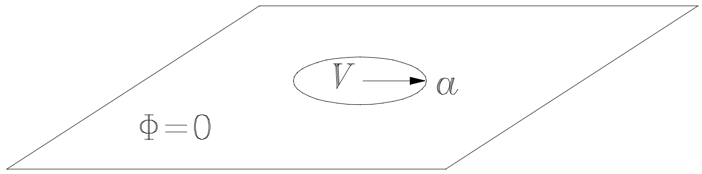


Figure 2-3: A large conducting plate is grounded except for a circular region which is at a potential V.

easily,

$$\begin{aligned}\Phi(z) &= \frac{zV}{2\pi}\int_0^a \frac{2\pi\rho' d\rho'}{(\rho'^2+z^2)^{3/2}} \\ &= V\left(1-\frac{z}{\sqrt{z^2+a^2}}\right), \quad z>0.\end{aligned} \tag{2.35}$$

The axial potential in the lower region $z<0$ can be found by observing the up-down symmetry and for both regions,

$$\Phi(z) = V\left(1-\frac{|z|}{\sqrt{z^2+a^2}}\right). \tag{2.36}$$

Then, the potential at arbitrary point (r,θ) is

$$\Phi(r,\theta) = \begin{cases} V\left[1-\frac{r}{a}\left|P_1(\cos\theta)\right| + \frac{1}{2}\left(\frac{r}{a}\right)^3\left|P_3(\cos\theta)\right| - \frac{3}{8}\left(\frac{r}{a}\right)^5\left|P_5(\cos\theta)\right| + \cdots\right], & r<a \\ \\ V\left[\frac{1}{2}\left(\frac{a}{r}\right)^2\left|P_1(\cos\theta)\right| - \frac{3}{8}\left(\frac{a}{r}\right)^4\left|P_3(\cos\theta)\right| + \cdots\right], & r>a \end{cases} \tag{2.37}$$

Note that at $r \gg a$, the potential is of dipole type,

$$\Phi(r\gg a) \propto \frac{1}{r^2}\left|\cos\theta\right|. \tag{2.38}$$

This problem should not be confused with the potential due to an isolated charged conducting disk which will be discussed later. The potential and electric field in the upper half region are identical to those realized by an ideally thin circular capacitor whose top plate is at a potential V and the lower plate at $-V$. The appearance of the dipole potential is thus an expected result. (For a thin capacitor with plate separation distance δ, the electric field between the plates diverges but the product $E\delta = 2V$ remains constant. Such a structure is called a double layer.)

2.3.2 Sphere

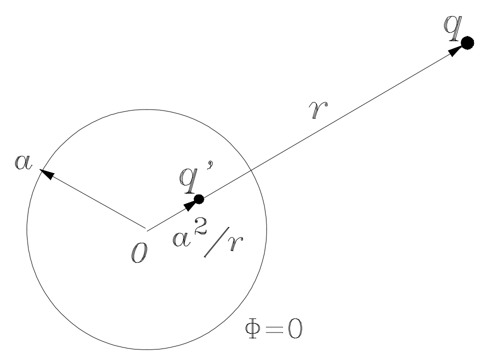


Figure 2-4: The image of charge q with respect to a grounded conducting sphere is $q' = -qa/r'$ located at $a^2\mathbf{r}'/r'^2$ where $\mathbf{r}'$ is the location of the charge q.

In finding a Green's function for a given surface shape, the method of images is most conveniently exploited. In the case of a sphere having a radius a, the Green's function can be found as a solution for the potential due to a charge q placed at a distance r' from the center of a *grounded* conducting sphere. In the case of a sphere having radius a, an image charge

$$q' = -\frac{a}{r'}q, \tag{2.39}$$

placed at

$$\mathbf{r}'' = \frac{a^2}{r'^2}\mathbf{r}', \tag{2.40}$$

together with the charge q, makes the surface potential vanish. This is illustrated in Fig.2-4. The potential due to charge q placed near a grounded conducting sphere is thus equivalent to that due to two charges, q and its image charge q', and is given by

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{\frac{a}{r'}}{|\mathbf{r}-\mathbf{r}''|}\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{(rr'/a)^2+a^2-2rr'\cos\gamma}}\right),\end{aligned} \tag{2.41}$$

which readily yields the Green's function for a sphere,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \left(\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}} - \frac{1}{\sqrt{(rr'/a)^2 + a^2 - 2rr' \cos\gamma}} \right). \tag{2.42}$$

Here γ is the angle between the two position vectors $\mathbf{r} = (r, \theta, \phi)$ and $\mathbf{r}' = (r', \theta', \phi')$. Its cosine value is

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi'). \tag{2.43}$$

The Green's function indeed vanishes on the sphere surface $r = a$ or $r' = a$. Again, the Green's function is invariant against coordinates exchange, $\mathbf{r} \leftrightarrow \mathbf{r}'$, that is, Green's functions are reciprocal.

For exterior ($r > a$) potential problems, the normal gradient $\partial G/\partial n$ is

$$\begin{aligned} \frac{\partial G}{\partial n} &= -\left.\frac{\partial G}{\partial r'}\right|_{r'=a} \\ &= \frac{1}{4\pi} \frac{a - \dfrac{r^2}{a}}{(r^2 + a^2 - 2ar\cos\gamma)^{3/2}}, \quad r > a \end{aligned} \tag{2.44}$$

and for interior ($r < a$) problems,

$$\begin{aligned} \frac{\partial G}{\partial n} &= +\left.\frac{\partial G}{\partial r'}\right|_{r'=a} \\ &= \frac{1}{4\pi} \frac{\dfrac{r^2}{a} - a}{(r^2 + a^2 - 2ar\cos\gamma)^{3/2}}, \quad r < a. \end{aligned} \tag{2.45}$$

Note that the normal coordinate n is directed *away* from the volume of interest. If the surface potential is specified as a function of θ' and ϕ', $\Phi_s(\theta', \phi')$, and there are no charges, the exterior potential at an arbitrary point $\mathbf{r} = (r, \theta, \phi)$ can be found from

$$\begin{aligned} \Phi(\mathbf{r}) &= -\oint \Phi_s \frac{\partial G}{\partial n} dS \\ &= \frac{1}{4\pi} \oint \frac{a(r^2 - a^2)}{(r^2 + a^2 - 2ar\cos\gamma)^{3/2}} \Phi_s(\theta', \phi') d\Omega' \\ &= \frac{a(r^2 - a^2)}{4\pi} \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \frac{\Phi_s(\theta', \phi')}{(r^2 + a^2 - 2ar\cos\gamma)^{3/2}}. \end{aligned} \tag{2.46}$$

Recalling the expansion of the function $1/|\mathbf{r} - \mathbf{r}'|$ in terms of the spherical harmonic functions

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi'), \quad r > r' \tag{2.47}$$

the exterior potential can be decomposed into multipole potentials,

$$\Phi(r,\theta,\phi)=\sum_{l,m}\left(\frac{a}{r}\right)^{l+1}Y_{lm}(\theta,\phi)\oint\Phi_s(\theta',\phi')Y_{lm}^*(\theta',\phi')d\Omega',\quad r>a. \tag{2.48}$$

The interior potential can be found using the expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{l,m}\frac{4\pi}{2l+1}\frac{r^l}{r'^{l+1}}Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi'),\quad r<r', \tag{2.49}$$

$$\Phi(r,\theta,\phi)=\sum_{l,m}\left(\frac{r}{a}\right)^{l}Y_{lm}(\theta,\phi)\oint\Phi_s(\theta',\phi')Y_{lm}^*(\theta',\phi')d\Omega',\quad r<a. \tag{2.50}$$

Example 2 ***Charge near a Floating Conducting Sphere***

To become familiar with the Green's function method, let us consider a somewhat trivial problem of finding the potential when a charge q is placed at a distance $\mathbf{d}$ from the center of a floating conducting sphere of radius a. The charge q and its image $q'=-\frac{a}{d}q$ at $(a/d)^2\mathbf{d}$ make the sphere potential 0 as we have just seen. However, since the floating sphere should carry no net charge, a charge $-q'=\frac{a}{d}q$ must be placed at the center of the sphere which raises the sphere potential to

$$\Phi_s=\frac{-q'}{4\pi\varepsilon_0 a}=\frac{q}{4\pi\varepsilon_0 d},\ d>a.$$

Therefore, the exterior potential can be found by summing contributions from q, its image q' and the charge $-q'$ at the center,

$$\Phi(r)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{d}|}-\frac{qa/d}{|\mathbf{r}-(a/d)^2\mathbf{d}|}+\frac{qa/d}{r}\right).$$

In this expression, the function

$$\frac{1}{4\pi}\left(\frac{1}{|\mathbf{r}-\mathbf{d}|}-\frac{a}{d}\frac{1}{|\mathbf{r}-(a/d)^2\mathbf{d}|}\right),$$

is the Green's function which vanishes on the sphere surface, $r=a$. The last term is in the form

$$\Phi_s\frac{a}{r},$$

where

$$\Phi_s=\frac{q}{4\pi\varepsilon_0 d},\ \text{(independent of } a\text{)}$$

is the surface potential. Indeed,

$$\begin{aligned}
-\oint \Phi_s \frac{\partial G}{\partial n} dS &= \frac{a(r^2-a^2)}{4\pi}\int_0^{\pi} \sin\theta' d\theta' \int_0^{2\pi} d\phi' \frac{\Phi_s(\theta',\phi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}} \\
&= \Phi_s \frac{a(r^2-a^2)}{2}\int_0^{\pi} \frac{1}{\left(r^2+a^2-2ar\cos\theta'\right)^{3/2}} \sin\theta' d\theta' \\
&= \Phi_s \frac{a}{r},
\end{aligned}$$

where θ' is measured from the direction of the vector $\mathbf{d}$. (This is allowed because of the symmetry.)

Example 3 ***Specified Potential on a Sphere Surface***

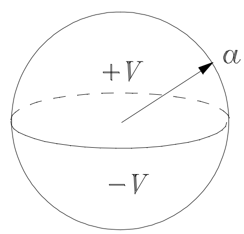


Figure 2-5: $\Phi_s = +V$ for $0 < \theta < \pi/2$, $-V$ for $\pi/2 < \theta < \pi$.

Let us find the potential outside a spherical shell of radius a whose top half is maintained at potential $+V$ and lower half at $-V$,

$$\Phi_s(\theta') = \begin{cases} +V, & 0 \le \theta' \le \dfrac{\pi}{2}, \\ -V, & \dfrac{\pi}{2} \le \theta' \le \pi, \end{cases} \tag{2.51}$$

as shown in Fig.2-5. Because of axial symmetry, only $m = 0$ terms survive the integration over the azimuthal angle ϕ'. Also, because of up-down antisymmetry, only odd l terms survive the integration over the polar angle θ'. Noting

$$\begin{aligned}
&\oint \Phi_s(\theta',\phi') Y_{l0}^*(\theta',\phi') d\Omega' \\
&= 2\pi \times 2V\sqrt{\frac{2l+1}{4\pi}}\int_0^1 P_l(\mu)d\mu, \quad l = 1,3,5,\cdots,
\end{aligned}$$

we readily find the exterior potential,

$$\Phi(r,\theta) = V\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \cdots\right], \quad r \ge a. \tag{2.52}$$

The interior potential is

$$\Phi(r,\theta) = V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \cdots\right], \quad r \le a. \tag{2.53}$$

The surface charge density on the sphere can be found from the normal component of the electric field,

$$\begin{aligned}\sigma &= -\varepsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=a+0} \\ &= \frac{\varepsilon_0 V}{a}\left[3P_1(\cos\theta) - \frac{7}{2}P_3(\cos\theta) + \cdots\right].\end{aligned}$$

The total surface charge on the upper hemisphere

$$q = 2\pi a^2 \int_0^{\pi/2} \sigma(\theta)\sin\theta d\theta,$$

simply diverges (albeit only logarithmically) and it is not possible to define the capacitance of the hemispheres. This is because of the assumption of ideally small gap separating the two hemispheres. If a small gap $\delta \ll a$ is assumed, a finite capacitance containing a factor $\ln(a/\delta)$ emerges.

2.3.3 Interior of Cylinder of Finite Length

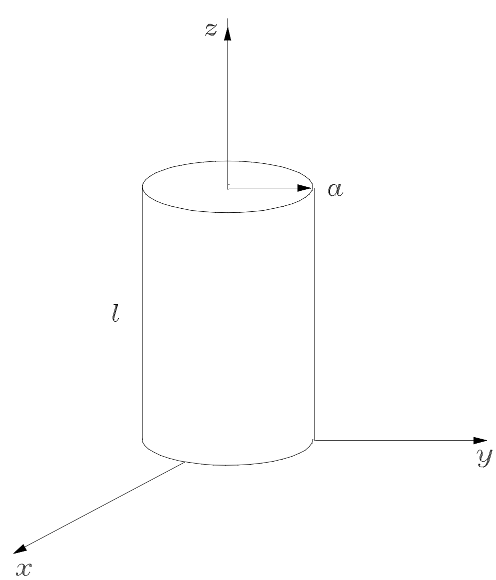


Figure 2-6: Cylinder of a finite length.

The Green's function for the interior of a cylinder of radius a and length l shown in Fig.2-6 can be found as a solution for the following singular Poisson's equation

$$\nabla^2 G = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right) G = -\frac{\delta(\rho - \rho')}{\rho}\delta(\phi - \acute{\phi}')\delta(z - z'), \tag{2.54}$$

with the boundary condition

$$G = 0, \;\; \rho = a, \;\; z = 0 \text{ and } l. \tag{2.55}$$

Since the Green's function should be periodic with respect to ϕ and should also be invariant with respect to exchange of ϕ and ϕ', the angular dependence can be assumed to be $\cos[m(\phi - \phi')]$ where m is an integer. Assuming the following separation of variables,

$$G(\mathbf{r}, \mathbf{r}') = \sum_m R_m(\rho) Z_m(z) \cos m(\phi - \phi'), \tag{2.56}$$

we see that the radial function $R_m(\rho)$ and the axial function $Z_m(z)$ satisfy, respectively,

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2\right) R_m(\rho) = 0, \tag{2.57}$$

$$\left(\frac{d^2}{dz^2} - k^2\right) Z_m(z) = 0, \tag{2.58}$$

where k^2 is a separation constant which can be either positive or negative.

Let us first consider the case $k^2 > 0$. Solutions for $R_m(\rho)$ which satisfies the boundary condition $R_m(\rho = a) = 0$ is the m-th order Bessel function,

$$R_{mn}(\rho, \rho') = J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right), \tag{2.59}$$

where x_{mn} is the n-th root of $J_m(x) = 0$. (The Bessel function of the second kind $N_m(x)$ is discarded because it diverges on the axis, $\rho = 0$.)

Solutions for the axial function $Z_m(z)$ are $e^{\pm kz}$ or $\sinh(kz)$ and $\cosh(kz)$. The boundary condition for $Z_m(z)$ is it vanish at $z = 0$ and l. Therefore, we can construct the axial function as follows,

$$Z_m(z, z') = \begin{cases} \sinh(k_{mn}z)\sinh[k_{mn}(l - z')], & 0 < z < z' < l, \\ \\ \sinh[k_{mn}(l - z)]\sinh(k_{mn}z'), & 0 < z' < z < l, \end{cases} \tag{2.60}$$

where $k_{mn} = x_{mn}/a$. A more fancy way to write $Z_m(z, z')$ is

$$Z_m(z, z') = \sinh[k_{mn}\min(z, z')]\sinh\{k_{mn}[l - \max(z, z')]\}. \tag{2.61}$$

The Green's function may thus be assumed in the form

$$G(r, r') = \sum_{m,n} A_{mn} R_{mn}(\rho, \rho') Z_{mn}(z, z') \cos[m(\phi - \phi')]. \tag{2.62}$$

The expansion coefficient A_{mn} can be determined from the discontinuity in the derivative of the axial function $Z_{mn}(z, z')$ at z',

$$\frac{d}{dz} Z_{mn}\bigg|_{z=z'+0} = -k_{mn} \cosh[k_{mn}(l - z')] \sinh(k_{mn} z'),$$

$$\frac{d}{dz} Z_{mn}\bigg|_{z=z'-0} = +k_{mn} \cosh(k_{mn} z') \sinh[(k_{mn}(l - z')].$$

Then, a singularity appears in the second order derivative,

$$\frac{d^2}{dz^2} Z_{mn} = -k_{mn} \sinh(k_{mn} l) \delta(z - z'), \tag{2.63}$$

which is compatible with the delta function in the RHS of the original singular Poisson's equation in Eq. (2.54). Eq. (2.54) now reduces to

$$\sum_{mn} A_{mn} k_{mn} \sinh(k_{mn} l) J_m(k_{mn}\rho) J_m(k_{mn}\rho') \cos m(\phi - \phi') = \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi'). \tag{2.64}$$

Multiplying both sides by $\rho' J_m(k_{mn}\rho') \cos m\phi'$ and integrating over ρ' and ϕ', we find

$$A_{0n} = \frac{1}{\pi a^2 k_{mn}} \frac{1}{J_{m+1}^2(k_{mn} a) \sinh(k_{mn} l)}, \quad m = 0, \tag{2.65}$$

$$A_{mn} = \frac{2}{\pi a^2 k_{mn}} \frac{1}{J_{m+1}^2(k_{mn} a) \sinh(k_{mn} l)}, \quad m \geq 1, \tag{2.66}$$

where use has been made of the following integral,

$$\int_0^a \rho J_m^2(k_{mn}\rho) d\rho = \frac{a^2}{2} J_{m+1}^2(k_{mn} a). \tag{2.67}$$

The final form of the desired Green's function is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\pi a} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right) Z_{mn}(z, z')}{x_{mn} J_{m+1}^2(x_{mn}) \sinh(k_{mn} l)} \cos[m(\phi - \phi')] \varepsilon_m, \tag{2.68}$$

where

$$\varepsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m \geq 1 \end{cases}$$

If one does not like the appearance of ε_m, the summation over m can be changed to from $-\infty$ to ∞,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\pi a} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right) Z_{mn}(z, z')}{x_{mn} J_{m+1}^2(x_{mn}) \sinh(k_{mn} l)} \cos[m(\phi - \phi')]. \tag{2.69}$$

If it is assumed that $k^2 = -\kappa^2 < 0$, appropriate general solutions to

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} - \kappa^2\right) R_m(\rho) = 0, \tag{2.70}$$

$$\left(\frac{d^2}{dz^2} + \kappa^2\right) Z_m(z) = 0, \tag{2.71}$$

are

$$R_m\left(\rho, \rho'\right) = \left[K_m\left(\kappa_m a\right) I_m\left(\kappa_m \rho'\right) - I_m\left(\kappa_m a\right) K_m\left(\kappa_m \rho'\right)\right] I_m\left(k_m \rho\right), \quad \rho < \rho' < a, \tag{2.72}$$

$$R_m\left(\rho, \rho'\right) = \left[K_m\left(\kappa_m a\right) I_m\left(\kappa_m \rho\right) - I_m\left(\kappa_m a\right) K_m\left(\kappa_m \rho\right)\right] I_m\left(k_m \rho'\right), \quad \rho' < \rho < a, \tag{2.73}$$

with $\kappa_m = m\pi/l$ and

$$Z_m\left(z\right) = \sin\left(\kappa_m z\right) \sin\left(\kappa_m z'\right), \tag{2.74}$$

from which the Green's function can be constructed. Remaining calculation is left for exercise. The reader should appreciate how a delta function $\delta\left(\rho - \rho'\right)$ appears from the term

$$\frac{d^2 R_m}{d\rho^2}. \tag{2.75}$$

One may wonder about Green's function for the *exterior* region of a cylinder of finite length. This problem appears to be a difficult one and analytical expressions are not available to the author's knowledge. It may be the case the problem can only be solved numerically.

2.3.4 Long Cylinder (3-Dimensional)

Three dimensional Green's function for a long cylinder satisfies

$$\nabla^2 G = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right) G = -\frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \acute{\phi}') \delta(z - z'), \tag{2.76}$$

which is to be solved for the boundary conditions

$$G(\rho = a) = 0, \quad G(z = \pm\infty) = 0. \tag{2.77}$$

Following the same procedure as in the preceding example, the interior solution for interior $\rho, \rho' < a$ may be assumed as

$$G(\mathbf{r}, \mathbf{r}') = \sum_{m,n} A_{mn} J_m(k_{mn}\rho) J_m(k_{mn}\rho') \cos[m(\phi - \phi')] \exp[-k_{mn}|z - z'|], \tag{2.78}$$

where

$$k_{mn} = \frac{x_{mn}}{a}, \tag{2.79}$$

and x_{mn} is the n-th root of $J_m(x) = 0$. Since

$$\frac{d^2}{dz^2} e^{-k_{mn}|z-z'|} = k_{mn}^2 e^{-k_{mn}|z-z'|} - 2k_{mn}\delta(z - z'), \tag{2.80}$$

$$\sum_{m=-\infty}^{\infty} \cos[m(\phi - \phi')] = 2\pi\delta(\phi - \phi'), \tag{2.81}$$

we readily find the interior Green's function (ρ, $\rho' < a$)

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{J_m(k_{mn}\rho) J_m(k_{mn}\rho')}{k_{mn} J_{m+1}^2(k_{mn}a)} \cos[m(\phi - \phi')] e^{-k_{mn}|z-z'|}. \tag{2.82}$$

For exterior of a long cylinder, solutions to the equation

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right) G = -\frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \acute{\phi}') \delta(z - z'),$$

can be found in terms of Fourier transform with respect to the z-coordinate. Let $G(\mathbf{r}, \mathbf{r}')$ be

$$G(\mathbf{r}, \mathbf{r}') = \sum_m e^{im(\phi - \phi')} \int R_m(\rho, \rho'; k) e^{ik(z - z')} dk. \tag{2.83}$$

The radial function $R_m(\rho, \rho'; k)$ satisfies

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} - k^2 \right) R_m(\rho, \rho'; k) = -\frac{\delta(\rho - \rho')}{2\pi\rho}. \tag{2.84}$$

Elementary solutions are the modified Bessel functions $I_m(k\rho)$ and $K_m(k\rho)$ and we can construct following solutions which remain bounded in the region $a < \rho < \infty$,

$$R_m(\rho, \rho'; k) = \begin{cases} A(k)I_m(k\rho) + B(k)K_m(k\rho), & a < \rho < \rho' < \infty, \\ \\ C(k)K_m(k\rho), & a < \rho' < \rho < \infty, \end{cases} \tag{2.85}$$

The boundary conditions are $R_m(\rho = a) = 0$ and $R_m(\rho)$ be continuous at $\rho = \rho'$,

$$A(k)I_m(ka) + B(k)K_m(ka) = 0, \tag{2.86}$$

$$A(k)I_m(k\rho') + B(k)K_m(k\rho') = C(k)K_m(k\rho'). \tag{2.87}$$

Then,

$$R_m(\rho, \rho'; k) = \begin{cases} A(k)\left(I_m(k\rho) - \dfrac{I_m(ka)}{K_m(ka)}K_m(k\rho)\right), & a < \rho < \rho' < \infty, \\ \\ A(k)\dfrac{1}{K_m(k\rho')}\left(I_m(k\rho') - \dfrac{I_m(ka)}{K_m(ka)}K_m(k\rho')\right)K_m(k\rho), & a < \rho' < \rho < \infty. \end{cases} \tag{2.88}$$

The unknown function $A(k)$ can be found from the discontinuity in the derivative at $\rho = \rho'$,

$$\frac{d^2}{d\rho^2}R_m(\rho, \rho'; k)\bigg|_{\rho=\rho'} = kA(k)\frac{K_m'(k\rho')I_m(k\rho') - K_m(k\rho')I_m'(k\rho')}{K_m(k\rho')}\delta(\rho - \rho') \tag{2.89}$$

$$= -\frac{A(k)}{K_m(k\rho')}\frac{\delta(\rho - \rho')}{\rho'}, \tag{2.90}$$

where again use has been made of the Wronskian of the modified Bessel functions,

$$I_m'(x)K_m(x) - I_m(x)K_m'(x) = \frac{1}{x}. \tag{2.91}$$

We thus find

$$A(k) = \frac{K_m(k\rho')}{2\pi}, \tag{2.92}$$

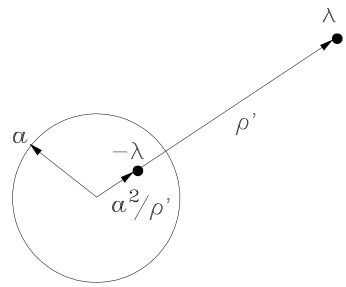
and $R_m(\rho, \rho'; k)$ reduces to

$$R_m(\rho, \rho'; k) = \begin{cases} \dfrac{1}{2\pi}K_m(k\rho')\left(I_m(k\rho) - \dfrac{I_m(ka)}{K_m(ka)}K_m(k\rho)\right), & a < \rho < \rho' < \infty, \\ \\ \dfrac{1}{2\pi}\left(I_m(k\rho') - \dfrac{I_m(ka)}{K_m(ka)}K_m(k\rho')\right)K_m(k\rho), & a < \rho' < \rho < \infty. \end{cases} \tag{2.93}$$

The exterior Green's function of a long cylinder is given by

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \sum_m e^{im(\phi-\phi')} \int R_m(\rho, \rho'; k) e^{ik(z-z')} dk. \tag{2.94}$$

2.3.5 Long Cylinder (2-Dimensional)



Cross-section of a long cylinder. $\pm\lambda$ are the line charge and its image, respectively, that together make the cylinder surface an equipotential surface,

$$\Phi(\rho = a) = \lambda/(2\pi\varepsilon_0) \ln(a/\rho').$$

For boundary value problems in which z-dependence is suppressed, it is convenient to formulate a two dimensional Green's function. Two dimensional Green's function for a long cylinder is to be found from

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta_2(\mathbf{r} - \mathbf{r}'), \tag{2.95}$$

where $\delta_2(\mathbf{r} - \mathbf{r}')$ is the two-dimensional delta function. In the cylindrical geometry, it is given by

$$\delta_2(\mathbf{r} - \mathbf{r}') = \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi'), \tag{2.96}$$

and the Green's function satisfies

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) G = -\frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi'). \tag{2.97}$$

In this case, the method of image can be exploited very conveniently. Let us consider a long line charge λ (C/m) placed at (ρ', ϕ') parallel to a long, grounded conducting cylinder of radius a. A negative line charge $-\lambda$ placed at (ρ'', ϕ') where

$$\rho'' = \frac{a^2}{\rho'}, \tag{2.98}$$

makes the cylinder surface an equipotential surface at a potential

$$\Phi_s = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{a}{\rho'}\right). \tag{2.99}$$

Since we are seeking a potential that vanishes on the cylinder surface $\rho = a$, the constant potential Φ_s can be subtracted from the potential due to two line charges λ and $-\lambda$,

$$\Phi(\mathbf{r}, \mathbf{r}') = -\frac{\lambda}{2\pi\varepsilon_0} \left[\ln\left|\mathbf{r} - \mathbf{r}'\right| - \ln\left|\mathbf{r} - \mathbf{r}''\right| + \ln\left(\frac{a}{\rho'}\right)\right], \tag{2.100}$$

where

$$\left|\mathbf{r} - \mathbf{r}'\right| = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}, \tag{2.101}$$

$$\begin{aligned} \left|\mathbf{r} - \mathbf{r}''\right| &= \sqrt{\rho^2 + \rho''^2 - 2\rho\rho'' \cos(\phi - \phi')} \\ &= \sqrt{\rho^2 + \left(\frac{a^2}{\rho'}\right)^2 - 2\frac{a^2\rho}{\rho'} \cos(\phi - \phi')}. \end{aligned} \tag{2.102}$$

The desired Green's function is

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi} \ln\left(\frac{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}}{\sqrt{(\rho\rho'/a)^2 + a^2 - 2\rho\rho' \cos(\phi - \phi')}}\right). \tag{2.103}$$

For exterior Dirichlet problems, the normal derivative at the cylinder surface is

$$\frac{\partial G}{\partial n} = -\left.\frac{\partial G}{\partial \rho'}\right|_{\rho'=a+0} = \frac{1}{2\pi} \frac{a - \dfrac{\rho^2}{a}}{\rho^2 + a^2 - 2a\rho\cos(\phi - \phi')}, \;\; \rho > a, \tag{2.104}$$

and for interior,

$$\frac{\partial G}{\partial n} = \left.\frac{\partial G}{\partial \rho'}\right|_{\rho'=a-0} = \frac{1}{2\pi} \frac{\dfrac{\rho^2}{a} - a}{\rho^2 + a^2 - 2a\rho\cos(\phi - \phi')}, \;\; \rho < a. \tag{2.105}$$

If the potential on a long cylindrical surface is specified as a function of the angle ϕ, $\Phi_s(\phi)$, the potential off the surface can be calculated from

$$\Phi(\rho, \phi) = -a \oint \Phi_s(\phi') \frac{\partial G}{\partial n} d\phi'. \tag{2.106}$$

For the interior $(\rho < a)$, the potential is given by

$$\begin{aligned}\Phi(\rho,\phi) &= \frac{1}{2\pi}\int_0^{2\pi}\Phi_s(\phi')\frac{a^2-\rho^2}{a^2+\rho^2-2a\rho\cos(\phi-\phi')}d\phi' \\ &= \frac{1}{2\pi}\int_0^{2\pi}\Phi_s(\phi')\left(1+2\sum_{m=1}^{\infty}\left(\frac{\rho}{a}\right)^m\cos[m(\phi-\phi')]\right)d\phi', \end{aligned} \tag{2.107}$$

where use is made of the following expansion,

$$\frac{a^2-\rho^2}{a^2+\rho^2-2a\rho\cos(\phi-\phi')} = 1+2\sum_{m=1}^{\infty}\left(\frac{\rho}{a}\right)^m\cos[m(\phi-\phi')].$$

Example 4

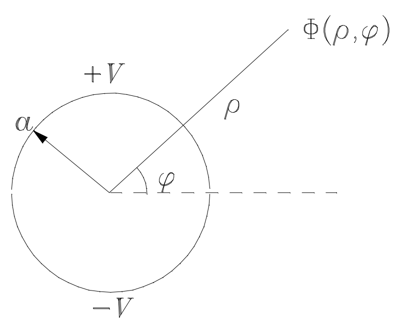


Figure 2-7: $\Phi = V$ for $0 < \phi < \pi$, $\Phi = -V$ for $-\pi < \phi < 0$ on the surface of a long cylinder. (Example of 2-D Green's function.)

As an example, let us consider a long conducting cylinder consisting of two equal troughs. The upper half in the region $0 < \phi < \pi$ is at a potential V and the lower half $-\pi < \phi < 0$ is at a potential $-V$ as shown Fig.2-7. The exterior potential $\rho > a$ is given by

$$\Phi(\rho,\phi) = V\frac{\rho^2-a^2}{2\pi}\left(\int_0^{\pi}\frac{1}{\rho^2+a^2-2a\rho\cos(\phi-\phi')}d\phi' - \int_{-\pi}^{0}\frac{1}{\rho^2+a^2-2a\rho\cos(\phi-\phi')}d\phi'\right). \tag{2.108}$$

The first integral can be effected by changing the variable from ϕ' to θ through $\phi' - \pi/2 = \theta$,

$$\begin{aligned}
&\int_{-\pi/2}^{\pi/2} \frac{1}{\rho^2 + a^2 + 2a\rho\sin(\theta - \phi)} d\theta \\
= \quad & \frac{2}{\rho^2 - a^2}\left[\tan^{-1}\left(\frac{(\rho^2 + a^2)\tan\left(\dfrac{\theta - \phi}{2}\right) + 2a\rho}{\rho^2 - a^2}\right)\right]_{-\pi/2}^{\pi/2} \\
= \quad & \frac{2}{\rho^2 - a^2}\left[\tan^{-1}\left(\frac{(\rho^2 + a^2)(\cot\phi - \tan\phi) + 2a\rho}{\rho^2 - a^2}\right) + \tan^{-1}\left(\frac{(\rho^2 + a^2)(\cot\phi + \tan\phi) + 2a\rho}{\rho^2 - a^2}\right)\right] \\
= \quad & \frac{2}{\rho^2 - a^2}\left[\tan^{-1}\left(\frac{2a\rho\sin\phi}{\rho^2 - a^2}\right) - \frac{\pi}{2}\right], \qquad (2.109)
\end{aligned}$$

where use has been made of the identities,

$$\tan\left(\frac{\pi}{4} \pm \frac{x}{2}\right) = \cot x \pm \tan x,$$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right),$$

$$\tan^{-1} x = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right).$$

Similarly, the second integral yields

$$\frac{2}{\rho^2 - a^2}\left[\tan^{-1}\left(\frac{2a\rho\sin\phi}{\rho^2 - a^2}\right) + \frac{\pi}{2}\right], \qquad (2.110)$$

and the potential becomes

$$\Phi(\rho, \phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2a\rho\sin\phi}{\rho^2 - a^2}\right), \quad \rho > a. \qquad (2.111)$$

The interior potential is

$$\Phi(\rho, \phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2a\rho\sin\phi}{a^2 - \rho^2}\right), \quad \rho < a. \qquad (2.112)$$

2.3.6 Wedge

A wedge is formed by two large plates intersecting at an angle γ as illustrated in Fig.2-8.

The potential due to a point charge q at (ρ', ϕ', z') with the boundary conditions $\Phi = 0$ at the plates $\phi = 0$ and $\phi = \gamma$, and $\rho = \infty$, $|z| = \infty$ essentially gives the Green's function. We thus seek

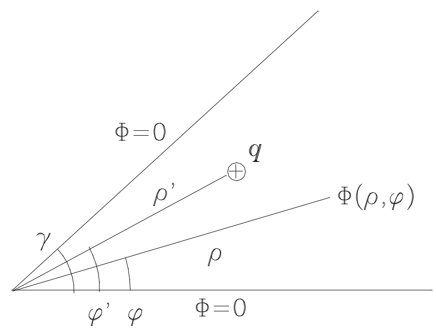


Figure 2-8: A wedge formed by two large conducting plates intersecting at an angle γ.

a solution to the Poisson's equation

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}\right)G=-\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')\delta(z-z'), \tag{2.113}$$

subject to the those boundary conditions. As in the case of 3-dimensional Green's function for a long cylinder, we Fourier transform the Green's function,

$$G(r,\phi,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(\rho,\phi,k)e^{ik(z-z')}dk. \tag{2.114}$$

The angular dependence of the Green's function can be assumed to be

$$\sin\left(\frac{m\pi}{\gamma}\phi\right)\sin\left(\frac{m\pi}{\gamma}\phi'\right), \tag{2.115}$$

which indeed vanishes at $\phi=0$ and $\phi=\gamma$. We thus assume

$$g(\rho,\phi,k)=\sum_m A_m R_m(\rho)\sin\left(\frac{m\pi}{\gamma}\phi\right)\sin\left(\frac{m\pi}{\gamma}\phi'\right), \tag{2.116}$$

to obtain

$$\sum_m A_m\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{1}{\rho^2}\left(\frac{m\pi}{\gamma}\right)^2-k^2\right]R_m(\rho)\sin\left(\frac{m\pi}{\gamma}\phi\right)\left(\sin\frac{m\pi}{\gamma}\phi'\right)=-\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi'). \tag{2.117}$$

The radial function can be composed of the modified Bessel functions,

$$R_m(\rho) = \begin{cases} I_{m\pi/\gamma}(k\rho)K_{m\pi/\gamma}(k\rho'), & \rho < \rho', \\ \\ I_{m\pi/\gamma}(k\rho')K_{m\pi/\gamma}(k\rho), & \rho' < \rho. \end{cases} \tag{2.118}$$

The derivative of the radial function $R_m(\rho)$ has discontinuity at $\rho = \rho'$, and the second order derivative yields

$$\frac{d^2R_m(\rho)}{d\rho^2} = -\frac{1}{\rho'}\delta(\rho - \rho'), \tag{2.119}$$

where the Wronskian of the modified Bessel functions,

$$I_\nu(x)K'_\nu(x) - I'_\nu(x)K_\nu(x) = -\frac{1}{x},$$

has been substituted. The expansion coefficient A_m is thus determined as

$$A_m = \frac{1}{\pi\gamma}, \tag{2.120}$$

and the desired Green's function is

$$G(\mathbf{r}, \mathbf{r}') = \frac{2}{\pi\gamma}\sum_m \int_0^\infty R_m(\rho, \rho')\cos[k(z - z')]dk\sin(\nu\phi)\sin(\nu\phi'), \tag{2.121}$$

where

$$\nu = \frac{m\pi}{\gamma}. \tag{2.122}$$

We will encounter an application of wedge potential in the section of inversion method later in this Chapter.

Example 5 *Line Charge parallel to a Long Dielectric Cylinder*

Consider a long line charge with charge density λ (C m^{-1}) placed parallel to a long dielectric cylinder of radius a. The distance between the line charge and the cylinder axis is b. The potential outside the cylinder can be sought by assuming an image line charge λ' at the position a^2/b and another image $-\lambda'$ at the center,

$$\Phi_{\rho>a}(\rho, \phi) = -\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\rho^2 + b^2 - 2\rho b\cos\phi\right) - \frac{\lambda'}{4\pi\varepsilon_0}\ln\left(\rho^2 + \left(\frac{a^2}{b}\right)^2 - 2\rho\frac{a^2}{b}\cos\phi\right) + \frac{\lambda'}{4\pi\varepsilon_0}\ln\rho^2,$$

where ϕ is measured relative to the location of line charge λ. The interior potential is free from singularity and may be assumed to be due to an image λ'' at the location of the line charge λ,

$$\Phi_{\rho<a}(\rho,\phi) = -\frac{\lambda''}{4\pi\varepsilon_0}\ln\left(\rho^2+b^2-2\rho b\cos\phi\right) + \frac{\lambda'}{4\pi\varepsilon_0}\ln b^2.$$

Note that the constant potential (the last term)

$$\frac{\lambda'}{4\pi\varepsilon_0}\ln b^2,$$

is needed to match same outer potential at $r = a$,

$$\Phi_{\rho>a}(r=a) = -\frac{\lambda+\lambda'}{4\pi\varepsilon_0}\ln\left(a^2+b^2-2ab\cos\phi\right) + \frac{\lambda'}{4\pi\varepsilon_0}\ln b^2.$$

The pertinent boundary conditions are E_ϕ and D_r be continuous at $r = a$ which yield

$$\lambda+\lambda' = \lambda'',$$

and

$$\varepsilon_0\left(\lambda-\lambda'\right) = \varepsilon\lambda''.$$

Then

$$\lambda' = \frac{\varepsilon_0-\varepsilon}{\varepsilon_0+\varepsilon}\lambda, \text{ and } \lambda'' = \frac{2\varepsilon_0}{\varepsilon_0+\varepsilon}\lambda.$$

The attracting force to act on the unit length of the line charge is given by

$$\frac{F}{l} = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{\lambda'}{b-(a^2/b)} - \frac{\lambda''}{b}\right) = \frac{\lambda^2}{2\pi\varepsilon_0}\frac{a^2}{b\,(b^2-a^2)}\frac{\varepsilon_0-\varepsilon}{\varepsilon_0+\varepsilon}, \text{ (N m}^{-1}\text{)}.$$

A magnetically dual problem is the case of long current I parallel to a magnetic cylinder with a permeability μ. The image currents are

$$I' = \frac{\mu-\mu_0}{\mu+\mu_0}I \text{ at } \rho = \frac{a^2}{b},$$

and

$$I'' = \frac{2\mu}{\mu+\mu_0}I \text{ at the axis.}$$

2.4 Other Useful Rectilinear Coordinates

The familiar three coordinate systems, cartesian, spherical, and cylindrical, are frequently used in analyzing potential problems. However, there are some 30 known coordinate systems developed for specific problems. For simple electrode shapes, potential problems can be rendered one dimensional by a suitable choice of coordinates. However, in some coordinates, solutions to Laplace equations are not always completely separable. We have encountered one such example in Chapter 1, the toroidal coordinates, in analyzing the potential due to a ring charge. In this section, some coordinate systems useful for potential problems will be introduced.

2.4.1 Oblate Spheroidal Coordinates (η, θ, ϕ)

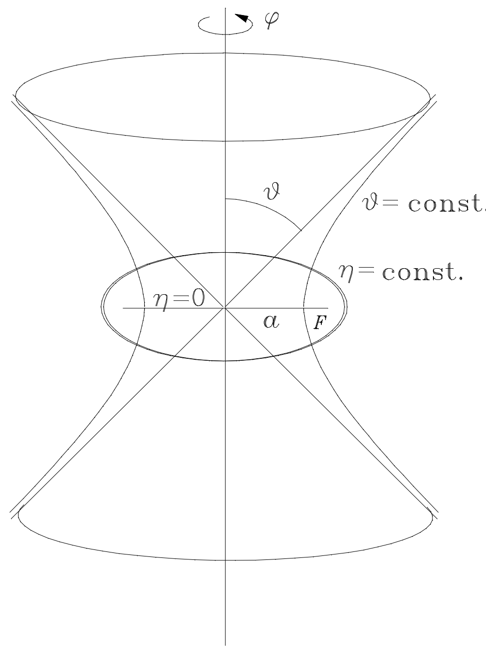


Figure 2-9: Oblate spheroidal coordinates (η, θ, ϕ). $\eta \to 0$ degenerates to a thin disk of radius a. $\theta =$ cons. describes the surface of a hyperboloid.

The oblate spherical coordinates (η, θ, ϕ) are related to the cartesian coordinates through the following transformation,

$$\begin{cases} x = a\cosh\eta\sin\theta\cos\phi \\ y = a\cosh\eta\sin\theta\sin\phi \\ z = a\sinh\eta\cos\theta \end{cases} \tag{2.123}$$

A surface of constant η is the surface of an oblate spheroid described by

$$\frac{x^2+y^2}{(a\cosh\eta)^2} + \frac{z^2}{(a\sinh\eta)^2} = 1, \tag{2.124}$$

as shown in Fig.2-9. In the limit of $\eta \to 0$, the surface degenerates to a thin disk of radius a with negligible thickness, and in the opposite limit $\eta \gg 1$, the surface approaches a sphere with a radius $r = a\cosh\eta \simeq a\sinh\eta$. This coordinate system is convenient if electrode shapes are an oblate sphere or disk. A surface of constant θ is a hyperboloid described by

$$\frac{x^2+y^2}{(a\sin\theta)^2} - \frac{z^2}{(a\sin\theta)^2} = 1. \tag{2.125}$$

The metric coefficients are

$$h_\eta = \sqrt{\left(\frac{\partial x}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2 + \left(\frac{\partial z}{\partial\eta}\right)^2} = a\sqrt{\cosh^2\eta - \sin^2\theta}, \tag{2.126}$$

$$h_\theta = \sqrt{\left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2} = h_\eta, \tag{2.127}$$

$$h_\phi = \sqrt{\left(\frac{\partial x}{\partial\phi}\right)^2 + \left(\frac{\partial y}{\partial\phi}\right)^2 + \left(\frac{\partial z}{\partial\phi}\right)^2} = a\cosh\eta\sin\theta. \tag{2.128}$$

The Laplace equation in the oblate spherical coordinates can thus be written down as

$$\frac{1}{h_\eta h_\theta h_\phi}\left\{\frac{\partial}{\partial\eta}\left(\frac{h_\theta h_\phi}{h_\eta}\frac{\partial\Phi}{\partial\eta}\right) + \frac{\partial}{\partial\theta}\left(\frac{h_\eta h_\phi}{h_\theta}\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{h_\eta h_\theta}{h_\phi}\frac{\partial\Phi}{\partial\phi}\right)\right\} = 0, \tag{2.129}$$

which reduces to

$$\frac{1}{\cosh^2\eta - \sin^2\theta}\left(\frac{\partial^2}{\partial\eta^2} + \tanh\eta\frac{\partial}{\partial\eta} + \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right)\Phi + \frac{1}{\cosh^2\eta\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0. \tag{2.130}$$

Assuming a separated solution $\Phi(\eta,\theta,\phi) = F_1(\eta)F_2(\theta)e^{im\phi}$, ($m$ = integer), we obtain

$$\left(\frac{d^2}{d\eta^2} + \tanh\eta\frac{d}{d\eta} - l(l+1) + \frac{m^2}{\cosh^2\eta}\right)F_1(\eta) = 0, \tag{2.131}$$

$$\left(\frac{d^2}{d\theta^2} + \cot\theta\frac{d}{d\theta} + l(l+1) - \frac{m^2}{\sin^2\theta}\right)F_2(\theta) = 0, \tag{2.132}$$

where $l(l+1)$ is a separation constant. Eq. (2.132) is the standard form of the Legendre equation and solutions for $F_2(\theta)$ are

$$F_2(\theta) = P_l^m(\cos\theta),\ Q_l^m(\cos\theta). \tag{2.133}$$

Eq. (2.131) can be rewritten as

$$\left(\frac{d^2}{d\eta^2} + \frac{\sinh\eta}{\cosh\eta}\frac{d}{d\eta} - l(l+1) + \frac{m^2}{1+\sinh^2\eta}\right)F_1(\eta) = 0, \tag{2.134}$$

which is also the Legendre equation with a variable $i\sinh\eta$. Therefore, solutions for $F_1(\eta)$ are

$$F_1(\eta) = P_l^m(i\sinh\eta),\ Q_l^m(i\sinh\eta), \tag{2.135}$$

and general solution to Laplace equation can be constructed from these elementary solutions.

If a point charge q is placed at $\left(\eta', \theta', \phi'\right)$, the potential in terms of the oblate spheroidal coordinates can be found as

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{q}{\varepsilon_0 a}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-|m|)!}{(l+|m|)!}\left\{\begin{array}{c} P_l^m(i\sinh\eta)\,Q_l^m(i\sinh\eta') \\ P_l^m(i\sinh\eta')\,Q_l^m(i\sinh\eta)\end{array}\right\} Y_{lm}(\theta,\phi)\,Y_{lm}^*\left(\theta',\phi'\right), \left\{\begin{array}{c}\eta<\eta' \\ \eta>\eta'\end{array}\right\}\end{aligned} \tag{2.136}$$

Derivation of this expression is left for exercise. The Wronskian of the Legendre functions,

$$P_l^m(x)\frac{d}{dx}Q_l^m(x) - Q_l^m(x)\frac{d}{dx}P_l^m(x) = \frac{1}{x^2-1}\frac{(l+|m|)!}{(l-|m|)!}, \tag{2.137}$$

should be useful. Furthermore, the Green's function for an oblate spheroidal surface described by $\eta = \eta_0$ can readily be worked out to be:
for $\eta_0 < \eta < \eta'$,

$$\begin{aligned}G\left(\mathbf{r},\mathbf{r}'\right) &= \frac{1}{a}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-|m|)!}{(l+|m|)!}P_l^m(i\sinh\eta)\,Q_l^m\left(i\sinh\eta'\right)Y_{lm}(\theta,\phi)\,Y_{lm}^*\left(\theta',\phi'\right) \\ &\quad -\frac{1}{a}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-|m|)!}{(l+|m|)!}\frac{P_l^m(i\sinh\eta_0)}{Q_l^m(i\sinh\eta_0)}Q_l^m(i\sinh\eta)\,Q_l^m\left(i\sinh\eta'\right)Y_{lm}(\theta,\phi)\,Y_{lm}^*\left(\theta',\phi'\right),\end{aligned} \tag{2.138}$$

and for $\eta_0 < \eta' < \eta$,

$$\begin{aligned}G\left(\mathbf{r},\mathbf{r}'\right) &= \frac{1}{a}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-|m|)!}{(l+|m|)!}P_l^m\left(i\sinh\eta'\right)Q_l^m(i\sinh\eta)\,Y_{lm}(\theta,\phi)\,Y_{lm}^*\left(\theta',\phi'\right) \\ &\quad -\frac{1}{a}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-|m|)!}{(l+|m|)!}\frac{P_l^m(i\sinh\eta_0)}{Q_l^m(i\sinh\eta_0)}Q_l^m(i\sinh\eta)\,Q_l^m\left(i\sinh\eta'\right)Y_{lm}(\theta,\phi)\,Y_{lm}^*\left(\theta',\phi'\right).\end{aligned} \tag{2.139}$$

Example 6 ***Charged Conducting Disk***

A thin disk of radius a is described by $\eta = 0$ in the oblate spherical coordinates. If a constant η surface is an equipotential surface, the potential off the surface is a function of η only, that is, the potential problem becomes one dimensional. This is the most advantageous merit of using a coordinate system most suitable for particular potential problems. The relevant solution which

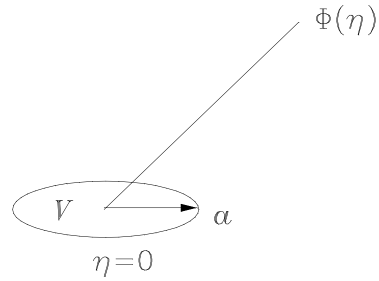


Figure 2-10: A charged conducting disk of radius a. A disk is described by $\eta = 0$, $0 \le \theta \le \pi$.

vanishes at $\eta = \infty$ is the lowest order Legendre function of the second kind,

$$\Phi(\eta) = AQ_0(i\sinh\eta) + B, \tag{2.140}$$

where A and B are constants. Since

$$Q_0(i\sinh\eta) = i\left[\tan^{-1}(\sinh\eta) - \frac{\pi}{2}\right] = -i\cot^{-1}(\sinh\eta), \tag{2.141}$$

and the boundary condition is

$$\Phi(\eta = 0) = V \text{ (disk potential)},$$

we readily find the potential at an arbitrary η,

$$\Phi(\eta) = \frac{2V}{\pi}\cot^{-1}(\sinh\eta). \tag{2.142}$$

Note that $\cot^{-1}(0) = \pi/2$. The far field potential at $\eta \gg 1$ or $r \gg a$ can be found from the asymptotic form of the function $\cot^{-1} x$,

$$\cot^{-1} x \simeq \frac{1}{x} - \frac{1}{3x^3} + \cdots, \; x \gg 1. \tag{2.143}$$

The leading far field potential is monopole as expected,

$$\Phi(\eta \gg 1) \simeq \frac{2V}{\pi}\frac{1}{\sinh\eta} \simeq \frac{2V}{\pi}\frac{a}{r}. \tag{2.144}$$

Comparing with the standard monopole potential

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}, \tag{2.145}$$

we readily find the total charge carried by the disk,

$$q = 8\varepsilon_0 aV,$$

and the self-capacitance of the disk,

$$C = 8\varepsilon_0 a, \ \text{(F)}. \tag{2.146}$$

This expression was first found by Cavendish.

The surface charge distribution on the disk is quite nonuniform because like charges repel each other. Charge is distributed in such a manner that the tangential electric field on the disk surface vanishes. The surface charge density can be found from the normal component of the electric field,

$$\sigma = \varepsilon_0 E_n = \varepsilon_0 E_\eta, \tag{2.147}$$

where

$$\begin{aligned} E_\eta &= -\frac{1}{h_\eta}\frac{\partial \Phi}{\partial \eta}\bigg|_{\eta=0} \\ &= \frac{2V}{\pi a}\frac{1}{|\cos\theta|}. \end{aligned} \tag{2.148}$$

Note that

$$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}. \tag{2.149}$$

The surface charge density diverges at the edge of the disk where $\theta = \pi/2$. The charge residing on the disk surface can be found from the following surface integral,

$$\begin{aligned} q &= \varepsilon_0 \int_0^\pi d\theta \int_0^{2\pi} d\phi \sigma h_\theta h_\phi \bigg|_{\eta=0} \\ &= \frac{2\varepsilon_0 aV}{\pi}\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \\ &= 8\varepsilon_0 aV. \end{aligned}$$

This is consistent with the charge found earlier using the monopole potential.

If one uses a coordinate system other than the oblate spheroidal system, solutions will be much more involved. Let us employ the cylindrical coordinates (ρ, ϕ, z). Because of axial symmetry, ϕ

dependence can be suppressed and we seek a solution in the form of Laplace transform,

$$\Phi(\rho, z) = \int_0^\infty \mathit{\Phi}(\rho, k) e^{-k|z|} dk. \tag{2.150}$$

The Laplace equation without ϕ dependence

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \Phi(\rho, z) = 0, \tag{2.151}$$

becomes

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + k^2 \right) \mathit{\Phi}(\rho, k) = 0, \tag{2.152}$$

which suggests that

$$\mathit{\Phi}(\rho, k) = A(k) J_0(k\rho). \tag{2.153}$$

The boundary conditions are:

$$\Phi(\rho \leq a,\ z = \pm 0) = V \text{ (constant)}.$$

The following integral has a peculiar property,

$$\int_0^\infty \frac{\sin ax}{x} J_0(bx) dx = \begin{cases} \dfrac{\pi}{2}, & \text{if} \quad a > b, \\ \\ \sin^{-1}(a/b), & \text{if} \quad a < b. \end{cases} \tag{2.154}$$

Exploiting this property, we can construct the following solution for the potential,

$$\Phi(\rho, z) = \frac{2V}{\pi} \int_0^\infty \frac{\sin ka}{k} J_0(k\rho) e^{-k|z|} dk. \tag{2.155}$$

The potential in the disk plane ($z = 0$) is

$$\Phi(\rho,\ z = 0) = \begin{cases} V, & \text{if} \quad \rho < a, \\ \\ \dfrac{2V}{\pi} \sin^{-1}(a/\rho), & \text{if} \quad \rho > a. \end{cases} \tag{2.156}$$

Example 7 ***Dipole Moment of a Conducting Disk in an External Electric Field***

If a thin conductor disk is placed perpendicular to an external field, the dipole moment is zero because of negligible thickness of the disk even though charge separation does take place in such a manner that disk surfaces are oppositely charged. The external electric field is little disturbed by the disk in this case. The maximum disturbance occurs when the disk surface is parallel to the

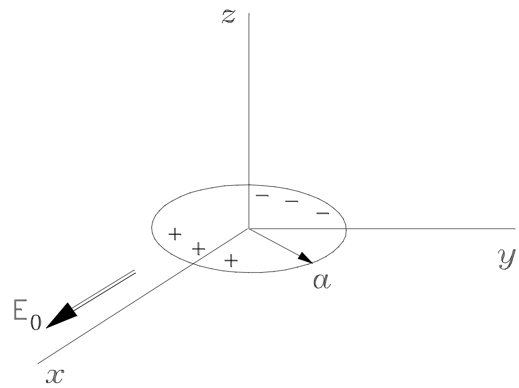


Figure 2-11: Conducting disk in an external electric field parallel to the disk surface.

field.

We assume a uniform external electric field in the $x-$direction and a thin conducting disk placed in the $x-y$ plane with its axis in the $z-$direction as shown in Fig.2-11. The potential associated with the external uniform electric field is

$$\begin{aligned}\Phi_0 &= -E_0 x \\ &= -E_0 a \cosh\eta \sin\theta \cos\phi. \end{aligned} \tag{2.157}$$

The "radial" function $\cosh\eta$ is actually $P_1^1(i\sinh\eta)$ and the presence of the disk should yield a perturbation proportional to the Legendre function of the second kind $Q_1^1(i\sinh\eta)$ since the perturbed potential should have the same angular dependence as $\Phi_0(\eta,\theta,\phi)$ to satisfy the boundary condition at the disk. Thus we assume

$$\Phi(\eta,\theta,\phi) = -E_0 a \cosh\eta \sin\theta\cos\phi + AQ_1^1(i\sinh\eta)\sin\theta\cos\phi, \tag{2.158}$$

where $Q_1^1(i\sinh\eta)$ is actually a real function,

$$Q_1^1(i\sinh\eta) = \cosh\eta\left(\cot^{-1}(\sinh\eta) - \frac{\sinh\eta}{\cosh^2\eta}\right). \tag{2.159}$$

The constant A can be determined from the boundary condition that the disk potential be zero, that is, $\Phi(\eta=0)=0$. We thus find

$$A = \frac{2}{\pi} a E_0,$$

and the potential becomes

$$\Phi(\eta,\theta,\phi) = -aE_0\left(\cosh\eta - \frac{2}{\pi}Q_1^1(i\sinh\eta)\right)\sin\theta\cos\phi. \tag{2.160}$$

Far away from the disk at $r \gg a$ or $\eta \gg 1$, the potential approaches

$$\lim_{\eta\gg 1}\Phi(\eta,\theta,\phi) \to -aE_0\cosh\eta\sin\theta\cos\phi + \frac{4E_0a^3}{3\pi}\frac{\sin\theta\cos\phi}{r^2}, \tag{2.161}$$

where the asymptotic form of $Q_1^1(i\sinh\eta)$,

$$Q_1^1(i\sinh\eta) \simeq \frac{2}{3}\frac{1}{\sinh^2\eta} = \frac{2}{3}\left(\frac{a}{r}\right)^2, \tag{2.162}$$

has been substituted. Comparing the dipole term in Eq. (2.161) with the standard dipole potential

$$\Phi_{\rm dipole} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}, \tag{2.163}$$

we can readily identify the dipole moment induced by the disk,

$$\mathbf{p} = 4\pi\varepsilon_0\frac{4a^3}{3\pi}\mathbf{E}_{0\parallel}, \tag{2.164}$$

where $\mathbf{E}_{0\parallel}$ is the component of the external electric field tangential to the disk surface. Note that the dipole moment is proportional to a^3. The moment is equally applicable for low frequency oscillating electric field as long as the wavelength associated with the oscillating filed is much longer than the disk radius, $ka = \frac{2\pi}{\lambda}a \ll 1$. A resultant scattering cross-section of a conducting disk (sphere too) placed in a low frequency electromagnetic wave is proportional to a^6.

Example 8 ***Leakage of Electric Field through a Small Hole in a Conducting Plate***

Consider a parallel plate capacitor whose grounded, lower plate has a small circular hole of radius a as shown in Fig.2-12. We wish to find how the hole perturbs the potential. This problem has important applications in analyzing leakage of microwaves through a small hole in waveguide walls.

The unperturbed electric field E_0 between the plates is assumed downward with a corresponding potential

$$\Phi_0(z) = \begin{cases} E_0 z, & z > 0 \\ 0, & z < 0 \end{cases} \tag{2.165}$$

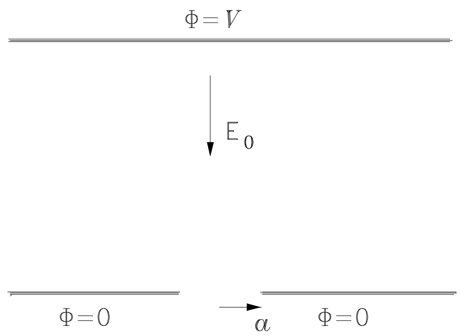


Figure 2-12: The lower plate of a parallel plate capacitor has a small hole of radius a. The electric field leaks throught the hole.

where $z = a\sinh\eta\cos\theta$. We note

$$\sinh\eta = -iP_1(i\sinh\eta). \tag{2.166}$$

Therefore, the perturbed potential can be sought in term of the Legendre function of the second kind $Q_1(i\sinh\eta)$ which is equivalent to

$$Q_1(i\sinh\eta) = \sinh\eta\cot^{-1}(\sinh\eta) - 1. \tag{2.167}$$

We thus assume the following form for the potential in both regions,

$$\Phi(\eta,\theta) = \begin{cases} aE_0\sinh\eta\cos\theta + A\left[\sinh\eta\cot^{-1}(\sinh\eta) - 1\right]\cos\theta, & 0 < \theta < \dfrac{\pi}{2} \\ \\ -A\left[\sinh\eta\cot^{-1}(\sinh\eta) - 1\right]\cos\theta, & \dfrac{\pi}{2} < \theta < \pi, \end{cases}$$

which ensures continuity of the potential at the hole ($\eta = 0$). Continuity of the normal component of the electric field at the hole requires

$$\left.\frac{\partial\Phi}{\partial\eta}\right|_{\substack{\eta=0\\ z=+0}} = \left.\frac{\partial\Phi}{\partial\eta}\right|_{\substack{\eta=0\\ z=-0}},$$

from which we readily find the constant A,

$$A = -\frac{aE_0}{\pi}.$$

In the region below the lower plate ($z < 0$), the potential is

$$\Phi(\eta, \theta) = \frac{aE_0}{\pi} \left[\sinh \eta \cot^{-1}(\sinh \eta) - 1 \right] \cos \theta, \ \frac{\pi}{2} < \theta < \pi. \tag{2.168}$$

Its asymptotic form is of dipole nature,

$$\Phi(r \gg a) \to -\frac{E_0 a^3}{3\pi} \frac{1}{r^2} \cos \theta > 0, \tag{2.169}$$

(note that $\cos \theta < 0$ in the region below the plate) and the effective dipole moment of the hole is

$$\mathbf{p} = \frac{4\pi\varepsilon_0 a^3}{3\pi} \mathbf{E}_0, \tag{2.170}$$

which is *downward*. The far-field potential in the upper region ($z > 0$) is

$$\Phi \simeq \Phi_0 + \frac{E_0 a^3}{3\pi} \frac{1}{r^2} \cos \theta, \tag{2.171}$$

in which the dipole term is due to an effective dipole moment *upward*. The potential at the center of the hole is

$$\Phi(\eta = 0, \theta = 0 \text{ or } \pi) = \frac{aE_0}{\pi}. \tag{2.172}$$

The results of this example, together with those of Example 14 in Chapter 3 (leakage of magnetic field through a hole in a superconducting plate), will have important implications on diffraction of electromagnetic waves by an aperture in a conducting plate. Since the effective dipoles are opposite to each other in the two regions $z > 0$ and $z < 0$, it follows in general that

$$E_z(-z) = -E_z(z),$$

that is, the electric field normal to the plate is an odd function of z. This means that the surface charges $\sigma = \varepsilon_0 \mathbf{n} \cdot \mathbf{E}$ (C/m^2) induced on both sides of the plate at $z = +0$ and $z = -0$ are identical, where $\mathbf{n}$ is the unit normal vector at the plate surface. (Note that $\mathbf{n}$ changes its sign from one side to other.) The component tangential to the plate,

$$\mathbf{E}_t = \mathbf{n} \times \mathbf{E},$$

is an even function of z,

$$\mathbf{E}_t(-z) = \mathbf{E}_t(z).$$

Of course, on the surface of the conducting plate, $\mathbf{E}_t$ vanishes but it does not in the hole. For magnetic fields resulting from a hole in an ideally conducting plate, we will see that the normal component should vanish at the plate surface

$$H_z = 0, \text{ at } z = \pm 0,$$

and off the plate, it is even with respect to z,

$$H_z(-z) = H_z(z),$$

while the tangential component $\mathbf{H}_t = \mathbf{n} \times \mathbf{H}$ is an odd function of z,

$$\mathbf{H}_t(-z) = -\mathbf{H}_t(z).$$

It follows that the surface currents

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H}, \text{ (A/m)}$$

on both surfaces of the plate are identical.

2.5 Method of Inversion

The method of inversion is useful when an electrode has a spherical shape, either complete spheres (e.g., two spheres touching) or incomplete sphere (e.g., spherical bowl, solid hemisphere, etc.). For a given sphere of radius a which we call inverting sphere, the inverted position of a point at $\mathbf{r}$ is defined by

$$\mathbf{r}_i = \frac{a^2}{r^2}\mathbf{r}. \tag{2.173}$$

(See Fig.2-13.) A sphere is inverted into another sphere. If the center of the inverting sphere is chosen on the surface of a sphere to be inverted, the inverted surface becomes a plane as shown in Fig.2-14. This is where the merit of method of inversion is found because potential problems of planar electrodes are often simpler than those involving spheres.

Consider a charge q placed at $\mathbf{r}' = (r', \theta', \phi')$. The potential at position $\mathbf{r} = (r, \theta, \phi)$ is

$$\Phi = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}}, \tag{2.174}$$

where

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi').$$

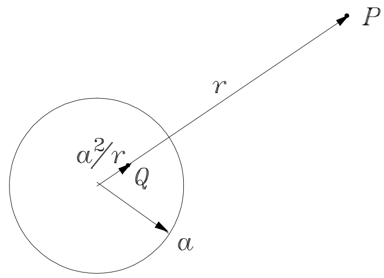


Figure 2-13: Point P at (r, θ, ϕ) is inverted with respect to the sphere of radius a to Q at $(a^2/r, \theta, \phi)$, *i.e.,* at the image position.

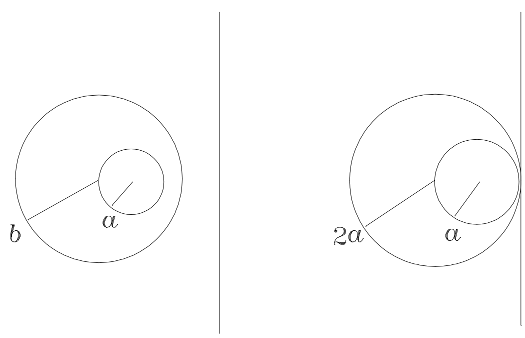


Figure 2-14: If an inverting sphere is centered on a surface of a sphere to be inverted, the sphere is inverted to an infinite plane.

In the inverted space with respect to a sphere of radius a, a charge q' will appear at

$$\left(\frac{a}{r'}\right)^2 \mathbf{r}', \tag{2.175}$$

and the position $\mathbf{r}$ is inverted to

$$\left(\frac{a}{r}\right)^2 \mathbf{r}. \tag{2.176}$$

The potential at the inverted position is

$$\begin{aligned} \Phi_i &= \frac{q'}{4\pi\varepsilon_0} \frac{1}{\sqrt{\dfrac{a^4}{r^2} + \dfrac{a^4}{r'^2} - 2\dfrac{a^4}{rr'}\cos\gamma}} \\ &= \frac{q'}{4\pi\varepsilon_0} \frac{rr'}{a^2} \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}}. \end{aligned} \tag{2.177}$$

In general, if a function $\Phi(r, \theta, \phi)$ satisfies the Laplace equation, the potential function

$$\frac{a}{r}\Phi\left(\frac{a^2}{r}, \theta, \phi\right), \tag{2.178}$$

also satisfies the Laplace equation.

It should be noted that an equipotential spherical surface is in general not inverted to an equipotential sphere. However, a spherical surface at zero potential is inverted to a zero potential spherical surface. Since the reference potential can be chosen arbitrarily without affecting the electric field, one can always choose the potential of an equipotential spherical surface at zero potential. For example, the potential of a charged conducting sphere of radius a is

$$\Phi_s = \frac{1}{4\pi\varepsilon_0}\frac{q}{a}, \tag{2.179}$$

relative to zero potential at infinity. However, we can subtract Φ_s from the potential everywhere and choose the sphere potential at zero and the potential at infinity as

$$\Phi_\infty = -\frac{1}{4\pi\varepsilon_0}\frac{q}{a}.$$

The electric field remains unchanged through uniform shift of the potential. If an inverting sphere is chosen in such a way that it has a radius $2a$ centered at the surface of the conducting sphere of radius a, the conducting sphere is inverted to an infinite plane touching the both spheres as shown in Fig. 2-15. Since the sphere potential is chosen at zero, the potential of the plane is also zero. The potential at infinity is inverted to

$$-\frac{1}{4\pi\varepsilon_0}\frac{q}{a} \times \frac{2a}{r} = -\frac{1}{4\pi\varepsilon_0}\frac{2q}{r}, \tag{2.180}$$

where r is the radial distance from the center of the inverting sphere with radius $2a$. This is a potential due to a point charge $-2q$. Therefore, a charge

$$-2q = -8\pi\varepsilon_0\Phi_s a, \tag{2.181}$$

appears at the center of the inverting sphere.

Example 9 ***Capacitance of Touching Spheres***

Using the method of inversion, we can find the capacitance of two conducting sphere touching each other as shown in Fig.2-15. The potential of the touching spheres is denoted by Φ_s. If the inverting sphere has radius $2a$ and its center at the touching point, the two spheres become two

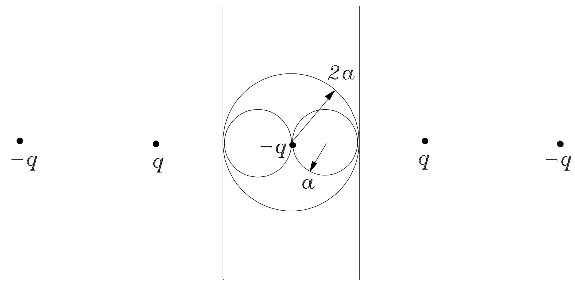


Figure 2-15: Touching spheres are inverted to parallel plates by a sphere of radius $2a$ centered at the touching point. Images appear in the inverted space.

parallel planes separated by a distance $4a$. A charge

$$-q = -8\pi\varepsilon_0 a\Phi_s, \tag{2.182}$$

appears at the midpoint between the plates after inversion which can be analyzed easily using the method of multiple images. The following image charges appear: q at $|z| = 4a$, $-q$ at $|z| = 8a$, q at $|z| = 12a, \cdots$. The amount of total charge on the surface of the original spheres can be found by re-inverting the image charges,

$$\begin{aligned} Q &= 2q\left(\frac{2a}{4a} - \frac{2a}{8a} + \frac{2a}{12a} - \cdots\right) \\ &= q\ln 2 \\ &= 8\pi\varepsilon_0 a\Phi_s \ln 2. \end{aligned} \tag{2.183}$$

Therefore, the self-capacitance of the touching spheres is

$$C = \frac{Q}{\Phi_s} = 8\pi\varepsilon_0 a \ln 2. \tag{2.184}$$

The potential $\Phi_i(\rho, z)$ in the inverted space shown in Fig.2-16can be found in the form of Fourier transform,

$$\Phi_i(\rho, z) = \int_0^\infty A(k)\sinh[k(2a - |z|)]J_0(k\rho)dk, \tag{2.185}$$

where $A(k)$ is a weighting function to be determined. It is noted that the elementary solution to the Laplace equation is

$$J_0(k\rho)e^{\pm kz}, \tag{2.186}$$

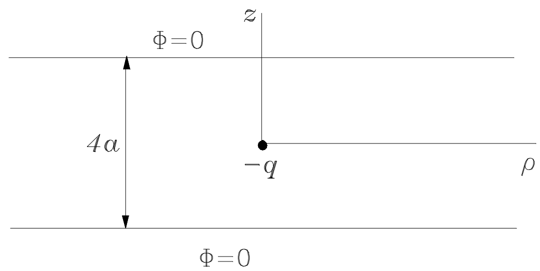


Figure 2-16: Geometry in the inverted space.

and the assumed form of the potential certainly satisfies the Laplace equation as well. The weighting function $A(k)$ can be determined by noting

$$\frac{d^2}{dz^2}\sinh[k(2a-|z|)] = -2k\cosh(2ak)\delta(z), \tag{2.187}$$

and

$$\int_0^\infty kJ_0(k\rho)dk = \frac{1}{\rho}\delta(\rho). \tag{2.188}$$

The charge density of the point charge q at the origin is

$$\rho_c = \frac{q}{2\pi\rho}\delta(\rho)\delta(z). \tag{2.189}$$

Then, $A(k)$ can be determined from the Poisson's equation

$$\nabla^2\Phi_i = -\frac{\rho_c}{\varepsilon_0}, \tag{2.190}$$

as

$$A(k) = -\frac{q}{4\pi\varepsilon_0}\frac{1}{\cosh(2ak)}, \tag{2.191}$$

and the potential in the inverted space is

$$\Phi_i(\rho,z) = -\frac{q}{4\pi\varepsilon_0}\int_0^\infty \frac{\sinh[k(2a-|z|)]}{\cosh(2ak)}J_0(k\rho)dk. \tag{2.192}$$

The potential in the original configuration can be found by reinverting Φ_i through the transformation

$$z \to \left(\frac{2a}{r}\right)^2 z, \quad \rho \to \left(\frac{2a}{r}\right)^2 \rho,$$

where

$$r^2 = \rho^2 + z^2,$$

is the distance from the center. The result is

$$\Phi(\rho, z) = -\frac{q}{4\pi\varepsilon_0}\frac{2a}{r}\int_0^\infty \frac{\sinh\left[k\left(2a - \left(\frac{2a}{r}\right)^2 |z|\right)\right]}{\cosh(2ak)} J_0\left[k\left(\frac{2a}{r}\right)^2 \rho\right] dk, \tag{2.193}$$

with $r^2 = \rho^2 + z^2$. Recalling that we have subtracted $\Phi_s = q/4\pi\varepsilon_0 a$ (the sphere potential) from the potential everywhere to make the sphere potential vanish, we finally obtain

$$\Phi(\rho, z) = \Phi_s\left[1 - \frac{(2a)^2}{r}\int_0^\infty \frac{\sinh\left[k\left(2a - \left(\frac{2a}{r}\right)^2 |z|\right)\right]}{\cosh(2ak)} J_0\left[k\left(\frac{2a}{r}\right)^2 \rho\right] dk\right]. \tag{2.194}$$

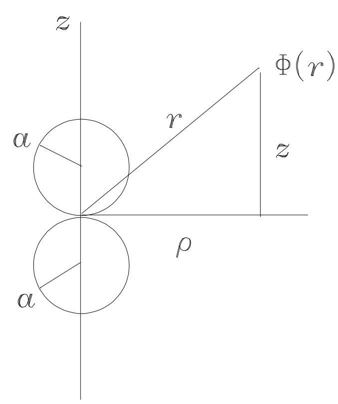


Figure 2-17: Geometry in the original space.

Example 10 ***Capacitance of Spherical Bowl***

As a second example, we consider a hollow spherical bowl of radius a with an angle 2θ subtended at the center shown in Fig.2-18. As inverting sphere, one can choose a sphere having a radius $2a\sin\theta$ centered at the edge of the bowl. After inversion, the bowl becomes a semi-infinite plane as shown and a charge $q = -8\pi\varepsilon_0 a\sin\theta\Phi_s$ will appear at the center of the inverting sphere. Potential problems involving a semi-infinite conducting plate can be analyzed as a limiting case of a wedge. For a charge q placed at (ρ', ϕ', z') near a wedge intersecting at an angle α, the potential is given

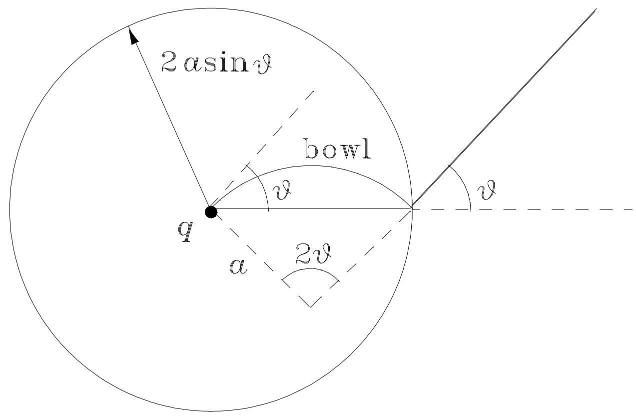


Figure 2-18: A bowl (radius a, center angle 2θ) is inverted to a semi-infinite plane by a sphere of radius $2a\sin\theta$ centered at the edge of the bowl.

by

$$\Phi(\rho,\phi,z) = \frac{2q}{\pi\varepsilon_0\alpha}\begin{cases} \sum_m \int_0^\infty I_\nu(k\rho)K_\nu(k\rho')\cos[k(z-z')]dk\sin(\nu\phi)\sin(\nu\phi'), & \rho<\rho' \\ \sum_m \int_0^\infty I_\nu(k\rho')K_\nu(k\rho)\cos[k(z-z')]dk\sin(\nu\phi)\sin(\nu\phi'), & \rho>\rho' \end{cases} \tag{2.195}$$

where $\nu = m\pi/\alpha$. Noting

$$\int_0^\infty I_\nu(k\rho)K_\nu(k\rho')\cos[k(z-z')]dk = \frac{1}{2\sqrt{2\rho\rho'}}\int_\eta^\infty \frac{e^{-\nu\zeta}}{\sqrt{\cosh\zeta-\cosh\eta}}d\zeta, \tag{2.196}$$

where

$$\cosh\eta = \frac{\rho^2+\rho'^2+(z-z')^2}{2\rho\rho'}, \tag{2.197}$$

and the sum formula

$$\sum_{m=1}^\infty p^m\cos(mx) = \frac{1}{2}\left(\frac{1-p^2}{1-2p\cos x+p^2}-1\right), \tag{2.198}$$

we see that the potential reduces to

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{q}{4\pi\varepsilon_0}\frac{1}{\alpha\sqrt{2\rho\rho'}}\int_\eta^\infty \sinh\left(\frac{\pi}{\alpha}\zeta\right)\\ &\times\left[\frac{1}{\cosh(\pi\zeta/\alpha)-\cos[\pi(\phi-\phi')/\alpha]}-\frac{1}{\cosh(\pi\zeta/\alpha)-\cos[\pi(\phi+\phi')/\alpha]}\right]\\ &\times\frac{1}{\sqrt{\cosh\zeta-\cosh\eta}}d\zeta. \end{aligned} \tag{2.199}$$

For a plate $\alpha = 2\pi$, and this becomes

$$\Phi(\mathbf{r}) = \frac{q}{4\pi^2\varepsilon_0}\left[\frac{1}{R}\cos^{-1}\left(-\frac{\cos[(\phi-\phi')/2]}{\cosh(\eta/2)}\right)-\frac{1}{R'}\cos^{-1}\left(-\frac{\cos[(\phi+\phi')/2]}{\cosh(\eta/2)}\right)\right], \tag{2.200}$$

where

$$\begin{aligned}R &= \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')},\\ R' &= \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi+\phi')}.\end{aligned}$$

The potential in the vicinity of the charge q can be found by letting $\rho' = \rho - r, \phi = \phi' = \pi - \theta, \eta = r/2a\sin\theta \ll 1$,

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} - \frac{q}{4\pi\varepsilon_0}\frac{1}{4\pi a\sin\theta}\left(1+\frac{\theta}{\sin\theta}\right), \tag{2.201}$$

where r is the distance from the charge. The correction due to the presence of the conducting plate is therefore

$$\Delta\Phi = -\frac{q}{4\pi\varepsilon_0}\frac{1}{4\pi a\sin\theta}\left(1+\frac{\theta}{\sin\theta}\right), \tag{2.202}$$

and in the physical space, the far field potential due to a charged conducting bowl is in the form

$$\begin{aligned}\Phi(r &\gg a) = -\frac{q}{4\pi\varepsilon_0}\frac{1}{2\pi r}\left(1+\frac{\theta}{\sin\theta}\right)\\ &= \frac{1}{\pi}\frac{a}{r}(\sin\theta+\theta)\,\Phi_s.\end{aligned} \tag{2.203}$$

Comparing with the standard monopole potential

$$\Phi = \frac{Q}{4\pi\varepsilon_0 r},$$

we finally find the capacitance of the bowl,

$$C = 4\varepsilon_0 a(\theta+\sin\theta). \tag{2.204}$$

For a sphere, $\theta = \pi$, we recover $C = 4\pi\varepsilon_0 a$. For a disk of radius R, $\theta \simeq \sin\theta \simeq R/a \ll 1$, and we also recover

$$C = 8\varepsilon_0 R.$$

The capacitance of a solid (or closed) hemisphere can be found in a similar manner and given by

$$C = 8\pi\varepsilon_0 a \left(1 - \frac{1}{\sqrt{3}}\right). \tag{2.205}$$

This is left for an exercise.

2.6 Numerical Methods

Analytic solutions in potential problems can only be found for a limited number of applications, and in practice, it is often necessary to resort to numerical analysis. In this section, we will estimate the capacitance of a square conductor plate of side a. Mathematically speaking, this problem constitutes an integral equation for the potential Φ, which is constant at the conductor,

$$\Phi = \frac{1}{4\pi\varepsilon_0} \oint \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dS' = -\frac{1}{4\pi} \oint \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial \Phi}{\partial n'} dS' = V = \text{ constant}, \tag{2.206}$$

where

$$\sigma = \varepsilon_0 E_n = -\varepsilon_0 \frac{\partial \Phi}{\partial n},$$

is the unknown surface charge density. The capacitance can be found from

$$C = \frac{1}{\Phi} \int \sigma dS. \tag{2.207}$$

As a very rough estimate, we recall that the capacitance of a circular disk of radius a is given by

$$C = 8\varepsilon_0 a, \tag{2.208}$$

and approximate the capacitance of the square plate by

$$C = 8\varepsilon_0 r_{\text{eff}} = 8\varepsilon_0 \times 0.564a, \tag{2.209}$$

where r_{eff} is the radius of a circular disk having the same area as the plate,

$$\pi r_{\text{eff}}^2 = a^2, \ r_{\text{eff}} = 0.564a.$$

The finite element numerical method given below yields $C \simeq 8\varepsilon_0 \times 0.547a$.

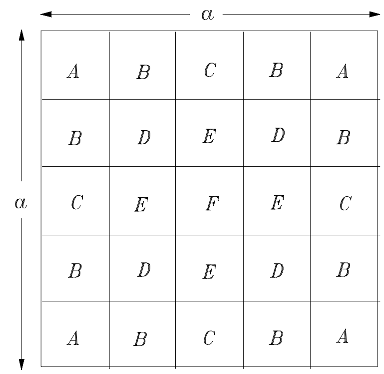


Figure 2-19: A square conducting plate of side a is divided into 25 sub-areas. Because of symmetry, the number of unknown potentials is reduced to 6.

The capacitance is the ratio between the total charge Q and the plate potential V, $C = Q/V$. We divide the plate into $n \times n$ sub-areas of equal size each with side a/n. Each sub-plate is at an equal potential V but charges on the sub-plates differ. To illustrate the procedure, we choose $n = 5$ (25 sub-areas) as shown in Fig. 2-19. Because of symmetry, there are 6 unknown charges to be found. The potential on each sub-plate can be calculated by summing contributions from charges on all sub-plates including the charge on itself. The self-potential of one unit can be estimated as follows. Consider a square plate of side δ carrying a uniform surface charge density σ (C/m^2). The potential at the center of the plate can be found from

$$\begin{aligned}\Phi &= \frac{\sigma}{4\pi\varepsilon_0}\int_{-\delta/2}^{\delta/2} dx \int_{-\delta/2}^{\delta/2} dy \frac{1}{\sqrt{x^2+y^2}} \\ &= \frac{\sigma}{4\pi\varepsilon_0} 4 \int_0^{\delta/2} \left[\ln\left(\sqrt{x^2+\delta^2}+\delta\right) - \ln x\right] dx \\ &= \frac{\sigma}{4\pi\varepsilon_0} 4\delta \ln\left(1+\sqrt{2}\right) \\ &= \frac{q}{4\pi\varepsilon_0 \delta} \times 4\ln\left(1+\sqrt{2}\right) = \frac{q}{4\pi\varepsilon_0\delta} \times 3.5255, \end{aligned} \tag{2.210}$$

where $q = \sigma\delta^2$ is the charge carried by the sub-plate. With this preparation, we can write down the potential of sub-plate A as follows:

$$\begin{aligned} 4\pi\varepsilon_0 \Phi_A \delta &= \left(3.5255 + \frac{2}{4} + \frac{1}{4\sqrt{2}}\right) q_A + \left(2 + \frac{2}{3} + \frac{2}{\sqrt{17}} + \frac{2}{5}\right) q_B \\ &\quad + \left(1 + \frac{2}{\sqrt{20}}\right) q_C + \left(\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{10}} + \frac{1}{3\sqrt{2}}\right) q_D + \left(\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{13}}\right) q_E + \frac{1}{2\sqrt{2}} q_F \\ &= 4.2023 q_A + 3.5517 q_B + 1.4472 q_C + 1.5753 q_D + 1.4491 q_E + 0.3536 q_F. \end{aligned} \tag{2.211}$$

Other potentials $\Phi_B \sim \Phi_F$, which are all equal, can be written down in a similar way and we obtain 6 simultaneous equations for $q_A \sim q_F$ which can be solved easily. A resultant total charge is

$$Q = 1.743 \times 4\pi\varepsilon_0 \Phi\delta,$$

and the capacitance is

$$\begin{aligned} C &\simeq 0.3486 \times 4\pi\varepsilon_0 a \\ &= 0.547 \times 8\varepsilon_0 a. \end{aligned} \tag{2.212}$$

Accuracy will improve if a larger number of sub-areas are used.

The method can be applied to estimate the capacitance of a conducting cube as well. With 150 sub-areas (25 sub-areas on each side), the following capacitance emerges,

$$C \simeq 0.65 \times 4\pi\varepsilon_0 a, \tag{2.213}$$

where a is the side of the cube. An estimate based on a sphere having the same surface area gives

$$C = 4\pi\varepsilon_0 r_{\text{eff}} = 0.69 \times 4\pi\varepsilon_0 a, \tag{2.214}$$

where

$$r_{\text{eff}} = \sqrt{\frac{6}{4\pi}} a = 0.69a.$$

A well known finite element method of solving the Laplace equation is based on the fact that the potential at the center of a cube may be approximated by the average of 6 surrounding potentials on each face of the cube,

$$\Phi_0 = \frac{1}{6}\sum_{i=1}^{6} \Phi_i.$$

This follows from the Taylor expansion of the potential,

$$\begin{aligned} \Phi(x \pm \delta, y, z) &= \Phi(x, y, z) \pm \delta\frac{\partial \Phi}{\partial x} + \frac{1}{2}\frac{\partial^2 \Phi}{\partial x^2} \pm \cdots, \\ \Phi(x, y \pm \delta, z) &= \Phi(x, y, z) \pm \delta\frac{\partial \Phi}{\partial y} + \frac{1}{2}\frac{\partial^2 \Phi}{\partial y^2} \pm \cdots, \\ \Phi(x, y, z \pm \delta) &= \Phi(x, y, z) \pm \delta\frac{\partial \Phi}{\partial z} + \frac{1}{2}\frac{\partial^2 \Phi}{\partial z^2} \pm \cdots, \end{aligned}$$

Adding these 6 equations, we find

$$\Phi(x \pm \delta, y, z) + \Phi(x, y \pm \delta, z) + \Phi(x, y, z \pm \delta) = 6\Phi(x, y, z) + \nabla^2\Phi + \mathcal{O}(\delta^4). \tag{2.215}$$

Therefore, if Φ satisfies the Laplace equation, $\nabla^2\Phi = 0$,

$$\Phi_{\text{ center}} \simeq \frac{1}{6}\sum_{i=1}^{6}\Phi_i, \tag{2.216}$$

valid to order δ^3.

For 2-dimensional problems in which z-dependence is suppressed, we have

$$\Phi_{\text{center}} \simeq \frac{1}{4}\sum_{i=1}^{4}\Phi_i. \tag{2.217}$$

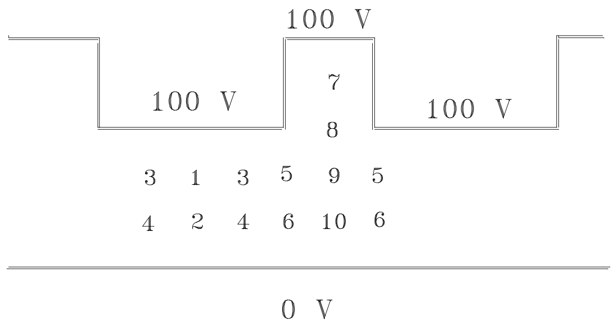
The equation can be applied to each sub-unit having a volume δ^3 (3-D) or area δ^2 (2-D). Resultant simultaneous equations can be solved numerically.

Example 11 ***Potential in a Long Cylinder***

Consider a conducting cylinder having a cross-section as shown in Fig. **??**. The periodic upper electrode is at a potential V and the flat lower electrode is grounded. In order to apply the finite element method, we divide the cross section into sub-sections and allocate 10 nodes points as shown. Applying Eq. (2.217) to the potentials $\Phi_i (i = 1 - 10)$, we obtain

$$\begin{aligned}
4\Phi_1 &= 100 + \Phi_2 + 2\Phi_3, \\
4\Phi_2 &= \Phi_1 + 2\Phi_4, \\
4\Phi_3 &= 100 + \Phi_1 + \Phi_4 + \Phi_5, \\
4\Phi_4 &= \Phi_2 + \Phi_3 + \Phi_6, \\
4\Phi_5 &= 100 + \Phi_4 + \Phi_6 + \Phi_9, \\
4\Phi_6 &= \Phi_4 + \Phi_5 + \Phi_{10}, \\
4\Phi_7 &= 300 + \Phi_8, \\
4\Phi_8 &= 200 + \Phi_7 + \Phi_9, \\
4\Phi_9 &= 2\Phi_5 + \Phi_8 + \Phi_{10}, \\
4\Phi_{10} &= 2\Phi_6 + \Phi_9.
\end{aligned}$$

Solutions are: $\Phi_1 = 63.7$, $\Phi_2 = 31.0$, $\Phi_3 = 61.8$, $\Phi_4 = 30.2$, $\Phi_5 = 53.5$, $\Phi_6 = 27.9$, $\Phi_7 = 97.1$, $\Phi_8 = 88.2$, $\Phi_9 = 55.7$, $\Phi_{10} = 27.9$ all in Volts. A larger number of node points will improve accuracy.



Cross-section of long cylinder with periodic anode structure.

Problems

2.1 A ring charge of total charge q and radius b is coaxial with a long grounded conducting cylinder of radius a ($< b$). Determine the potential everywhere.

2.2 A ring charge of total charge q and radius b is coaxial with a long uncharged dielectric cylinder of permittivity ε and radius a. Determine the potential everywhere.

2.3 A charge q is placed at an axial distance b from a conducting disk of radius a. Determine the potential everywhere. Consider two cases, (a) the disk is grounded, and (b) the disk is floating.

2.4 Show that a charge q at a distance d from the center of a floating conducting spherical shell of radius a raises the sphere potential to

$$\Phi_s = \frac{q}{4\pi\varepsilon_0 d}, \text{ if } d > a \text{ (}q\text{ outside the sphere),}$$

or

$$\Phi_s = \frac{q}{4\pi\varepsilon_0 a}, \text{ if } d < a \text{ (}q\text{ inside).}$$

2.5 A large grounded conducting plate has a hemispherical bob of radius a. A charge q is placed at an axial distance d from the center of the bob. Find the force on the charge.

2.6 Show that the capacitance per unit length of a parallel wire transmission line with a common wire radius a and separation distance d is

$$\frac{C}{l} = \frac{\pi\varepsilon_0}{\ln\left(\dfrac{d + \sqrt{d^2 - 4a^2}}{2a}\right)}.$$

2.7 A coaxial cable having inner and outer radii a and b is bent to form a thin toroidal capacitor with a major radius R ($\gg a$, b). Find the capacitance.

2.8 Show that the mutual capacitance between conducting spheres of radii a and b separated by a large distance $d \gg a, b$ is approximately given by

$$C_{ab} \simeq 4\pi\varepsilon_0 \frac{ab}{d},$$

and that the capacitance of the sphere of radius a is affected by the sphere of radius b as

$$C_{aa} \simeq 4\pi\varepsilon_0 a \left(1 + \frac{ab}{d^2}\right).$$

2.9 Rigorous analysis of potential problems involving two conducting spheres can be made by using the bispherical coordinates defined by

$$\begin{aligned} x &= \frac{a\sin\theta\cos\phi}{\cosh\eta - \cos\theta}, \\ y &= \frac{a\sin\theta\sin\phi}{\cosh\eta - \cos\theta}, \\ z &= \frac{a\sinh\eta}{\cosh\eta - \cos\theta}. \end{aligned}$$

$\eta =$ constant surface is a sphere described by

$$x^2 + y^2 + (z - a\text{cotanh}\eta)^2 = \left(\frac{a}{\sinh\eta}\right)^2,$$

and $\theta =$ constant surface is

$$(\rho - a\text{cotan}\theta)^2 + z^2 = \left(\frac{a}{\sin\theta}\right)^2,$$

which is spindle-like shape.

(a) Finding the metric coefficients $h_\eta, h_\theta,$ and h_ϕ, show that the Laplace equation in the bispherical coordinates is

$$\frac{\partial}{\partial\eta}\left(\frac{1}{\cosh\eta - \cos\theta}\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cosh\eta - \cos\theta}\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta(\cosh\eta - \cos\theta)}\frac{\partial^2\Phi}{\partial\phi^2} = 0.$$

(b) Show that the general solution to the Laplace equation is in the form

$$\Phi(\eta,\theta,\phi) = \sqrt{\cosh\eta - \cos\theta}\sum_{l,m}(A_{lm}e^{(l+\frac{1}{2})\eta} + B_{lm}e^{-(l+\frac{1}{2})\eta})P_l^m(\cosh\eta)e^{im\phi}.$$

As in the oblate spheroidal coordinates, in this coordinate system too, the Laplace equation is not separable.

2.10 Using the inversion method, show that the capacitance of a solid or closed conducting hemisphere of radius a is given by

$$C = 8\pi\varepsilon_0 a\left(1 - \frac{1}{\sqrt{3}}\right).$$

2.11 Find a 2D Green's function for the interior of long cylinder having a semicircular cross-section of radius a.

Hint: Assume

$$G(\mathbf{r},\mathbf{r}') = \begin{cases} \sum_m A_m(\rho/\rho')^m \sin(m\phi)\sin(m\phi'), & \rho < \rho' < a, \\ \sum_m \left[B_m(\rho/\rho')^m + C_m(\rho'/\rho)^m\right]\sin(m\phi)\sin(m\phi'), & \rho' < \rho < a, \end{cases}$$

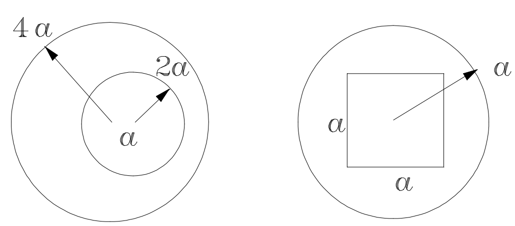
where $\mathbf{r} = (\rho, \phi)$, $\mathbf{r}' = (\rho', \phi')$.

2.12 A conducting disk of radius a is placed parallel to an external electric field E_0. The dominant perturbation to the potential is dipole as shown in Example 6. What is the leading higher order correction?

2.13 Cylindrical capacitors have cross-sections as shown. Estimate graphically the capacitance per unit length for each. For (a), analytic expression for the capacitance is

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\dfrac{\rho_1^2 + \rho_2^2 - d^2}{2\rho_1\rho_2}\right)},$$

where $\rho_1 = 4a, \rho_2 = 2a$, and $d = a$. For (b), one has to resort to numerical analysis for an exact value.



(a) (b)

2.14 Find numerically the capacitance of a conducting cube of side a. What do you estimate for the lower and upper bounds of the capacitance?

2.15 Derive Eq. (**??**), the expression for the potential due to a point charge in the oblate spheroidal coordinates (η, θ, ϕ). The charge is at (η', θ', ϕ').

2.16 Derive Eqs. (2.138) and (2.139), the Green's function for an oblate spheroid described by $\eta = \eta_0$ (const.)

2.17 The prolate spheroidal coordinates (η, θ, ϕ) is convenient to solve potential problems involving prolate spheroids (sphere elongated along the z axis). The coordinate transformation is defined by

$$
\begin{aligned}
x &= a \sinh\eta \sin\theta \cos\phi, \\
y &= a \sinh\eta \sin\theta \sin\phi, \\
z &= a \cosh\eta \cos\theta.
\end{aligned}
$$

In the limit of $\eta \to 0$, $\eta = \text{const.}$ surface describes a thin rod having a length $2a$, and in the limit $\eta \to \infty$, it approaches a sphere with radius $a\cosh\eta \simeq a\sinh\eta$. Show that the metric coefficients are:

$$
\begin{aligned}
h_\eta &= a\sqrt{\sinh^2\eta + \cos^2\theta} = h_\theta, \\
h_\phi &= a \sinh\eta \cos\theta.
\end{aligned}
$$

Then, show that general solution of Laplace's equation $\nabla^2\Phi = 0$ in the prolate spheroidal coordinates is in the form

$$\Phi(\eta, \theta, \phi) = \sum_{l,m} \left[A_{lm} P_l^m(\cosh\theta) + B_{lm} Q_l^m(\cosh\eta)\right]\left[C_{lm} P_l^m(\cos\theta) + D_{lm} Q_l^m(\cos\theta)\right] e^{im\phi}.$$

In the lowest order $l = 0$, possible one dimensional solutions are

$$\Phi(\eta) = Q_0(\cosh\eta) = \ln\coth\left(\frac{\eta}{2}\right),$$

$$\Phi(\theta) = Q_0(\cos\theta) = \ln\cot\left(\frac{\theta}{2}\right).$$