# Chapter 10

# Electromagnetism and Relativity

## 10.1 Introduction

"If I am moving at a large velocity along a light wave, what propagation velocity should I measure?" This was a question young Einstein asked himself and in 1905, he published a monumental paper on special relativity which formulated how to transform coordinates, velocity and electromagnetic fields between two inertial frames. Einstein postulated that:

1. all physical laws remain intact in any inertial frames, and

2. the light velocity $c$ is invariant against inertial coordinate transformation.

Postulate 1 means that, for example, the Maxwell's equations in a moving frame remain formally identical to those in the laboratory frame, provided the spatial coordinates, time, and electromagnetic fields are appropriately transformed. Electromagnetic fields appear and disappear as we change observing frame. For example, if a charge is moving in the laboratory frame, we observe both electric and magnetic fields. In the frame of the moving charge, the current and magnetic field evidently disappear. However, the electromagnetic fields (primed quantities) in the charge frame are subject to the transformation

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \mathbf{V} \times \mathbf{B}_{\perp}\right), \tag{10.1}$$

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \quad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \mathbf{V} \times \mathbf{E}_{\perp}\right), \tag{10.2}$$

where $\parallel$ and $\perp$ are relative to the direction of the velocity $\mathbf{V}$. Since in this example, $\mathbf{B}_{\parallel} = 0$ and $\mathbf{B}_{\perp} = \mathbf{V} \times \mathbf{E}_{\perp}$ in the laboratory frame, the magmatic field in the frame of the moving charge indeed vanishes consistent with our intuition. The pertinent static Maxwell's equations are satisfied in both frames

$$\text{laboratory frame: } \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{10.3}$$

$$\text{in the frame of moving charge: } \nabla' \cdot \mathbf{E}' = \frac{\rho'}{\varepsilon_0}, \ \nabla' \times \mathbf{B}' = 0, \ (\mathbf{J}' = 0). \tag{10.4}$$

Here the primed operators and quantities are those in the moving frame which are subject to the Lorentz transformation.

As shown in Chapter 8, electromagnetic fields due to a charged particle moving at an arbitrary velocity can be correctly formulated by the Lienard-Wiechert potentials which had been discovered prior to the theory of relativity. Electromagnetic disturbances propagate at the velocity $c$ regardless of the velocity of the charge, just as sound waves emitted by a moving source propagate at a sound velocity independent of the source velocity. A major difference between sound waves and electromagnetic waves occurs for stationary source and moving observer. For sound waves, if an observer is approaching a source at a velocity $V_O$, the apparent sound velocity becomes $c_s + V_O$ because both $c_s$ and $V_O$ are well defined with respect to the medium of sound waves, namely, air. In electromagnetic waves that can propagate in vacuum, there is no preferred inertial frame to define velocities and a moving observer will measure the same propagation velocity $c$ regardless of the relative velocity between two inertial frames. Of course, the frequency and wavelength are Doppler shifted but the product $\lambda\nu = \lambda'\nu' = c$ or the ratio $\omega/k = \omega'/k' = c$ remains invariant.

## 10.2 CGS-ESU System

In this Chapter, the CGS-ESU (Electro-Static Unit) unit system is used so that electromagnetic fields $\mathbf{E}$ (statvolt/cm $\simeq$ 300 volt/cm = $3 \times 10^4$ volt/m) and $\mathbf{B}$ (gauss = $10^{-4}$ T) have the same dimensions. In CGS-ESU, the Coulomb's law is adopted to connect the mechanical world and electromagnetic world. (Recall that in SI, the magnetic force is adopted to define 1 ampere current which in CGS-USU is $3 \times 10^9$ stat-ampere.) If two equal charges separated by 1 cm exert a force of 1 dyne (= erg/cm = $10^{-7}$J/$10^{-2}$ m = $10^{-5}$ N) on each other, the charge is defined to be 1 ESU $\simeq 3 \times 10^9$ C. The Coulomb's law in CGS-ESU system is

$$\text{Coulomb's law: } F = \frac{q_1 q_2}{r^2} \text{ (dyne).} \tag{10.5}$$

The electronic charge is $e = 4.8 \times 10^{-10}$ ESU (= $1.6 \times 10^{-19}$ C). The potentials $\Phi$ and $\mathbf{A}$ also have common dimensions (statvolt) in CGS-ESU. This is particularly convenient in theoretical electrodynamics because the fields $\mathbf{E}$ (polar vector) and $\mathbf{B}$ (axial or pseudo vector) are in fact components of a unified $4 \times 4$ field tensor and the potentials $\Phi$, $\mathbf{A}$ form a four vector ($\Phi$, $\mathbf{A}$).The Maxwell's equations in this unit system are:

$$\nabla \cdot \mathbf{E} = 4\pi\rho,\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{10.6}$$

$$\nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{10.7}$$

and the relationships between the fields and potentials are

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t},\ \mathbf{B} = \nabla \times \mathbf{A}. \tag{10.8}$$

The wave equations for the potentials are modified as

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -4\pi\rho, \tag{10.9}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}, \tag{10.10}$$

subject to the Lorentz gauge,

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0. \tag{10.11}$$

Electromagnetic force in the CGS-ESU system is

$$\mathbf{F}\text{ (dyne)} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right),\ \mathbf{f}\text{ (dyne/cm}^3\text{)} = \rho\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B}, \tag{10.12}$$

the Poynting flux is

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B},\ (\text{erg cm}^{-2}\text{ sec}^{-1}), \tag{10.13}$$

the flux of momentum is

$$\frac{1}{4\pi}\mathbf{E}\times\mathbf{B},\ (\text{dyne cm}^{-2}), \tag{10.14}$$

and the momentum density is

$$\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B},\ (\text{dyne sec cm}^{-3}). \tag{10.15}$$

Electromagnetic energy density is

$$\frac{1}{8\pi}\left(E^2 + B^2\right),\ (\text{erg cm}^{-3}). \tag{10.16}$$

The vacuum impedance for a plane wave is unity (dimensionless),

$$\mathbf{B} = \frac{c}{\omega}\mathbf{k}\times\mathbf{E},\ |\mathbf{B}| = |\mathbf{E}|. \tag{10.17}$$

In CGS-ESU system, macroscopic proportional constants inevitably have unfamiliar units. For example, the capacitance has dimensions of length (cm) as seen from its definition,

$$[C] = \frac{[q]}{[\Phi]} = \text{ length}. \tag{10.18}$$

The conductivity $\sigma$ relates the electric field and current density, $\mathbf{J} = \sigma\mathbf{E}$, and has dimensions of frequency, $\text{sec}^{-1}$, since

$$[\sigma] = \frac{[J]}{[E]} = \frac{[q]\ \text{cm}^{-2}\ \text{sec}^{-1}}{[q]\ \text{cm}^{-2}} = \text{sec}^{-1}.$$

## 10.3 Lorentz Transformation

The null result of Michelson-Morley's extensive interference experiments to detect the ether velocity was explained by Lorentz who assumed that a moving object contracts in the direction of its velocity by a factor $\gamma$,

$$L' = \frac{1}{\gamma} L_0 = \sqrt{1-\beta^2} L_0. \tag{10.19}$$

This was followed by the finding by Lorentz and Poincaré that if the spatial coordinates, time and electromagnetic fields are all transformed according to what is known as Lorentz transformation, the Maxwell's equations remain intact. If a relative velocity $V$ is assumed in the $x$ direction, the laboratory coordinates $(ct, x, y, z)$ and coordinates in the moving frame $(ct', x', y', z')$ may be assumed to be related through a linear transformation,

$$\begin{aligned} t' &= \gamma_0 (t - aVx) \\ x' &= \gamma_1 (x - Vt) \\ y' &= y, \; z' = z, \end{aligned}$$

provided that the two coordinate systems coincide at $t = 0$. The invariance of the coordinates perpendicular to the relative velocity, $y = y'$ and $z = z'$, follows from isotropy of space which is implicitly assumed. For light pulse emitted at $t = t' = 0$ and $x = x' = 0$ in the positive $x$ direction (same direction as $V$),

$$x' = ct', \; x = ct,$$

which yield

$$c = \frac{\gamma_1}{\gamma_0} \frac{c - V}{1 - acV}, \tag{10.20}$$

while for light pulse emitted in the negative $x$ direction

$$x' = -ct, \; x = -ct,$$

or

$$c = \frac{\gamma_1}{\gamma_0} \frac{c + V}{1 + acV} \tag{10.21}$$

From Eqs. (10.20) and (10.21), we find

$$\gamma_0 = \gamma_1 = \gamma \text{ and } a = \frac{1}{c^2}. \tag{10.22}$$

To determine $\gamma$, consider a light pulse emitted along the $y'$ axis ($x' = 0$, that is, $x = Vt$) in the moving frame,

$$y' = ct' = c\gamma \left( t - \frac{1}{c^2} Vx \right) = c\gamma \left( 1 - \frac{V^2}{c^2} \right) t.$$

In the laboratory frame, light propagation is tilted due to the relative motion between the two coordinates, $(ct)^2 = y^2 + (Vt)^2$ or

$$y = \sqrt{c^2 - V^2}t.$$

Since $y' = y$, we find

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{10.23}$$

Desired transformation between $(x, t)$ and $(x', t')$ is

$$\begin{aligned} x' &= \gamma(x - Vt), \\ t' &= \gamma\left(t - \frac{V}{c^2}x\right), \end{aligned}$$

and the four dimensional coordinates in the laboratory frame $(ct, x, y, z)$ and those in the moving frame $(ct', x', y', z')$ are related through

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\frac{V}{c}\gamma & 0 & 0 \\ -\frac{V}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{10.24}$$

Its inverse transformation can be found by replacing $V$ with $-V$,

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma & \frac{V}{c}\gamma & 0 & 0 \\ \frac{V}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} \tag{10.25}$$

Graphically, Lorentz transformation may be visualized as contraction in the $(ct, x)$ plane with a quasi angle $\psi$ defined by

$$\cosh\psi = \gamma, \;\; \sinh\psi = \beta\gamma, \;\; \tanh\psi = \beta,$$

as illustrated in Fig. (10-1). Note that the coordinates $(ct', x')$ are not orthonormal if those in the laboratory frame are so chosen. Since

$$\tanh(\psi_1 + \psi_2) = \frac{\tanh\psi_1 + \tanh\psi_2}{1 + \tanh\psi_1 \tanh\psi_2} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2},$$

sum of two velocities cannot exceed $c$.

Time dilation and length contraction can be visualized as follows. A clock stationary at $x = 0$ in the laboratory frame moves along the vertical $ct$ axis $(x = 0)$ as time elapses. 1 second in the laboratory frame appears as $\gamma$ second in the moving frame. If seen from the moving frame, the clock is moving and a moving clock ticks slower. If a clock is stationanry at $x' = 0$ in the moving

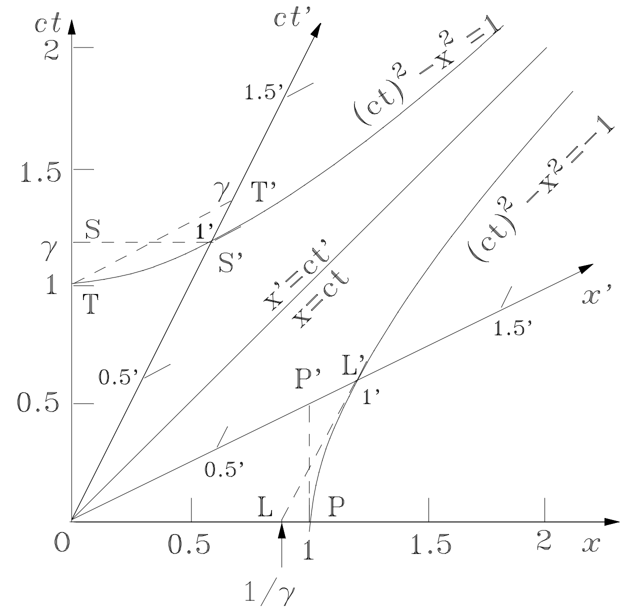



Figure 10-1: Graphical representation of Lorentz transformation for the case $\beta = 0.5$. Hyperbolic curves show $(ct)^2 - x^2 = 1$ (interval of time-like events) and $(ct)^2 - x^2 = -1$ (length of space-like object).

frame, it "travels" along the $t'$ axis ($x\prime = 0$). In the laboratory frame, 1 second in the moving frame appears as $\gamma$ second. In both cases, a moving clock ticks slower.

Likewise, a stick one meter long in the moving frame is contarcted by a factor $\gamma$ if seen from the laboratory frame. Note that length measurements should be done for a common time in both frames.

It is convenient to introduce a metric tensor defined by

$$g_{ij} = g^{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{10.26}$$

A contravariant vector (vector in ordinary sense) $x^i = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ can be converted to a covariant vector through

$$x_i = g_{ij}x^j = (x^0, -x^1, -x^2, -x^3) \tag{10.27}$$

so that

$$s^2 = x^i x_i. \tag{10.28}$$

Note that we follow Einstein's convention, that is, repeated subscripts and superscripts mean summation is to be taken,

$$g_{ij}x^j = \sum_{j=0}^{3} g_{ij}x^j, \tag{10.29}$$

$$x^i x_i = x^0 x_0 + x^1 x_1 + x^2 x_3 + x^4 x_4. \tag{10.30}$$

$s^2$ can be either positive or negative. The Lorentz transformation for the coordinates

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

can be written in the form

$$x'^i = L^i{}_j x^j, \tag{10.31}$$

where $L^i{}_j$ is the Lorentz transformation mixed tensor,

$$L^i{}_j = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10.32}$$

Its inverse tensor is

$$\left(L^i{}_j\right)^{-1} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10.33}$$

A four dimensional vector that transforms according to the Lorentz transformation law is called a four-vector. The "position" vector $x^i$ evidently forms a four vector. For an object moving at a velocity $\mathbf{v}$ in the laboratory frame, the velocity four vector is

$$v^i = \frac{dx^i}{d\tau} = \gamma\left(c, v_x v_y, v_z\right), \tag{10.34}$$

where

$$d\tau = \frac{1}{\gamma}dt = \sqrt{1-\beta^2}dt, \tag{10.35}$$

is the *proper* time as measured in the moving frame of the object. This is the celebrated time dilation effect. The magnitude of the four velocity is constant,

$$v^i v_i = c^2. \tag{10.36}$$

The momentum four vector is defined by

$$p^i = \left(\frac{\mathcal{E}}{c}, \mathbf{p}\right) = \left(\frac{\mathcal{E}}{c}, p_x, p_y, p_z\right), \tag{10.37}$$

where

$$\mathcal{E} = \gamma mc^2, \tag{10.38}$$

is the energy and $\mathbf{p}$ is the momentum of mass of a particle having a mass $m$,

$$\mathbf{p} = \gamma m\mathbf{v}. \tag{10.39}$$

The magnitude of this four vector is also invariant,

$$p^i p_i = \left(\frac{E}{c}\right)^2 - p^2 = \gamma^2(1-\beta^2)(mc)^2 = (mc)^2. \tag{10.40}$$

**Example 1** *How are the velocity and acceleration transformed?*

For a relative velocity $\mathbf{V}$ between two inertial frames, spatial coordinates are transformed as

$$\mathbf{r}'_{\parallel} = \gamma(\mathbf{r}_{\parallel} - \mathbf{V}t),\ \mathbf{r}'_{\perp} = \mathbf{r}_{\perp}, \tag{10.41}$$

where $\parallel$ and $\perp$ indicate components parallel and perpendicular to the velocity $\mathbf{V}$. Since the time is transformed as

$$dt' = \gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{V}}{c^2}\right)dt, \tag{10.42}$$

the velocity in the direction of the relative velocity is transformed as

$$\mathbf{v}'_{\parallel} = \frac{d\mathbf{r}'_{\parallel}}{dt'} = \frac{\mathbf{v}_{\parallel} - \mathbf{V}}{1 - \frac{\mathbf{v}\cdot\mathbf{V}}{c^2}}, \tag{10.43}$$

and the normal component as

$$\mathbf{v}'_{\perp} = \frac{d\mathbf{r}'_{\perp}}{dt'} = \frac{\mathbf{v}_{\perp}}{\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{V}}{c^2}\right)}. \tag{10.44}$$

For example, if two particles are approaching each with a velocity $v$ relative to the laboratory frame, the relative velocity in the frame of either particle is

$$\frac{2v}{1+(v/c)^2},$$

which cannot exceed $c$.Note that $2v$ itself can exceed $c$.However, a relative velocity between two objects is meaningful only if it is measured in rest frame of either object. $2v$ pertains to an observer in the laboratory frame in which both objects are moving. Therefore, $2v$ itself is not a very meaningful velocity. As an example, consider head on collision of two protons. We assume each proton has a velocity $v = 0.9c$ relative to the laboratory frame. In the rest frame of either proton, the two protons approach with a relative velocity

$$V = \frac{2v}{1+(v/c)^2} = 0.9945c,$$

and the kinetic energy available for nuclear interaction is

$$(\gamma - 1)mc^2 = 8 \text{ GeV}.$$

To find transformation of acceleration, we note the acceleration parallel to the relative velocity is transformed as

$$\mathbf{a}'_{\parallel} = \frac{d\mathbf{v}'_{\parallel}}{dt'} = \frac{\mathbf{a}_{\parallel}}{\gamma^3 \left(1 - \frac{\mathbf{v}\cdot\mathbf{V}}{c^2}\right)^3}, \tag{10.45}$$

and the normal component as

$$\mathbf{a}'_{\perp} = \frac{d\mathbf{v}'_{\perp}}{dt'} = \frac{\mathbf{a}_{\perp}}{\gamma^2 \left(1 - \frac{\mathbf{v}\cdot\mathbf{V}}{c^2}\right)^2} + \frac{\mathbf{v}_{\perp}}{\gamma^2 \left(1 - \frac{\mathbf{v}\cdot\mathbf{V}}{c^2}\right)^3} \frac{\mathbf{a}\cdot\mathbf{V}}{c^2}. \tag{10.46}$$

Note that transformation of the normal acceleration involves parallel acceleration as well.

The inverse transformation for the acceleration is

$$\mathbf{a}_{\parallel} = \frac{d\mathbf{v}_{\parallel}}{dt} = \frac{\mathbf{a}'_{\parallel}}{\gamma^3 \left(1 + \frac{\mathbf{v}'\cdot\mathbf{V}}{c^2}\right)^3} \tag{10.47}$$

$$\mathbf{a}_{\perp} = \frac{\mathbf{a}'_{\perp}}{\gamma^2 \left(1 + \frac{\mathbf{v}'\cdot\mathbf{V}}{c^2}\right)^2} - \frac{\mathbf{v}'_{\perp}}{\gamma^2 \left(1 + \frac{\mathbf{v}'\cdot\mathbf{V}}{c^2}\right)^3} \frac{\mathbf{a}'\cdot\mathbf{V}}{c^2}. \tag{10.48}$$

In an instantaneously rest frame of a particle, $\mathbf{v}' = \mathbf{V}$, and $\mathbf{v}_{\perp} = 0$. Then

$$\mathbf{a}'_{\parallel} = \gamma^3 \frac{1}{\left(1 - \frac{V^2}{c^2}\right)^3} \mathbf{a}_{\parallel} = \gamma^3 \mathbf{a}_{\parallel}, \tag{10.49}$$

$$\mathbf{a}'_{\perp} = \gamma^2 \mathbf{a}_{\perp}. \tag{10.50}$$

The current density $\mathbf{J}$ and charge density $\rho$ form the following four vector

$$J^i = (c\rho, \mathbf{J}) = (c\rho, J_x, J_y, J_z), \tag{10.51}$$

where

$$\rho(\mathbf{v}) = en(\mathbf{v}) = e\frac{n_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \tag{10.52}$$

$$\mathbf{J} = en(\mathbf{v})\mathbf{v}, \tag{10.53}$$

$$n(\mathbf{v}) = \frac{n_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \tag{10.54}$$

is the density of charged particles corrected for length contraction in the direction of the velocity $\mathbf{v}$ and $n_0$ is the charge density in the rest frame, $\mathbf{v} = 0$. The magnitude of the current four vector is

$$J^i J_i = (c\rho_0)^2 = \text{ const.} \tag{10.55}$$

where $\rho_0$ is the proper charge density in the rest frame of the charge.

In Lorentz gauge, the potentials satisfy the decoupled wave equations,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -4\pi\rho, \tag{10.56}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}. \tag{10.57}$$

Therefore, a resultant four vector potential is

$$A^i = (\Phi, \mathbf{A}) = (\Phi, A_x, A_y, A_z),$$

which satisfies the wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) A_i = \frac{4\pi}{c} J_i.$$

Noting

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} = g^{ii}\partial_i\partial_i = \partial^i\partial_i,$$

the wave equation can readily be Lorentz transformed as

$$\partial'^j\partial'_j A'_i = \frac{4\pi}{c} J'_i$$

since $\partial^i\partial_i$ is Lorentz invariant. The electromagnetic wave equation is thus Lorentz invariant which guarantees the constancy of the wave propagation velocity $c$.

## 10.4 Transformation of Electromagnetic Fields

The electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, do not form four vectors. This is due to different vectorial nature of the respective fields. The electric field is a polar vector (or true vector) because it changes the sign if coordinates are reversed, $\mathbf{r} \to -\mathbf{r}$. In contrast, the magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c}\int \frac{(\mathbf{r} - \mathbf{r}') \times \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV',$$

remains unchanged against coordinate inversion since both $\mathbf{r}-\mathbf{r}'$ and $\mathbf{J}(\mathbf{r})$ change sign. The magnetic field is an axial vector (or pseudo vector). Rather they are components of an antisymmetric pseudo tensor,

$$B^{ij} = \begin{bmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{bmatrix}$$

The conventional magnetic Lorentz force

$$\mathbf{f}_m = \frac{1}{c}\mathbf{J}\times\mathbf{B}$$

now takes a form

$$f_m^i = \frac{1}{c}J_j B^{ij},\ (i,\ j=1,2,3)$$

where

$$J_j = (-J_x, -J_y, -J_z)$$

is the covariant current density. Combining the electric field and magnetic field into a single field tensor

$$F^{ij} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix},$$

the conventional electromagnetic force

$$\mathbf{f} = \rho\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B}$$

can be rewritten as

$$f^i = \frac{1}{c}F^{ij}J_j,\ i=1,2,3;\ j=0,1,2,3.$$

The component $f^0$

$$f^0 = \frac{1}{c}\mathbf{E}\cdot\mathbf{J}$$

indicates the work done by the electromagnetic field. A resultant force four vector is

$$f^j = \left(\frac{1}{c}\mathbf{E}\cdot\mathbf{J},\ \mathbf{f}\right).$$

Using the field tensor in Eq. (), we now reformulate Maxwell's equations as follows. Since

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\nabla A^0 - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\ \mathbf{B} = \nabla\times\mathbf{A},$$

the field tensor can be written as

$$F^{ij} = \partial^i A^j - \partial^j A^i$$

where $\partial^i$ is the contravariant derivative

$$\partial^i = \frac{\partial}{\partial x_i} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right).$$

For example,

$$F^{ii} = \partial^i A^i - \partial^i A^i = 0, \text{ (no summation here)}$$

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{\partial \Phi}{\partial x} = -E_x,$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = -B_z,$$

and so on. Differentiating $F^{ij} = \partial^i A^j - \partial^j A^i$ with respect to $x^i$ covariantly, we obtain

$$\partial_i F^{ij} = \partial_i \partial^i A^j - \partial_i \partial^j A^i = \partial_i \partial^i A^j - \partial^j \partial_i A^i.$$

The first term in the RHS is

$$\partial_i \partial^i A^j = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) A^j = \frac{4\pi}{c} J^j,$$

while the second term vanishes because of our choice of Lorentz gauge,

$$\partial_i A^i = \frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0.$$

Thus

$$\partial_i F^{ij} = \frac{4\pi}{c} J^j.$$

For $j = 0$, noting $J^0 = c\rho$, we recover Gauss' law,

$$\partial_i F^{ij} = \nabla \cdot \mathbf{E} = 4\pi\rho.$$

For $j = 1, 2, 3$, we also recover $j$-th component of generalized Ampere's law,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}.$$

Another identity satisfied by $F^{ij}$ is

$$\partial^i F^{jk} + \partial^j F^{ki} + \partial^k F^{ij} = 0,$$

as can be readily checked by substituting $F^{ij} = \partial^i A^j - \partial^j A^i$. When $i = 0, j = 1, k = 2$, Eq. () yields

$$\partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} = 0,$$

or

$$\frac{1}{c}\frac{\partial}{\partial t}(-B_z) - \frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}(-E_x) = 0,$$

that is, we recover Faraday's law

$$(\nabla \times \mathbf{E})_z = -\frac{1}{c}\frac{\partial B_z}{\partial t}.$$

Furthermore, for $i = 1, j = 2, k = 3$, we recover

$$\nabla \cdot \mathbf{B} = 0.$$

In order to find how the field tensor $F^{ij}$ is Lorentz transformed, let us consider an arbitrary contravariant vector $B^j$ defined by

$$B^j = F^{ij} A_i.$$

After Lorentz transformation, this becomes

$$B'^j = F'^{ij} A'_i,$$

where

$$B'^j = L^j_{\ k} B^k, \quad A'_i = L_i^{\ k} A_k.$$

Then,

$$L^j_{\ k} B^k = F'^{ij} L_i^{\ m} A_m$$

$$L^j_{\ k} F^{lk} A_l = F'^{ij} L_i^{\ l} A_l$$

Since $A_l$ is arbitrary, we obtain

$$F'^{ij} L_i^{\ m} = L^j_{\ n} F^{mn}.$$

Multiplying both sides by $L^r_{\ m}$ and noting

$$L_i^{\ m} L^r_{\ m} = \delta_i^r,$$

we find

$$F'^{ij} = L^i_{\ m} L^j_{\ n} F^{mn} = L^i_{\ m} F^{mn} L^n_{\ j}.$$

For example,

$$\begin{aligned}
F'^{01} &= L^0_{\ m} L^1_{\ n} F^{mn} \\
&= L^0_{\ 0} L^1_{\ 1} F^{01} + L^0_{\ 1} L^1_{\ 0} F^{10} \\
&= -\gamma^2 E_x + \gamma^2 \beta^2 E_x \\
&= -E_x = F^{01},
\end{aligned}$$

that is, the electric field parallel to the relative velocity $V$ is invariant. For $F'^{02}$, we find

$$\begin{aligned} F'^{02} &= L^0_0 L^2_2 F^{02} + L^0_1 L^2_2 F^{12} \\ &= -\gamma(E_y - \beta B_z), \end{aligned}$$

and so on. The overall result is

$$F'^{ij} = \begin{bmatrix} 0 & -E_x & -\gamma(E_y - \beta B_z) & -\gamma(E_z + \beta B_y) \\ E_x & 0 & -\gamma(B_z - \beta E_y) & \gamma(B_y + \beta E_z) \\ \gamma(E_y - \beta B_z) & \gamma(B_z - \beta E_y) & 0 & -B_x \\ \gamma(E_z + \beta B_y) & -\gamma(B_y + \beta E_z) & B_x & 0 \end{bmatrix}$$

For a relative velocity in an arbitrary direction, the electromagnetic fields are transformed according to

$$\begin{aligned} \mathbf{E}'_\parallel &= \mathbf{E}_\parallel, \quad \mathbf{E}'_\perp = \boldsymbol{\gamma}(\mathbf{E}_\perp + \boldsymbol{\beta} \times \mathbf{B}_\perp), \\ \mathbf{B}'_\parallel &= \mathbf{B}_\parallel, \quad \mathbf{B}'_\perp = \boldsymbol{\gamma}(\mathbf{B}_\perp - \boldsymbol{\beta} \times \mathbf{E}_\perp), \end{aligned}$$

where $\parallel$ and $\perp$ indicate components parallel and perpendicular to the relative velocity $\mathbf{V}$.

The invariance of $B_x$, the magnetic field parallel to $V$, can be seen from the following observation. $B_x$ appears only in $F^{23} = -F^{32}$. Since the component $F^{ij}$ transforms similar to the coordinates $x^i$ and $x^j$, $B_x$ trarnsforms as $y$ and $z$ which are invariant. Therefore, $B_x$ does not change through Lorentz transformation. Likewise, $F^{02} = -E_y$ transforms as $x^0 = ct$ and $x^2 = y$,

$$E'_y = \gamma(E_y - \beta B_z),$$

$F^{03} = -E_z$ as

$$E'_z = \gamma(E_z + \beta B_y),$$

and so on. $F^{01} = -E_x = -F^{01}$ is invariant since the Lorentz transformation corresponds to rotation in the $(ct, x)$ plane and $F^{00} = F^{11} = 0, F^{01} = -F^{10}$ form an antisymmetric tensor.

**Example 2** *A charge $e$ is moving at a velocity $V$ along the $x$ axis. Find the electric field and compare it with the field expected from the Lienard-Wirchert potentials.*

In the frame of the moving charge, the scalar potential and electric field are

$$\Phi' = \frac{e}{r'}, \ \mathbf{E}' = \frac{e\mathbf{r}'}{r'^3},$$

where

$$r' = \sqrt{x'^2 + y'^2 + z'^2}.$$

The vector potential in the moving frame is zero, $\mathbf{A}' = 0$. The $x$ component of the electric field is invariant,

$$
\begin{aligned}
E_x &= E'_x \\
&= \frac{ex'}{r'^3} \\
&= e\frac{\gamma(x - Vt)}{[\gamma^2(x - Vt)^2 + y^2 + z^2]^{3/2}} \\
&= e(1 - \beta^2)\frac{x - Vt}{\left[(x - Vt)^2 + (1 - \beta^2)(y^2 + z^2)\right]^{3/2}}.
\end{aligned}
$$

Similarly,

$$
E_y = \gamma E'_y = e(1 - \beta^2)\frac{y}{\left[(x - Vt)^2 + (1 - \beta^2)(y^2 + z^2)\right]^{3/2}},
$$

$$
E_z = \gamma E'_z = e(1 - \beta^2)\frac{z}{\left[(x - Vt)^2 + (1 - \beta^2)(y^2 + z^2)\right]^{3/2}}.
$$

Therefore, the electric field in the laboratory frame is

$$
\mathbf{E} = e(1 - \beta^2)\frac{(x - Vt)\mathbf{e}_x + \mathbf{y} + \mathbf{z}}{\left[(x - Vt)^2 + (1 - \beta^2)(y^2 + z^2)\right]^{3/2}}.
$$

Equivalence of this expression to the Coulomb field emerging from the Lienard-Wiechert potentials,

$$
\mathbf{E} = e(1 - \beta^2)\left.\frac{\mathbf{n} - \boldsymbol{\beta}}{\kappa^3 R^2}\right|_{\rm ret},
$$

can be readily proven by noting

$$
\left.\frac{\mathbf{n} - \boldsymbol{\beta}}{1 - \mathbf{n}\cdot\boldsymbol{\beta}}\right|_{\rm ret} = \frac{(x - Vt)\mathbf{e}_x + \mathbf{y} + \mathbf{z}}{\left[(x - Vt)^2 + (1 - \beta^2)(y^2 + z^2)\right]^{1/2}},
$$

where "ret" means the retarded time. Denoting the angle between the $x$ axis and the position vector $\mathbf{R} = (x - Vt)\mathbf{e}_x + \mathbf{y} + \mathbf{z}$ by $\theta$, we find

$$
\mathbf{E} = \frac{e\mathbf{R}}{R^3}\frac{1 - \beta^2}{(1 - \beta^2\sin^2\theta)^{3/2}}.
$$

The Coulomb field is "radial" with respect to the present location of the charge. At $\theta = 0$,

$$
E_\parallel = \frac{e}{R^2}(1 - \beta^2) = \frac{e}{R^2}\frac{1}{\gamma^2},
$$

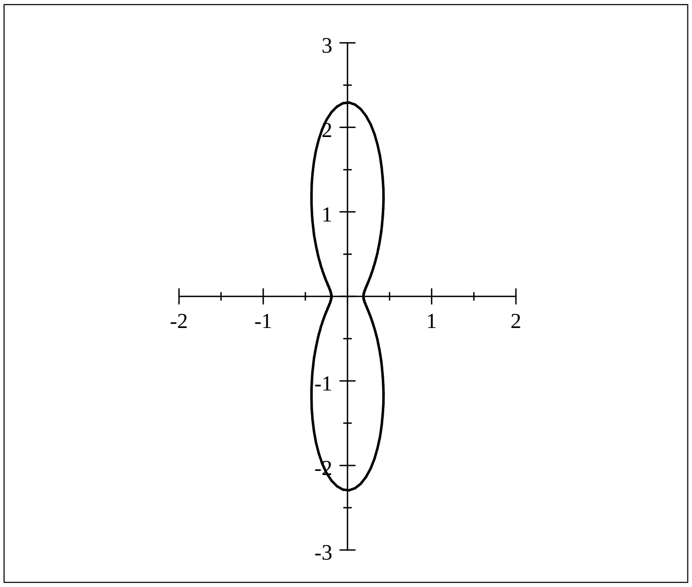
and at $\theta = \pi/2$,

$$
E_\perp = \frac{e}{R^2}\gamma.
$$

In highly relativistic case, the field is dominated by components perpendicular to the velocity.

Angular dependence of the electric field for the case $\beta = 0.9$ is shown below in polar plot.

$$\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}$$




**Example 3** *Show that the quantities $E^2 - B^2$ and $\mathbf{E}\cdot\mathbf{B}$ are invariant in Lorentz transformation.*

Since

$$E^2 = E_\parallel^2 + E_\perp^2,$$

and

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel, \quad \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \boldsymbol{\beta}\times\mathbf{B}_\perp),$$

we find

$$E'^2 = E_\parallel^2 + \gamma^2(\mathbf{E}_\perp + \boldsymbol{\beta}\times\mathbf{B}_\perp)^2.$$

Similarly,

$$B'^2 = B_\parallel^2 + \gamma^2(\mathbf{B}_\perp - \boldsymbol{\beta}\times\mathbf{E}_\perp)^2.$$

Then

$$\begin{aligned}
E'^2 - B'^2 &= E_\parallel^2 - B_\parallel^2 + \gamma^2(\mathbf{E}_\perp + \boldsymbol{\beta}\times\mathbf{B}_\perp)^2 - \gamma^2(\mathbf{B}_\perp - \boldsymbol{\beta}\times\mathbf{E}_\perp)^2 \\
&= E_\parallel^2 - B_\parallel^2 + \gamma^2(1-\beta^2)(E_\perp^2 - B_\perp^2) + 2\gamma^2[\mathbf{E}_\perp\cdot(\boldsymbol{\beta}\times\mathbf{B}_\perp) + \mathbf{B}_\perp\cdot(\boldsymbol{\beta}\times\mathbf{E}_\perp)] \\
&= E^2 - B^2.
\end{aligned}$$

For $\mathbf{E}\cdot\mathbf{B}$,

$$\begin{aligned}
\mathbf{E}'\cdot\mathbf{B}' &= E'_{\parallel}B'_{\parallel} + \mathbf{E}'_{\perp}\cdot\mathbf{B}'_{\perp} \\
&= E_{\parallel}B_{\parallel} + \gamma^2(\mathbf{E}_{\perp}+\boldsymbol{\beta}\times\mathbf{B}_{\perp})\cdot(\mathbf{B}_{\perp}-\boldsymbol{\beta}\times\mathbf{E}_{\perp}) \\
&= E_{\parallel}B_{\parallel} + \gamma^2(1-\beta^2)\mathbf{E}_{\perp}\cdot\mathbf{B}_{\perp} \\
&= \mathbf{E}\cdot\mathbf{B}.
\end{aligned}$$

Note that

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}).$$

The invariance of $\mathbf{E}\cdot\mathbf{B}$ means that if $\mathbf{E}$ and $\mathbf{B}$ are normal to each other in one reference frame, they remain so in any other frames. If $\mathbf{E}$ or $\mathbf{B}$ is zero in one reference frame, in other frames they are normal to each other. Furthermore, if the fields in the laboratory frame are $\mathbf{E}$ and $\mathbf{B}$, there exists a frame moving at a velocity

$$\frac{\mathbf{V}}{1-(V/c)^2} = c\frac{\mathbf{E}\times\mathbf{B}}{E^2+B^2},$$

wherein electric and magnetic fields are parallel to each other.

## 10.5 Energy and Momentum Tensor

As shown in Chapter 1 and 3, the electromagnetic force

$$\mathbf{f} = \rho\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B},$$

can be expressed as the divergence of the Maxwell's stress tensor $T^{ij}$,

$$f_i = -\partial_j T^{ij} = -\frac{1}{4\pi}\partial_j\left(\frac{1}{2}(E^2+B^2)\delta ij - E_iE_j - B_iB_j\right),\ (i,j=1,2,3).$$

The time component of the force four vector was

$$f_0 = \frac{1}{c}\mathbf{J}\cdot\mathbf{E}.$$

Since

$$J\cdot E = -\frac{1}{8\pi}\frac{\partial}{\partial t}(E^2+B^2) - \nabla\cdot\mathbf{S},$$

where $S$ is the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B},$$

$f^0$ can be written as

$$f^0 = -\frac{1}{8\pi c}\frac{\partial}{\partial t}(E^2+B^2) - \frac{1}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{B}),$$

and a resultant four dimensional Maxwell's stress tensor is

$$T^{\mu\nu} = \begin{bmatrix} \frac{1}{8\pi}(E^2+B^2) & \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^x & \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^y & \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^z \\ \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^x & T^{11} & T^{12} & T^{13} \\ \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^y & T^{21}(=T^{12}) & T^{22} & T^{23} \\ \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^z & T^{31}(=T^{13}) & T^{32}=(T^{23}) & T^{33} \end{bmatrix}$$

The quaintly

$$\frac{1}{c}\mathbf{S} = \frac{1}{4\pi}\mathbf{E}\times\mathbf{B},$$

is the momentum flux density and thus

$$\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B},$$

is the momentum density of electromagnetic fields. The angular momentum density is accordingly given by

$$\frac{1}{4\pi c}\mathbf{r}\times(\mathbf{E}\times\mathbf{B}),$$

and the total electromagnetic angular momentum is

$$\frac{1}{4\pi c}\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV.$$

In four dimensional form, the angular momentum tensor can thus be defined by

$$K^{ij} = \frac{1}{c}\int (x_i T^{ik} - x_j T^{ik})d\sigma_k,$$

where $d\sigma_k$ is the "area" element in the four dimensional (hyper) space having the dimensions of volume (cm$^3$).

## 10.6 Relativistic Mechanics

In terms of the velocity four vector

$$v^i = \gamma(c, \mathbf{v}) = \gamma(c, v_x, v_y, v_z),$$

and the field tensor

$$F^{ij} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

the force four vector to act on a charged particle having a charge $e$ can be formulated as

$$F^i = \frac{e}{c} F^{ij} v_j,$$

where

$$v_j = \gamma(c, -\mathbf{v}) = \gamma(c, -v_x, -v_y, -v_z),$$

is the covariant four velocity. $F^i$ reduces to

$$F^i = \gamma(e\boldsymbol{\beta} \cdot \mathbf{E}, \mathbf{F}),$$

where

$$\mathbf{F} = e(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}),$$

is the space component of the electromagnetic force. Note that

$$\boldsymbol{\beta} \cdot \mathbf{F} = e\boldsymbol{\beta} \cdot \mathbf{E},$$

reflecting the fact that the magnetic field does not do any work on charged particles.

The magnitude of the velocity four vector is constant,

$$v^i v_i = \gamma^2 (1 - \beta^2) c^2 = c^2.$$

The corresponding momentum four vector

$$p^i = \left( \frac{\mathcal{E}}{c}, \mathbf{p} \right),$$

also has a constant magnitude,

$$p^i p_i = \left( \frac{\mathcal{E}}{c} \right)^2 - p^2 = (mc)^2,$$

where

$$\mathcal{E}^2 = (cp)^2 + (mc^2)^2,$$

is the energy of the particle.

The Lagrangian in nonrelativistic mechanics,

$$L = \frac{1}{2} m v^2 + e(\boldsymbol{\beta} \cdot \mathbf{A} - \Phi),$$

can be readily generalized as

$$\begin{aligned} L &= -mc^2 \sqrt{1 - \beta^2} + e(\boldsymbol{\beta} \cdot \mathbf{A} - \Phi) \\ &= -mc^2 \sqrt{1 - \beta^2} + \mathbf{v} \cdot (\mathbf{P} - \gamma m \mathbf{v}) - e\Phi, \end{aligned}$$

where

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{v}} = \gamma m\mathbf{v} + \frac{e}{c}\mathbf{A} = \mathbf{p}+\frac{e}{c}\mathbf{A},$$

is the canonical momentum. The equation of motion can be derived from Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) = \nabla L.$$

Noting

$$\begin{aligned}\nabla(\mathbf{v}\cdot\mathbf{A}) &= \mathbf{v}\cdot\nabla\mathbf{A} + \mathbf{A}\cdot\nabla\mathbf{v} + \mathbf{v}\times\nabla\times\mathbf{A} + \mathbf{A}\times\nabla\times\mathbf{v} \\ &= \mathbf{v}\cdot\nabla\mathbf{A} + \mathbf{v}\times\nabla\times\mathbf{A},\end{aligned}$$

since the velocity should be fixed in carrying out spatial differentiation, we find

$$\frac{d}{dt}(\mathbf{p} + \frac{e}{c}\mathbf{A}) = \frac{e}{c}(\mathbf{v}\cdot\nabla\mathbf{A} + \mathbf{v}\times\nabla\times\mathbf{A}) - e\nabla\Phi.$$

Noting

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{A}, \quad \nabla\times\mathbf{A} = \mathbf{B}, \quad \mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},$$

we recover the familiar equation of motion for a charged particle,

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{v}) = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right).$$

Since the momentum $p$ and energy $\mathcal{E}$ are related through

$$\mathbf{p} = \frac{\mathbf{v}}{c^2}\mathcal{E},$$

the acceleration $\mathbf{a} = d\mathbf{v}/dt$ can be readily found,

$$\mathbf{a} = \frac{e}{m\gamma}\left[\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B} - \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})\right]. \tag{10.58}$$

**Example 4** *Analyze the motion of an electron in the Coulomb field of a heavy ion carrying a charge $Ze$.*

Since the electric field is static, the energy of electron is conserved,

$$\mathcal{E} = c\sqrt{p^2 + (mc)^2} - \frac{Ze^2}{r} = \mathcal{E}_0 \text{ (const.)} \tag{10.59}$$

By suitable choice of coordinates, the problem can be made two dimensional so that electron motion is confined in the plane $(r,\theta)$. The momentum can be decomposed into

$$p^2 = p_r^2 + \left(\frac{L_\theta}{r}\right)^2, \tag{10.60}$$

where $L_\theta = rp_\theta$, the angular momentum, is also conserved. Then,

$$c\sqrt{p_r^2 + \left(\frac{L_\theta}{r}\right)^2 + (mc)^2} - \frac{Ze^2}{r} = \mathcal{E}_0. \tag{10.61}$$

In nonrelativistic limit $v \ll c$, we have

$$\frac{1}{2m}\sqrt{p_r^2 + \left(\frac{L_\theta}{r}\right)^2} - \frac{Ze^2}{r} = \text{ const.} \tag{10.62}$$

In the limit $r \to 0$, the LHS diverges. (Note that in nonrelativistic limit, the momentum is bounded.) This means that in nonrelativistic limit, the electron cannot approach the ion indefinitely. In relativistic case, however, the LHS of Eq. () remains finite when $r \to 0$ provided $p_r \to \infty$.

The electron trajectory can be found by differentiating Eq. (10.61) with respect to time,

$$\frac{d}{dt}(m\gamma\dot{r}) - \frac{L_\theta^2}{m\gamma}\frac{1}{r^3} + \frac{Ze^2}{r^2} = 0. \tag{10.63}$$

$L_\theta =$ const. means

$$m\gamma r^2\dot{\theta} = L_\theta = \text{ const.}$$

Then time derivative can be converted into angular derivative through

$$\frac{d}{dt} = \frac{L_\theta}{m\gamma r^2}\frac{d}{d\theta} \tag{10.64}$$

and Eq. () reduces to

$$\frac{d^2}{d\theta^2}\frac{1}{r} + \left(1-\rho^2\right)\frac{1}{r} = \frac{Ze^2}{L_\theta^2 c^2}\mathcal{E}_0, \tag{10.65}$$

where

$$\rho = \frac{Ze^2}{cL_\theta}. \tag{10.66}$$

When $\rho < 1$, the solution for $r(\theta)$ is quasi-oscillatory,

$$r(\theta) = \frac{b}{1 + a\cos(\sqrt{1-\rho^2}\theta)}, \tag{10.67}$$

where $a$ is an integration constant determined by the initial condition and

$$b = \frac{(cL_\theta)^2 - (Ze^2)^2}{Ze^2\mathcal{E}_0}. \tag{10.68}$$

Since $\sqrt{1-\rho^2}$ is in general an irrational number, the oscillation is quasi-periodic without closed orbits. The electron cannot approach the ion indefinitely. If $\rho > 1$, or $Ze^2 > L_\theta$, $\cos(\sqrt{1-\rho^2}\theta)$ in Eq. (10.67) becomes $\cosh(\sqrt{\rho^2-1}\theta)$ and in this case the electron can collapse on the ion.

**Example 5** *Find the power radiated by a charge explicitly in terms of the external electric and magnetic fields.*

In nonrelativistic limit, the radiation power is given by the Larmor's formula,

$$P = \frac{2}{3}\frac{(e\mathbf{a})^2}{c^3}. \tag{10.69}$$

This is still applicable in an instantaneous rest frame of the charged particle,

$$P = \frac{2}{3}\frac{(e\mathbf{a}')^2}{c^3}, \tag{10.70}$$

where $\mathbf{a}'$ is the acceleration in that frame. According to Example 1, the acceleration in the laboratory frame $\mathbf{a}$ and that in the instantaneous rest frame of the charge $a'$ are related through

$$\mathbf{a}' = \gamma^3\mathbf{a}_\parallel + \gamma^2\mathbf{a}_\perp, \tag{10.71}$$

as can be found by choosing $\mathbf{v} = \mathbf{V}$. Since

$$\mathbf{a}_\parallel = \frac{e}{m\gamma^3}\mathbf{E}_\parallel,\ \mathbf{a}_\perp = \frac{e}{m\gamma}(\mathbf{E}_\perp + \boldsymbol{\beta}\times\mathbf{B}), \tag{10.72}$$

we find for the radiation power

$$\begin{aligned} P &= \frac{2}{3}\frac{(e\mathbf{a}')^2}{c^3} \\ &= \frac{2}{3}\frac{e^4}{m^2c^3}\left[E_\parallel^2 + \gamma^2(\mathbf{E}_\perp + \boldsymbol{\beta}\times\mathbf{B})^2\right] \\ &= \frac{2}{3}\frac{e^4}{m^2c^3}\gamma^2\left[(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B})^2 - (\boldsymbol{\beta}\cdot\mathbf{E})^2\right]. \end{aligned} \tag{10.73}$$

If the electric field is parallel to the velocity and there is no magnetic field as in linear accelerators, the power becomes independent of the particle energy,

$$P = \frac{2}{3}\frac{e^4}{m^2c^3}E_\parallel^2.$$

In the absence of the electric field, we obtain

$$P = \frac{2}{3}\frac{e^4}{m^2c^3}\gamma^2(\boldsymbol{\beta}\times\mathbf{B})^2.$$

## 10.7 Radiation Damping

Consider an electron subject to an acceleration $a$ for a duration $T$. In nonrelativistic limit, the electron acquires a kinetic energy of order

$$\mathcal{E}_{kin} = \frac{1}{2}m(aT)^2,$$

while the amount of energy radiated is

$$\mathcal{E}_{rad} = \frac{2}{3}\frac{(ea)^2}{c^3}T.$$

It is reasonable to conjecture that the radiation energy should not exceed the kinetic energy, $E_{rad} < E_{kin}$ for otherwise, one must wonder where the radiation energy comes from. This imposes a limit on $T$,

$$T \gtrsim \frac{e^2}{mc^3} = \tau \simeq 10^{-23} \text{ sec},$$

where $\tau$ is approximately the transit time of light over the classical electron radius,

$$r_e = \frac{e^2}{mc^2} = 2.85 \times 10^{-13} \text{ cm}.$$

(To realize such a time scale in cyclotron motion in a magnetic field, we need an unrealistically large magnetic field, $B \simeq 10^{12}$ T.) In this time scale, radiation reaction on dynamics of charged particle is expected to become significant. However, the uncertainty principle imposes a more strict constraint,

$$\mathcal{E}\Delta t \gtrsim \hbar.$$

If we choose the self energy of the electron for $\mathcal{E} = mc^2$, we find

$$\Delta t \gtrsim \frac{\hbar}{mc^2} = \frac{\hbar c}{e^2}\frac{e^2}{mc^3} = 137\tau,$$

where

$$\frac{e^2}{\hbar c} = \frac{1}{137},$$

is the fine structure constant. Therefore, in nonrelativistic cases, quantum mechanical effects become important well before recoil force due to radiation need to be considered.

However, in radiation from a charge undergoing harmonic motion, radiation reaction is well defined and has been observed experimentally. Recoil force exerted by radiation may be estimated from energy balance,

$$\begin{aligned}\int_0^T F \cdot v dt &= -\frac{2}{3}\frac{e^2}{c^3}\int_0^T a^2 dt \\ &= \frac{2}{3}\frac{e^2}{c^3}\int_0^T v\frac{d^2v}{dt^2}dt.\end{aligned}$$

Then

$$F = \frac{2}{3}\frac{e^2}{c^3}\frac{d^2 v}{dt^2},$$

and the equation of motion of harmonic oscillator is modified as

$$\ddot{x} - \tau\dddot{x} + \omega_0^2 x = -\frac{eE}{m}e^{-i\omega t}.$$

Solution for $x(t)$ is

$$x(t) = \frac{eE}{m}\frac{e^{-i\omega t}}{\omega^2 + i\tau\omega^3 - \omega_0^2},$$

which remains bounded at the resonance $\omega \simeq \omega_0$ due to radiation damping. The shift in the resonance frequency is given by

$$\Delta\omega \simeq -\frac{5}{8}\omega_0^3\tau^2.$$