Energy in Standing Waves

Standing waves are formed by sinusoidal waves propagating in opposite directions and thus no energy is carried. Rather, energy is confined between nodes (zero displacement points). We consider a standing wave

$$\xi_0 \sin (kx) \cos (\omega t), \ 0 < x < L$$

in a string of length $L$, mass density $\rho_l$ (kg/m) and tension $T$ (N). The wave lengths allowed are $\lambda_n = nL/2$ and $k_n = 4\pi/nL$ where $n$ is integer. The wave velocity is

$$c_w = \sqrt{\frac{T}{\rho_l}} \text{ m/s}$$

The kinetic energy density is

$$\text{KED} = \frac{1}{2} \rho_l \left( \frac{\partial \xi}{\partial t} \right)^2 = \frac{1}{2} \rho_l \omega^2 \xi_0^2 \sin^2 (kx) \sin^2 (\omega t) \text{ J/m}$$

Its spatial average is

$$\frac{1}{2} \times \text{peak} = \frac{1}{4} \rho_l \omega^2 \xi_0^2 \sin^2 (\omega t)$$

and the total kinetic energy is

$$\text{KE} = \frac{1}{4} L \rho_l \omega^2 \xi_0^2 \sin^2 (\omega t)$$

The potential energy density is

$$\text{PED} = \frac{1}{2} T \left( \frac{\partial \xi}{\partial x} \right)^2 \text{ (N = J/m)}$$

$$= \frac{1}{2} T k^2 \xi_0^2 \cos^2 (kx) \cos^2 (\omega t)$$

$$= \frac{1}{2} \rho_l \omega^2 \xi_0^2 \cos^2 (kx) \cos^2 (\omega t)$$

The spatial average is

$$\frac{1}{4} \rho_l \omega^2 \xi_0^2 \cos^2 (\omega t)$$

and the total potential energy is

$$\text{PE} = \frac{1}{4} L \rho_l \omega^2 \xi_0^2 \cos^2 (\omega t)$$

The total energy is

$$\frac{1}{4} L \rho_l \omega^2 \xi_0^2 \sin^2 (\omega t) + \frac{1}{4} L \rho_l \omega^2 \xi_0^2 \cos^2 (\omega t) = \frac{1}{4} L \rho_l \omega^2 \xi_0^2 \text{ (J)}$$

Beat
When two sinusoidal signals with slightly different frequencies $f$ and $f + \Delta f$ are superposed, the amplitude is modulated and a beat with a frequency $\Delta f$ is detected.

\[
A \sin(2\pi ft) + A \sin[2\pi (f + \Delta f) t] \\
= 2A \sin\left(2\pi \frac{f + \Delta f}{2} t\right) \cos\left(2\pi \frac{\Delta f}{2} t\right) \\
\approx 2A \sin(2\pi ft) \cos\left(2\pi \frac{\Delta f}{2} t\right)
\]

The signal intensity is

\[
4A^2 \sin^2(2\pi ft) \cos^2\left(2\pi \frac{\Delta f}{2} t\right) = 2A^2 \sin^2(2\pi ft) [1 + \cos(2\pi \Delta ft)]
\]

where use is made of

\[
\cos^2 x = \frac{1 + \cos 2x}{2}
\]

The intensity is modulated at the frequency difference $\Delta f$. The figure below shows addition of waves with frequencies 10 Hz and 11 Hz.

\[
sin (2\pi \times 10t)
\]

\[
sin (2\pi \times 11t)
\]

\[
sin (2\pi \times 10t) + sin (2\pi \times 11t)
\]
Phase and Group Velocities

Superposition of two waves having slightly different frequencies and wavenumbers

\[ A \sin (kx - \omega t) + A \sin [(k + \Delta k) x - (\omega + \Delta \omega) t] \]

\[ \approx 2A (kx - \omega t) \cos \left( \frac{\Delta k}{2} x - \frac{\Delta \omega}{2} t \right) \]

Fine ripples propagate at

Phase velocity \( \frac{\omega}{k} \)

The envelope propagates at

Group velocity \( \frac{\Delta \omega}{\Delta k} \rightarrow \frac{d\omega}{dk} \)

In non-dispersive waves, the phase and group velocities are the same. Examples are light wave in vacuum, sound waves. Examples of dispersive waves:

Water wave \( \omega = \sqrt{gk} \), \( \frac{\omega}{k} = \sqrt{\frac{g}{k}} \), \( \frac{d\omega}{dk} = \frac{1}{2} \sqrt{\frac{g}{k}} \)

Electromagnetic waves in plasma \( \omega^2 = (ck)^2 + \omega_p^2 \)

\( \frac{\omega}{k} = c \frac{1}{\sqrt{1 - (\omega_p^2/\omega^2)}} > c ; \frac{d\omega}{dk} = c \sqrt{1 - (\omega_p^2/\omega^2)} < c \)

Sound Waves in a Gas

When a gas is compressed, the density increases. At the same time, the temperature increases adiabatically because thermal conduction in a gas is slow. The equation of state is

\[ PV^\gamma = \text{const.} \]

where \( \gamma \) is the ratio of specific heats,

\[ \gamma = \frac{f + 2}{f} \]

with \( f \) the number of degree of freedom. For monoatomic gases (He, Ar, etc.) \( f = 3 \) and \( \gamma = \frac{5}{3} \). For diatomic gases, \( f = 5 \) and \( \gamma = \frac{7}{5} \).

Equation of state is

\[ dPV^\gamma + P\gamma V^{\gamma-1} = 0 \]

\[ \frac{\Delta P}{P} = -\gamma \frac{\Delta V}{V} \]

Then the pressure wave is

\[ p = -\gamma P \frac{\partial \xi}{\partial x} \]

and the equation of motion is

\[ \rho \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial}{\partial x} p = \gamma P \frac{\partial^2 \xi}{\partial x^2} \]
The sound speed is given by
\[ c_s = \sqrt{\frac{\gamma P}{\rho_v}} = \sqrt{\frac{\gamma RT}{M_{\text{mol}}}} \]
where \( R = 8.3 \) J/K is the gas constant and \( M_{\text{mol}} \) is the molar mass of the gas. The molar mass of air is 0.029 kg. Note that the sound speed is independent of the gas density. At 20°C, the sound speed is
\[ c_s = \sqrt{\frac{1.4 \times 8.3 \times 293}{0.029}} = 343 \text{ m/s} \]

Doppler shift
Doppler effect is apparent change in the frequency (or wavelength) due to the motion of sound source and observer. If the source is moving, the wavelength changes,
\[ \lambda' = \frac{c_s - V_s}{c_s} \lambda, \quad f' = \frac{c_s}{c_s - V_s} f \]
Waves in front of the source are squeezed (shorter wavelength \( \rightarrow \) higher frequency).

If the observer is moving, the sound speed changes from \( c_s \) to \( c_s + V_o \). Then
\[ f'' = \frac{c_s + V_o}{c_s} f \]