Chapter 8

Nonlinear Effects

8.1 Introduction

A plasma is inherently a nonlinear medium. We have discussed numerous instabilities assuming that the amplitude of growing perturbations is small. As the amplitude becomes sufficiently large, the linearization procedure breaks down. Nonlinear effects tend to limit the growth of instabilities (nonlinear saturation). Also, through mode coupling, energy transfer occurs toward modes which are linearly stable. Nonlinear effects also occur when a large amplitude plasma wave is excited by an external means. For example, the dispersion in the ion acoustic wave can be counter-balanced by nonlinearity and an ion acoustic soliton (pulse-like solitary perturbation) can propagate without appreciable deformation.

In this chapter, some typical nonlinear plasma effects will be discussed. In a few cases, analytic solutions can be worked out. However, majority of problems require numerical analysis.

8.2 Frequency Shift of Langmuir Mode with Finite Amplitude

This nonlinear problem has a long history. Intuitively, one may imagine that the frequency of Langmuir mode should increase with the amplitude because the oscillating kinetic energy acquired by the electrons effectively increases the electron temperature. The temperature increase may be estimated as

$$\Delta T_e \simeq A \left(\frac{\Delta n}{n_0} \right)^2 T_e,$$

where $\Delta n/n_0$ is the relative density perturbation and A is a numerical factor. Therefore, according to the dispersion relation

$$\omega \simeq \omega_{pe} + \frac{3}{2}\frac{k^2}{\omega_{pe} m}\left(T_e + \Delta T_e\right),$$

the frequency increase is expected to be of the order of

$$\Delta\omega \simeq A\left(\frac{k}{k_{De}}\right)^2\left(\frac{\Delta n}{n_0}\right)^2\omega_{pe}.$$

In a cold plasma ($T_e = 0$), no frequency shift is expected.

Let us first use the quasilinear theory which is the lowest order nonlinear theory. The quasilinear theory is able to describe time evolution of the particle distribution function when plasma waves are excited either spontaneously (instability) or externally. The Vlasov equation for the electrons

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}E\frac{\partial f}{\partial v} = 0, \tag{8.1}$$

may be linearized by assuming $f = f_0(v,t) + \sum_k f_k \; e^{i(kx-\omega_k t)}$ where the Fourier component f_k corresponds to that of the electric field which is similarly decomposed as $E = \sum_k E_k e^{i(kx-\omega_k t)}$. Substituting the linear solution for f_k

$$f_k = \frac{e/m}{i(kv-\omega_k)}E_k\frac{\partial f_0}{\partial v}, \tag{8.2}$$

into the original Vlasov equation, we find the following evolution equation for f_0

$$\frac{\partial f_0}{\partial t} = \frac{1}{2}\left(\frac{e}{m}\right)^2\sum_k |E_k|^2\frac{\partial}{\partial v}\left(\frac{\gamma_k}{(kv-\omega_k)^2+\gamma_k^2}\frac{\partial f_0}{\partial v}\right), \tag{8.3}$$

where the factor 1/2 is due to averaging, γ_k is the growth rate and the mode frequency ω_k has been replaced with a complex frequency, $\omega_k \to \omega_k + i\gamma_k$. The field energy is assumed to be in the form

$$|E_k(t)|^2 = E_k^2(0)e^{2\int^t \gamma(t)dt}$$

where $\gamma(t)$ is a slowly varying function of time. Eq. (8.3) can be integrated by noting

$$\frac{\partial}{\partial t}|E_k|^2 = 2\gamma_k|E_k|^2,$$

$$f_0(v,t) = f_0(v,0) + \frac{1}{4}\left(\frac{e}{m}\right)^2\sum_k |E_k|^2\frac{\partial}{\partial v}\left(\frac{1}{(kv-\omega_{kr})^2+\gamma_k^2}\frac{\partial f_0}{\partial v}\right). \tag{8.4}$$

If the initial distribution is Maxwellian, the kinetic energy of electrons after excitation of single Langmuir wave with wavenumber k and $\omega/k \gg v_{Te}$ (electron thermal velocity) can be readily calculated as

$$\frac{m}{2}\int v^2 f_0(v,t)dv = \frac{T_e}{2}+\frac{|E_k|^2}{16\pi n_0}+\frac{9}{4}\frac{e^2k^2|E_k|^2T_e}{m^2\omega_{pe}^4}$$
$$= \frac{T_e}{2}+\frac{|E_k|^2}{16\pi n_0}+\frac{9}{4}\left(\frac{\Delta n}{n_0}\right)^2 T_e, \tag{8.5}$$

where the relation between the electric field and density perturbation $kE_k = 4\pi e\Delta n$ has been used in the last term. The second term in the RHS,

$$\frac{|E_k|^2}{16\pi n_0}$$

is the average wave energy shared by each electron. (Recall that in Langmuir mode, equal amount of energy is shared by the electric field and electrons.) The last term may be regarded as an effective increase in the electron temperature due to excitation of the Langmuir mode and the quasilinear estimate for the frequency shift is

$$\Delta\omega_{QL} \simeq \frac{27}{4}\left(\frac{k}{k_{De}}\right)^2\left(\frac{\Delta n}{n_0}\right)^2\omega_{pe}. \tag{8.6}$$

In quasilinear theory, higher order nonlinearity is evidently ignored. In particular, effects of virtual modes due to mode coupling are absent. Virtual modes do not satisfy the dispersion relation and thus cannot exist as natural (weakly damped) modes. In a sense, they are forced to exist due to nonlinear mode coupling and provide a temporary energy reservoir. However, they can influence the process of energy transfer between different modes. For simplicity, we employ hydrodynamic equations to see how harmonic modes appear in the case of nonlinear Langmuir mode.

The set of equations to describe electrostatic electron oscillations is

$$\begin{aligned}\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv) &= 0,\\ \left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)v &= -eE-\frac{1}{n}\frac{\partial p}{\partial x},\\ \frac{\partial E}{\partial x} &= -4\pi e(n-n_0),\end{aligned} \tag{8.7}$$

where n is the electron density, n_0 unperturbed ion density, v electron velocity, p electron pressure, and E electric field. For the pressure variation accompanying rapid oscillation, we assume the following adiabatic law ($\gamma_e = 3$),

$$\frac{p}{p_0} = \left(\frac{n}{n_0}\right)^3. \tag{8.8}$$

Introducing the following normalizations,

$$\omega_{pe}t \to t,\ k_{De}x \to x,\ v/v_{Te} \to v,\ n/n_0 \to 1+n,\ \frac{4\pi e n_0}{k_{De}}E \to E,$$

we rewrite the set as

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial v}{\partial x} &= -\frac{\partial}{\partial x}(nv), \\ \frac{\partial v}{\partial t} + 3\frac{\partial n}{\partial x} + E &= -v\frac{\partial v}{\partial x} - 3n\frac{\partial n}{\partial x}, \\ \frac{\partial E}{\partial x} + n &= 0. \end{aligned} \tag{8.9}$$

The set is arranged so that the terms in the LHS are linear and those in the RHS are nonlinear. We attempt to solve the nonlinear equations using the method of multiple time scales.

In the lowest order, the nonlinear terms are ignored. The linear density perturbation is assumed to be

$$n_l = a\cos(kx-\omega t) = \frac{1}{2}a\left(e^{i(kx-\omega t)} + \text{c.c.}\right),$$

where $a = \Delta n/n_0$ is the normalized amplitude of the density perturbation and c.c. means complex conjugate. Then, the linear velocity and electric field perturbations should be proportional to the density perturbation, and the LHS of Eq. (8.9) can be written in the matrix form

$$\begin{pmatrix} -i\omega & ik & 0 \\ 3ik & -i\omega & 1 \\ 1 & 0 & ik \end{pmatrix} \begin{pmatrix} n_l \\ v_l \\ E_l \end{pmatrix} = 0. \tag{8.10}$$

From the vanishing determinant of the matrix, the familiar linear dispersion relation emerges,

$$\omega^2 = 1 + 3k^2 \text{ or } \omega^2 = \omega_{pe}^2 + \frac{3T_e}{m}k^2. \tag{8.11}$$

The velocity and field perturbations in terms of the density perturbation are given by

$$v_l = \frac{\omega}{k}n_l, \quad E_l = \frac{i}{k}n_l. \tag{8.12}$$

The right-hand eigenvector of the matrix

$$\mathbf{M} = \begin{pmatrix} -i\omega & ik & 0 \\ 3ik & -i\omega & 1 \\ 1 & 0 & ik \end{pmatrix},$$

is thus

$$\mathbf{R} = \begin{pmatrix} k \\ \omega \\ i \end{pmatrix},$$

and the left-hand eigenvector is

$$\mathbf{L} = (\omega, k, i).$$

We now proceed to find nonlinear corrections to the dispersion relation. The amplitude a is assumed to be small being of order ε. The parameter ε is called book keeping parameter in nonlinear perturbation analysis. The perturbed quantities are now assumed to be in the form,

$$\begin{aligned} n &= \frac{1}{2}a(e^{i\theta} + e^{-i\theta}) + \varepsilon n_1 + \varepsilon^2 n_2 + \cdots \\ v &= \frac{1}{2}\frac{\omega}{k}a(e^{i\theta} + e^{-i\theta}) + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots \\ E &= \frac{i}{2k}a(e^{i\theta} + e^{-i\theta}) + \varepsilon E_1 + \varepsilon^2 E_2 + \cdots \end{aligned} \tag{8.13}$$

where $\theta = kx - \omega t$. The amplitude and frequency are expected to change due to nonlinearity. Therefore, the time derivative may also be expanded in terms of ε as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + \cdots \tag{8.14}$$

Substituting n, v, E in Eq. (8.13) into Eq. (8.9), we find in order ε,

$$\begin{aligned} \frac{\partial n_1}{\partial \tau} + \frac{\partial v_1}{\partial x} &= -\frac{1}{2}\frac{\partial a}{\partial \tau_1}e^{i\theta} - \frac{1}{2}i\omega a^2 e^{2i\theta} + \text{c.c.}, \\ \frac{\partial v_1}{\partial \tau} + 3\frac{\partial n_1}{\partial x} + E_1 &= -\frac{\omega}{2k}\frac{\partial a}{\partial \tau_1}e^{i\theta} - \frac{i}{4k}(\omega^2 + 3k^2)a^2 e^{2i\theta} + \text{c.c.}, \\ \frac{\partial E_1}{\partial x} + n_1 &= 0. \end{aligned} \tag{8.15}$$

Note the appearance of higher harmonic terms in the RHS, $e^{2i\theta} = e^{2i(kx-\omega t)}$, due to the nonlinear terms nv and n^2. The equations describe harmonic oscillators (nonlinearly) driven by oscillating sources. If the frequency of a driver matches the frequency of harmonic oscillators, often secular amplitude increase occurs. In the present problem, we do not expect such secularity because the plasma is isolated from external driving sources. The driving terms in the above set of equations are created internally by the plasma itself.

Let us further assume that n_1, v_1 and E_1 consist of terms proportional to $e^{im\theta}$, where m is an integer, $m = 0, \pm 1, \pm 2, \cdots$,

$$\begin{aligned} n_1 &= n_1^{(0)} + n_1^{(1)} e^{i\theta} + n_1^{(2)} e^{2i\theta} + \cdots + \text{ c.c.}, \\ v_1 &= v_1^{(0)} + v_1^{(1)} e^{i\theta} + v_1^{(2)} e^{2i\theta} + \cdots + \text{ c.c.}, \\ E_1 &= E_1^{(0)} + E_1^{(1)} e^{i\theta} + E_1^{(2)} e^{2i\theta} + \cdots + \text{ c.c.} \end{aligned} \tag{8.16}$$

The terms oscillating with the fundamental frequency, $n_1^{(1)} e^{i\theta}, v_1^{(1)} e^{i\theta}, E_1^{(1)} e^{i\theta}$ are actually redundant and can be ignored. For the second harmonic terms, we find

$$\begin{pmatrix} -i2\omega & i2k & 0 \\ 6ik & -i2\omega & 1 \\ 1 & 0 & i2k \end{pmatrix} \begin{pmatrix} n_1^{(2)} \\ v_1^{(2)} \\ E_1^{(2)} \end{pmatrix} = -\frac{ia^2}{4k} \begin{pmatrix} 2k\omega \\ \omega^2 + 3k^2 \\ 0 \end{pmatrix} \tag{8.17}$$

The determinant of the matrix does not vanish because the second harmonic is not a natural mode but virtual (it does not satisfy the dispersion relation). Noting

$$\begin{vmatrix} -i2\omega & i2k & 0 \\ 6ik & -i2\omega & 1 \\ 1 & 0 & i2k \end{vmatrix} = -6ik, \tag{8.18}$$

Eq. (8.17) can be solved as

$$\begin{pmatrix} n_1^{(2)} \\ v_1^{(2)} \\ E_1^{(2)} \end{pmatrix} = \frac{a^2}{4k} \begin{pmatrix} 2k(1+4k^2) \\ \omega(1+8k^2) \\ i(1+4k^2) \end{pmatrix}. \tag{8.19}$$

The nonoscillating dc components, $n_1^{(0)}$ and $E_1^{(0)}$ must vanish because we do not expect creation of uniform increase (or decrease) in the plasma density and dc electric field. Thus

$$n_1^{(0)} = E_1^{(0)} = 0. \tag{8.20}$$

However, the dc velocity component $v_1^{(0)}$ may exist because the plasma oscillation deforms the electron distribution function. This can be found from the quasilinear equation for the electron momentum,

$$\begin{aligned} V_e &= \int v f_0(v,t) dv \\ &= \frac{1}{4}\left(\frac{e}{m}\right)^2 |E_k|^2 \int v \frac{\partial}{\partial v}\left(\frac{1}{(kv-\omega_k)^2+\gamma_k^2}\frac{\partial f_0}{\partial v}\right) dv \\ &\simeq \frac{1}{2}\left(\frac{e}{m}\right)^2 \frac{k}{\omega^3} |E_k|^2 \\ &\simeq \frac{1}{2}\frac{\omega}{k}\left(\frac{\Delta n}{n_0}\right)^2 = \frac{1}{2}\frac{\omega}{k} a^2. \end{aligned} \tag{8.21}$$

This drift velocity of electrons is the result of momentum transfer (current drive) from the Langmuir mode to electrons. To avoid the Doppler shift due to the drift velocity V_e, we have to ride on the frame moving at the velocity V_e. Then, in this frame, electrons drift with a velocity,

$$v_1^{(0)} = -V_e = -\frac{1}{2}\frac{\omega}{k} a^2.$$

In summary, nonlinear corrections to order ε are

$$\begin{aligned} n_1 &= \tfrac{1}{2}(1+4k^2)a^2 e^{2i\theta} + \text{c.c.}, \\ v_1 &= -\frac{\omega}{2k}|a|^2 + \frac{\omega}{4k}(1+8k^2)a^2 e^{2i\theta} + \text{c.c.}, \\ E_1 &= \frac{i}{4k}(1+4k^2)a^2 e^{2i\theta} + \text{c.c.} \end{aligned} \tag{8.22}$$

We now proceed to order ε^2. In order ε^2, Eq. (8.9) yields for terms proportional to $e^{i\theta}$,

$$\begin{aligned} \frac{\partial n_2}{\partial \tau} + \frac{\partial v_2}{\partial x} &= -\frac{\partial a}{\partial \tau_2} e^{i\theta} - i\frac{\omega}{4}(1+16k^2)|a|^2 a e^{i\theta}, \\ \frac{\partial v_2}{\partial \tau} + 3\frac{\partial n_2}{\partial x} + E_2 &= -\frac{\omega}{k}\frac{\partial a}{\partial \tau_2} e^{i\theta} + \frac{i}{4k}(1-11k^2)|a|^2 a e^{i\theta}, \\ \frac{\partial E_2}{\partial \tau} + n_2 &= 0 \end{aligned} \tag{8.23}$$

This can be written in the matrix form

$$\mathbf{M}\cdot \begin{pmatrix} n_2 \\ v_2 \\ E_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial a}{\partial \tau_2} - i\dfrac{\omega}{4}(1+16k^2)|a|^2 a \\ -\dfrac{\omega}{k}\dfrac{\partial a}{\partial \tau_2} + \dfrac{i}{4k}(1-11k^2)|a|^2 a \\ 0 \end{pmatrix}.$$

Multiplying the left-hand eigenvector $\mathbf{L} = (\omega, k, i)$, we find

$$\mathbf{L}\cdot\mathbf{M} = 0 = \mathbf{L}\cdot \begin{pmatrix} -\dfrac{\partial a}{\partial \tau_2} - i\dfrac{\omega}{4}(1+16k^2)|a|^2 a \\ -\dfrac{\omega}{k}\dfrac{\partial a}{\partial \tau_2} + \dfrac{i}{4k}(1-11k^2)|a|^2 a \\ 0 \end{pmatrix}.$$

This compatibility condition readily yields for $\partial a/\partial \tau_2$,

$$\frac{\partial a}{\partial \tau_2} = -\frac{15i}{4} k^2 \left| a \right|^2 a. \tag{8.24}$$

Letting $a(\tau_2) = |a(\tau_2)| \, e^{i\phi(\tau_2)}$, we see that the amplitude is not affected by nonlinearity in order ε^2 but the frequency is up-shifted by

$$\Delta\omega_{MC} = \frac{15}{4} k^2 \left| a \right|^2, \text{ or } \Delta\omega_{MC} = \frac{15}{4} \left(\frac{\Delta n}{n_0} \right)^2 \left(\frac{k}{k_{De}} \right)^2 \omega_{pe},$$

which is comparable with the frequency shift estimated from quasilinear theory. The total shift is

$$\Delta\omega = \Delta\omega_{QL} + \Delta\omega_{MC} = \frac{21}{2} \left(\frac{\Delta n}{n_0} \right)^2 \left(\frac{k}{k_{De}} \right)^2 \omega_{pe}. \tag{8.25}$$

8.3 Nonlinear Suppression of Landau Damping and Growth

Landau damping is caused by resonant wave-particle interaction. Particles moving at velocities close to the phase velocity of a wave can interact strongly with the wave because they essentially experience dc electric field riding on the wave frame. However, as the wave amplitude increases, those resonant particles are trapped in the potential well associated with the wave and energy exchange between the wave and resonant particles become ineffective. In the case of Landau damping, the damping rate approaches zero and in the case of inverse Landau damping, wave growth ceases.

As shown in Chapter 7, a resonant particle moving at the phase velocity of the wave, $V = \omega/k$, is most efficiently accelerated by the electric field. This can be seen from the solution for the particle velocity in an electric field $E_0 \sin(kx - \omega t)$,

$$v(x,t) = -\frac{\dfrac{e}{m}E_0}{kV - \omega} \left[\cos(kx - \omega t) - \cos[k(x - Vt)] \right], \tag{8.26}$$

which has been obtained with the initial condition $v(x, t = 0) = 0$. In the limit $V \rightarrow \omega/k$, the velocity becomes secular

$$v(x,t) = \frac{e}{m} E_0 t \sin(kx - \omega t), \tag{8.27}$$

indicating a free fall of the charge in the electric field. The amplitude of the velocity increases with time in a secular manner and at some instant, linearization based on the assumption of small perturbation should break down. One such measure may be found from the transit distance,

$$\frac{1}{2} \frac{eE_0}{m} t^2.$$

If the distance travelled by the charge becomes comparable with the wavelength, the field changes its direction and the initial linear acceleration should cease. Therefore, the upper bound for the time below which linear approximation is valid can be estimated as

$$t < \tau = \sqrt{\frac{m}{eE_0 k}}.$$

In the case of Landau damping, the initial damping rate γ (< 0) remains valid well before τ. After τ, Landau damping becomes suppressed and steady oscillation persists.

In the case of inverse Landau damping (Landau growth), the initial exponential amplitude increase with a linear growth rate $\gamma(> 0)$ ceases when the field amplitude becomes large enough so that

$$\frac{1}{\tau} = \sqrt{\frac{eE_0 k}{m}} \simeq \gamma. \tag{8.28}$$

Let us apply this rough estimate to the case of beam-plasma instability. The maximum linear growth rate is

$$\gamma = \frac{\sqrt{3}}{2}\left(\frac{n_b}{2n_0}\right)^{1/3}\omega_{pe}, \tag{8.29}$$

where n_b is the electron beam density which is assumed to be much smaller than the background electron density n_0. The maximum growth rate occurs at the resonance, $kV = \omega_{pe}$, where V is the beam velocity. The condition $\gamma\tau = 1$ yields

$$\frac{E_0^2}{8\pi} \simeq \frac{9}{64}\left(\frac{n_b}{2n_0}\right)^{1/3} n_b m V^2. \tag{8.30}$$

This agrees with the earlier estimate for the saturation level in chapter 8 except for numerical factors. However, it should be cautioned that this method does not work always. For example, if the method is applied to the Buneman instability, the field amplitude at saturation is grossly underestimated because the phase velocity of the Buneman instability is too remotely separated from the electron drift velocity. We will analyze nonlinear saturation of the Buneman instability separately in the following section.

8.4 Saturation of the Buneman Instability-Quasilinear Analysis

Although the Buneman instability appears similar to the electron beam plasma instability, one distinct feature is that the phase velocity of the instability is much smaller than the electron drift velocity,

$$\frac{\omega_r}{k} \simeq \frac{1}{2}\left(\frac{m}{2M}\right)^{1/3} V \ll V,$$

which makes the estimate of the amplitude saturation based on the electron trapping mechanism inapplicable. The method presented here is essentially quasilinear. One distinction is that the complex mode frequency is allowed to vary with time and its time evolution followed.

The linear stage of the Buneman instability is described by the dispersion relation

$$1 = \left(\frac{\omega_{pi}}{\omega}\right)^2 + \frac{\omega_{pe}^2}{(\omega - kV)^2}, \tag{8.31}$$

provided the electron thermal velocity is negligible compared with the drift velocity V. As the instability grows, the drift velocity decreases because the drift energy is expended for wave excitation. At the same time, the electrons are "heated" by the instability although heating is not genuine but has meaning of increase in the electron sloshing energy associated with the instability.

The Vlasov equation for the electrons is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}E\frac{\partial f}{\partial v} = 0. \tag{8.32}$$

After linearization and Fourier decomposition, the perturbed distribution function can be readily found,

$$f_k = \frac{\frac{e}{m}E_k}{i(kv - \omega_k)}\frac{\partial f_0}{\partial v}. \tag{8.33}$$

Substituting this back into the original Vlasov equation, we see that the zero-th order distribution function now becomes time-dependent and obeys the following evolution equation,

$$\frac{\partial f_0}{\partial t} = \left(\frac{e}{m}\right)^2 \sum_k |E_k|^2 \,\mathrm{Re}\, \frac{\partial}{\partial v}\left(\frac{i}{\omega_k - kv}\frac{\partial f_0}{\partial v}\right). \tag{8.34}$$

In the case of the Buneman instability, the distribution function is characterized by a large drift velocity, and the instability itself by a large growth rate. Noting

$$\mathrm{Re}\,\frac{i}{\omega_k - kv} = \frac{\gamma_k}{(kv - \omega_{kr})^2 + \gamma_k^2}, \tag{8.35}$$

and approximating f_0 by

$$f_0(v) = \delta(v - V),$$

we find that the drift velocity and an effective electron temperature evolve according to

$$\frac{dV(t)}{dt} = \int v\frac{\partial f_0}{\partial t}dv = -\left(\frac{e}{m}\right)^2 \sum_k |E_k|^2 \frac{k(kV - \omega_{kr})\gamma_k}{[(kV - \omega_{kr})^2 + \gamma_k^2]^2}, \tag{8.36}$$

$$\frac{dT_e}{dt} = \frac{1}{2}m\int [v - V(t)]^2 \frac{\partial f_0}{\partial t}dv = \frac{e^2}{m}\sum_k |E_k|^2 \frac{\gamma_k}{(kV - \omega_{kr})^2 + \gamma_k^2}. \tag{8.37}$$

Since the mode with the maximum growth rate is expected to dominate, we consider only the dominant mode with $k = \omega_{pe}/V$. Then after integration over time, we find

$$V(t) = V - \frac{e^2}{2m^2} \frac{k(kV - \omega_r)}{[(kV - \omega_r)^2 + \gamma^2]^2} E^2(t), \tag{8.38}$$

$$T_e(t) = \frac{e^2}{2m} \frac{E^2(t)}{(kV - \omega_r)^2 + \gamma^2}, \tag{8.39}$$

where both the frequency $\omega_r(t)$ and the growth rate $\gamma(t)$ are now time dependent. The electron term in the dispersion relation is modified as

$$\frac{\omega_{pe}^2}{[\omega(t) - kV(t)]^2 - 3k^2 v_{Te}^2(t)}, \tag{8.40}$$

where $\omega(t) = \omega_r(t) + i\gamma(t)$ and v_{Te} is an effective thermal velocity defined by

$$v_{Te}^2(t) = \frac{T_e(t)}{m}. \tag{8.41}$$

The ion term for time dependent frequency should be generalized as

$$\omega_{pi}^2 \exp\left(i \int^t \omega(t')dt'\right) \int^t (t' - t) \exp\left(-i \int^{t'} \omega(t'')dt''\right) dt'.$$

The desired nonlinear dispersion relation is therefore

$$\begin{aligned} 1 &= \omega_{pi}^2 \exp\left(i \int^t \omega(t')dt'\right) \int^t (t' - t) \exp\left(-i \int^{t'} \omega(t'')dt''\right) dt' \\ &+ \frac{\omega_{pe}^2}{[\omega(t) - kV(t)]^2 - 3k^2 v_{Te}^2(t)}. \end{aligned} \tag{8.42}$$

Differentiating once with respect to time, we obtain the following integro-differential equation for $\omega(t)$,

$$\begin{aligned} \frac{d\omega}{dt} &= k\frac{dV}{dt} + \frac{3}{2}\frac{k^2}{\omega - kV}\frac{dv_{Te}^2}{dt} \\ &+ i\frac{\omega}{2(\omega - kV)}\left(\frac{[(\omega - kV)^2 - 3k^2 v_{Te}^2(t)]^2}{\omega_{pe}^2} - (\omega - kV)^2 + 3k^2 v_{Te}^2\right) \\ &- \frac{m}{2M}\frac{[(\omega - kV)^2 - 3k^2 v_{Te}^2(t)]^2}{\omega - kV} \exp\left(i \int^t \omega(t')dt'\right) \int^t \exp\left(-i \int^{t'} \omega(t'')dt''\right) dt' \end{aligned} \tag{8.43}$$

Further differentiation eliminates the integral part and yields a second order nonlinear differential equation for $\omega = \omega_r + i\gamma$.

Equation (8.43) has been solved numerically by Ishihara et al. Major findings made in numerical analysis are as follows. Deviation form the initial linear growth occurs when the field energy reaches a small fraction of the electron kinetic energy,

$$\frac{E^2}{8\pi} \simeq \left(\frac{m}{M}\right)^{1/3} n_0 m V^2. \tag{8.44}$$

By the time the field intensity has acquired this value, coherent ion plasma oscillation at $\omega \simeq \omega_{pi}$ (ion plasma frequency) starts. Electron trapping sets in well after the breakdown of the linear growth which indicates that electron trapping is not the cause of saturation of the Buneman instability. The coherent ion plasma oscillation acquires an energy approximately given by

$$\frac{E^2}{8\pi} \simeq 0.11 \ n_0 m V^2, \tag{8.45}$$

which is insensitive to the electron/ion mass ratio. The coherent ion oscillation is eventually thermalized through phase mixing and the original electron kinetic energy is completely thermalized. Efficient ion heating revealed in the studies by Ishihara et al. is in contrast to the earlier view that ion heating in the Buneman instability is ineffective.

In practice, it is not easy to realize conditions suitable for the Buneman instability. Even if the condition $V \gg v_{Te}$ is created, it is short-lived because of the rapid growth of the instability. In experiments, what can be observed is the aftermath of the instability, namely, efficient turbulent heating of ions as well as electrons and strong ion plasma oscillation at the ion plasma frequency, $\omega = \omega_{pi}$.

8.5 Oscillating Two-Stream Instability

The oscillating two-stream instability occurs if a sufficiently large amplitude oscillating dipole (non-propagating) electric field is applied to a plasma. It is not a nonlinear instability. However, the methodology to analyze the instability is similar to that in nonlinear decay instabilities and for this reason it is included in this Chapter.

Suppose that a collisionless plasma is placed in an oscillating electric field,

$$E_0 \sin \omega_0 t. \tag{8.46}$$

If the field intensity is large enough, the amplitude of oscillating electron drift velocity can exceed the electron thermal velocity,

$$\frac{e}{m}\frac{E_0}{\omega_0} \gg v_{Te},$$

and the thermal effects of electrons may be ignored. (Laser pulses having intensities of the order of 10^{20} W/cm^2 are becoming available these days. In such strong laser field, the amplitude of oscillating electron velocity becomes relativistic. This is an emerging new area of plasma physics.) The equation of motion for the electron

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) v = -\frac{e}{m}E \tag{8.47}$$

may be linearized as

$$\frac{\partial v_e}{\partial t} + v_0(t)\frac{\partial v_e}{\partial x} = -\frac{e}{m}E_1, \tag{8.48}$$

where

$$v_0(t) = \frac{e}{m\omega_0}E_0 \cos\omega_0 t, \tag{8.49}$$

is the oscillating electron drift velocity and E_1 is the perturbed electric field. We assume the perturbed electron velocity and electric filed have spatial dependence proportional to e^{ikx}. Then Eq. (8.48) reduces to

$$\frac{\partial v_e}{\partial t} + ikv_0(t)v_e = -\frac{e}{m}E_1. \tag{8.50}$$

The continuity equation for the electrons

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0 \tag{8.51}$$

can be linearized in a similar manner,

$$\frac{\partial n_e}{\partial t} + ik\left[v_0(t)n_e + v_e n_0\right] = 0. \tag{8.52}$$

Substituting

$$v_e = -\frac{1}{ikn_0}\left(\frac{\partial}{\partial t} + ikv_0(t)\right) n_e \tag{8.53}$$

into Eq. (8.52) and introducing a scalar potential $E_1 = -ik\phi$, we obtain the following equation for the density perturbation,

$$\left(\frac{\partial}{\partial t} + ikv_0(t)\right)^2 n_e = \frac{en_0}{m}k^2\phi. \tag{8.54}$$

For the ions, the oscillating drift velocity can be ignored because of their large mass, and the perturbed ion density n_i is related to the potential through

$$\frac{\partial^2 n_i}{\partial t^2} = -\frac{en_0}{M}k^2\phi. \tag{8.55}$$

Finally, the Poisson's equation relates the potential to the electron and ion density perturbations,

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi e(n_i - n_e), \tag{8.56}$$

or

$$k^2\phi = 4\pi e(n_i - n_e). \tag{8.57}$$

Therefore, we obtain the following two simultaneous equations for n_e and n_i,

$$\left(\frac{\partial}{\partial t} + ikv_0(t)\right)^2 n_e = \omega_{pe}^2(n_i - n_e), \tag{8.58}$$

$$\frac{\partial^2 n_i}{\partial t^2} = \omega_{pi}^2(n_e - n_i). \tag{8.59}$$

To solve these equations, it is convenient to introduce a new electron density $\nu_e(t)$ through

$$n_e(t) = \nu_e(t)\exp\left(-i\int^t kv_0(t)dt'\right). \tag{8.60}$$

Then,

$$\frac{\partial n_e}{\partial t} = \left(\frac{\partial \nu_e}{\partial t} - ikv_0(t)\right)\exp\left(-i\int^t kv_0(t)dt'\right), \tag{8.61}$$

and Eqs. (8.58) and (8.59) reduce to

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{pe}^2\right)\nu_e = \omega_{pe}^2 n_i \exp\left(i\int^t kv_0(t)dt'\right), \tag{8.62}$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{pi}^2\right)n_i = \omega_{pi}^2 \nu_e \exp\left(-i\int^t kv_0(t)dt'\right). \tag{8.63}$$

Since

$$v_0(t) = \frac{eE_0}{m\omega_0}\cos\omega_0 t,$$

$$\int^t v_0(t')dt' = \frac{eE_0}{m\omega_0^2}\sin\omega_0 t, \tag{8.64}$$

and noting the expansion

$$e^{ia\sin\omega_0 t} = \sum_n J_n(a)e^{in\omega_0 t},$$

we may rewrite Eqs. (8.62) and (8.63) as

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{pe}^2\right)\nu_e = \omega_{pe}^2 n_i \sum_n J_n(a)e^{in\omega_0 t}, \tag{8.65}$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{pi}^2\right)n_i = \omega_{pi}^2 \nu_e \sum_m J_m(a)e^{-ima\omega_0 t}, \tag{8.66}$$

where

$$a = \frac{keE_0}{m\omega_0^2}, \tag{8.67}$$

is a dimensionless quantity to measure the intensity of the oscillating field. a is usually a small quantity. In laser-plasma interaction, the frequency of the oscillating field is close to the local electron plasma frequency because only under such situation, high electric field can be realized. Thus we retain only the lowest order terms, $m = n = 0, \pm 1$. Letting

$$\begin{aligned} \nu_e(t) &= \nu_1(t)e^{i\omega_0 t} + \nu_0(t) + \nu_{-1}(t)e^{-i\omega_0 t}, \\ n_i(t) &= n_i e^{i\omega t}, \nu_n(t) = \nu_n e^{i\omega t}, \end{aligned}$$

we obtain from Eq. (8.65)

$$\begin{aligned} \left[\omega_{pe}^2 - (\omega + \omega_0)^2\right]\nu_1 &= \omega_{pe}^2 J_1(a) n_i, \\ (\omega_{pe}^2 - \omega^2)\nu_0 &= \omega_{pe}^2 J_0(a) n_i, \\ \left[\omega_{pe}^2 - (\omega - \omega_0)^2\right]\nu_1 &= \omega_{pe}^2 J_{-1}(a) n_i. \end{aligned}$$

Substituting these into Eq. (8.66), we finally obtain the following dispersion relation,

$$\omega_{pi}^2 - \omega^2 = \omega_{pi}^2\omega_{pe}^2 \left(\frac{J_1^2(a)}{\omega_{pe}^2 - (\omega+\omega_0)^2} + \frac{J_0^2(a)}{\omega_{pe}^2 - \omega^2} + \frac{J_{-1}^2(a)}{\omega_{pe}^2 - (\omega-\omega_0)^2}\right). \tag{8.68}$$

Approximating the Bessel functions by

$$J_0^2(a) \simeq 1 - \frac{a^2}{2}, \quad J_{\pm 1}^2(a) \simeq \frac{a^2}{4},$$

and assuming $\omega_0 \simeq \omega_{pe} \gg \omega_{pi}$ reduces Eq. (8.68) to

$$4X^4 - \left[\left(1 - \frac{\omega_0^2}{\omega_{pe}^2}\right)^2 + \frac{3}{2}\frac{m}{M}a^2\right]X^2 - \frac{ma^2}{2M}\left(1 - \frac{\omega_0^2}{\omega_{pe}^2}\right) = 0. \tag{8.69}$$

Solutions for X^2 are

$$X^2 = \frac{1}{8}\left(\Delta^2 + \frac{3ma^2}{2M} \pm \sqrt{\left(\Delta^2 + \frac{3ma^2}{2M}\right)^2 + \frac{8ma^2}{M}\Delta}\right), \tag{8.70}$$

where

$$\Delta = 1 - \left(\frac{\omega_0}{\omega_{pe}}\right)^2. \tag{8.71}$$

When $\Delta > 0$, the plasma is over-dense ($\omega_0 < \omega_{pe}$). The instability is purely growing ($\omega_r = 0$) and the maximum growth rate occurs when $\partial X^2/\partial\Delta = 0$, or

$$2\Delta^3 - 3\varepsilon a^2\Delta - 2\varepsilon a^2 = 0,$$

where $\varepsilon = m/M$. Then

$$\Delta \simeq \left(\varepsilon a^2\right)^{1/3},$$

and the maximum growth rate is

$$\gamma_{\max} \simeq \frac{1}{2}\left(\varepsilon a^2\right)^{1/3}\omega_{pe}. \tag{8.72}$$

Note that the growth rate has the same dependence on the mass ratio as the Buneman instability, $\gamma \propto (m/M)^{1/3}$. Fig. 9.1 shows the normalized growth rate γ/ω_{pe} as a function of $\Delta = 1-(\omega_0/\omega_{pe})^2$ (>0) when $\varepsilon a^2 = 10^{-5}$. The case $\Delta < 0$ (under dense plasma) is shown in Fig. 9.2. When $\Delta < 0$, the instability becomes convective characterized by a finite frequency.

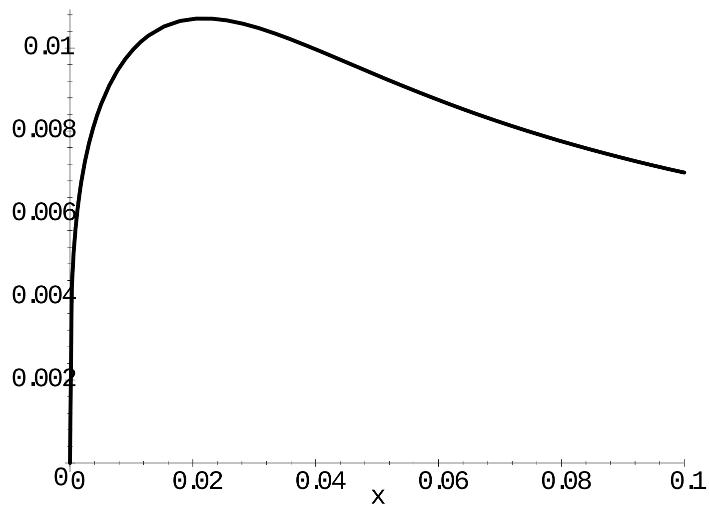

Figure 8.1: Growth rate of the oscillating two-stream instability. $x > 0$.

8.6 Decay Instability of Langmuir Mode; Langmuir Soliton

In section 9.2, it was shown that the Langmuir mode excited by a tenuous electron beam becomes saturated and exhibits oscillation at the bounce frequency of the trapped beam electrons. The oscillation eventually ceases due to thermalization of the trapped electrons and the state of constant amplitude Langmuir oscillation is established. However, if the amplitude is large enough, a nonlinear instability known as decay instability can occur. In the decay instability of Langmuir mode, the energy associated with a particular Langmuir mode with frequency ω_{L1} and wavenumber k_{L1} is

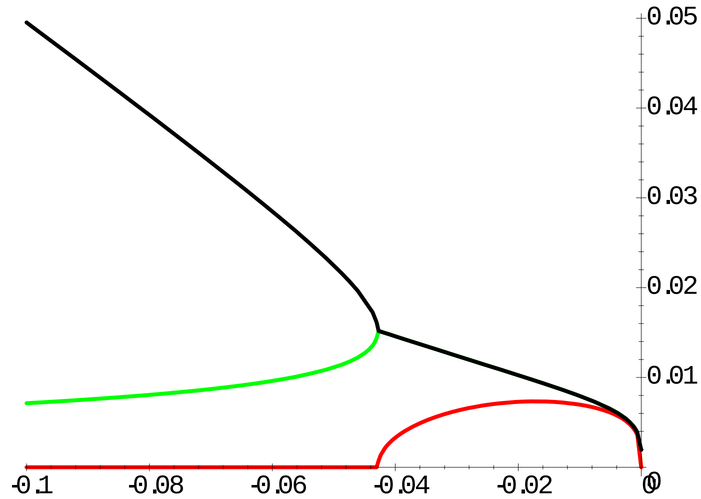

Figure 8.2: Frequency (upper curves with branches) and growth rate (semi-elliptic curve) of the oscillating two-stream instability when $x < 0$.

split into another Langmuir mode (ω_{L2}, k_{L2}) at a lower frequency and ion acoustic mode (ω_s, k_s). Among the three modes, the following energy and momentum conservation laws are to be satisfied,

$$\omega_{L1} = \omega_{L2} + \omega_s, \tag{8.73}$$

$$k_{L1} = k_{L2} + k_s. \tag{8.74}$$

It is evident that the decay instability is prohibited if the electrons are cold because if so, the Langmuir frequency is constant $(\omega = \omega_{pe})$ regardless of the wavenumber k. The dispersion relation of the Langmuir mode with a finite electron temperature is

$$\omega_L^2 = \omega_{pe}^2 + \frac{3T_e}{m}k^2, \tag{8.75}$$

while that of the ion acoustic mode in the long wavelength regime $(k \ll k_{De})$ is

$$\omega_s = \sqrt{\frac{T_e + 3T_i}{M}}k.$$

For a given Langmuir frequency ω_{L1}, it is possible to find another Langmuir mode and ion acoustic mode satisfying the decay conditions given in Eqs. (8.73) and (8.74).

Hydrodynamic equation of motion for the electrons is

$$mn\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} = -\boldsymbol{\nabla}p_e + en\boldsymbol{\nabla}\phi, \tag{8.76}$$

where p_e is the electron pressure and ϕ is a scalar potential. Nonlinearity appears in $n\mathbf{v}, n\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$ and $n\boldsymbol{\nabla}\phi$. However, nonlinearity in the first two terms can be ignored because decay instability within the Langmuir branch itself is prohibited. Then, for high frequency electron motion, Eq. (8.76) can be approximated by

$$mn_0\frac{\partial \mathbf{v}_e}{\partial t} = -\gamma_e T_e \boldsymbol{\nabla} n_e + en_e\boldsymbol{\nabla}\phi, \tag{8.77}$$

where $\gamma_e = 3$ is the electron adiabatic coefficient for high frequency phenomena. The divergence of this equation is

$$mn_0\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{v}_e = -\gamma_e T_e\nabla^2 n_e + e\boldsymbol{\nabla}\cdot(n_e\boldsymbol{\nabla}\phi), \tag{8.78}$$

while the linearized equation of continuity is given by

$$\frac{\partial n_e}{\partial t} + n_e\boldsymbol{\nabla}\cdot\mathbf{v}_e = 0. \tag{8.79}$$

The potential ϕ has three components,

$$\phi = \phi^{L0}_{k+k'} + \phi^L_k + \phi^s_{k'}, \tag{8.80}$$

where $\phi^{L0}_{k+k'}$ is the large amplitude Langmuir mode, ϕ^L_k is the Langmuir mode produced through the decay instability and $\phi^s_{k'}$ is the accompanying ion acoustic mode. The electron density has the same components,

$$n_e = n_0 + n^{L0}_{k+k'} + n^L_k + n^s_{k'}. \tag{8.81}$$

The term $\nabla\cdot(n_e\nabla\phi)$ in Eq. (8.78) then yields

$$\nabla\cdot(n_e\nabla\phi) = -n_0k^2\phi^L_k - k(k+k')n^s_{-k'}\phi^{L0}_{k+k'} + kk'n^{L0}_{k+k'}\phi^s_{-k'}. \tag{8.82}$$

For the ion acoustic mode, the density and potential are related through

$$n^s_{-k'} = \frac{e\phi^s_{-k'}}{T_e}n_0, \tag{8.83}$$

and for the Langmuir mode through

$$(k+k')^2\phi^{L0}_{k+k'} = -4\pi e n^{L0}_{k+k'}. \tag{8.84}$$

Therefore, the ratio between the second and third terms in the RHS of Eq. (8.82) is of order $(k_{De}/k)^2$ and thus for long wavelength modes we are considering, the last term can be ignored. Substituting

$$\nabla\cdot(n_e\nabla\phi) \simeq -n_0k^2\phi^L_k - k(k+k')n^s_{-k'}\phi^{L0}_{k+k'}$$

into Eq. (8.78), we obtain

$$\left(\omega_k^2 - \omega_L^2\right)\phi_k^L = \frac{e\omega_{pe}^2}{T_e}\frac{k+k'}{k}\phi_{k+k'}^{L0}\phi_{-k'}^s, \tag{8.85}$$

where

$$\omega_L^2 = \omega_{pe}^2 + \frac{3T_e}{m}k^2. \tag{8.86}$$

For low frequency ion acoustic regime, the electron nonlinearity mainly comes from the term $\mathbf{v}\cdot\nabla\mathbf{v}$ because beat between the two high frequency Langmuir modes $\phi_{k+k'}^{L0}$ and ϕ_{-k}^L excites the ion acoustic mode $\phi_{k'}^s$. Again taking the divergence of the equation of motion for the electrons, we find

$$\left(\omega_{k'}^2 - \omega_s^2\right)\phi_{k'}^s = -\frac{e\omega^2}{m\omega_{pe}^2}k(k+k')\phi_{k+k'}^{L0}\phi_{-k}^L. \tag{8.87}$$

Eqs. (8.85) and (8.87) constitute two simultaneous equations for ϕ_L and ϕ_s. Let us consider the case in which a Langmuir mode with a frequency-wavenumber combination (ω_{pe}, k_0) decays into another Langmuir mode with $(\omega_{pe}-\omega_s, -k_0)$ and ion acoustic mode with $(\omega_s, 2k_0)$. Noting $k+k' = k_0, k = -k_0$ and $k' = 2k_0$, and $\omega_k = \omega_L + i\gamma$, $\omega_{k'} = \omega_s + i\gamma$ where γ is the growth rate of the decay instability assumed to be small $\gamma \ll \omega_s \ll \omega_{pe}$, we find

$$2i\gamma\omega_{pe}\phi^L = -\omega_{pe}^2\frac{e\phi_0}{T_e}\phi^s, \tag{8.88}$$

$$2i\gamma\omega_s\phi^s = \frac{e\omega_s^2}{m\omega_{pe}^2}k_0^2\phi^L. \tag{8.89}$$

Therefore, the growth rate of the decay instability may be estimated from

$$\gamma = \frac{1}{2}\sqrt{\frac{k_0^2 T_e}{m}\frac{\omega_s}{\omega_{pe}}}\frac{e\phi_0}{T_e}. \tag{8.90}$$

Of course, for the instability to occur, the growth rate must exceed the intrinsic (linear) Landau damping. The decay instability transfers the energy of the Langmuir mode to the ion acoustic mode which is Landau damped particularly in a plasma with ion temperature comparable with the electron temperature. Steady state of Langmuir mode cannot be maintained unless energy is continuously fed from an external source.

The inverse process, i.e., upconversion of large amplitude ion acoustic mode ω_s to two Langmuir modes ω_L and $-\omega_L + \omega_s$ is stable. This is left for an exercise.

If the intensity of Langmuir mode is further increased, the analysis presented above based on weak plasma turbulence tends to break down. The intensity of Langmuir wave becomes spatially nonuniform through development of a modulational instability. Coupling to the ion acoustic mode still plays an essential role, although, unlike the decay instability, the modulational instability is spatially localized phenomenon and the concept of well defined wave propagation is to be abandoned.

The linear equation to describe Langmuir wave is

$$\frac{\partial^2 E}{\partial t^2} + \left(\omega_{pe}^2 - \frac{3T_e}{m}\frac{\partial^2}{\partial x^2}\right)E = 0. \tag{8.91}$$

If a modulational instability develops, the electron density is modulated,

$$n_e = n_0 + n_s(x,t), \tag{8.92}$$

where $n_s(x,t)$ indicates the density perturbation associated with the modulational instability. Then Eq. (8.91) is modified as

$$\frac{\partial^2 E}{\partial t^2} + \omega_{pe}^2\left(1 + \frac{n_s}{n_0}\right)E - \frac{3T_e}{m}\frac{\partial^2}{\partial x^2}E = 0. \tag{8.93}$$

Let us assume that the amplitude of the electric field $E(x,t)$ is also modulated,

$$E(x,t) = E_0(x,t)e^{i(k_0 x - i\omega_0 t)}, \tag{8.94}$$

where ω_0 is the usual Langmuir frequency,

$$\omega_0^2 = \omega_{pe}^2 + \frac{3T_e}{m}k_0^2 \simeq \omega_{pe}^2.$$

Since

$$\frac{\partial}{\partial t}E_0(x,t)e^{i(k_0 x - i\omega_0 t)} = \left(-i\omega_0 E_0 + \frac{\partial E_0}{\partial t}\right)e^{i(k_0 x - i\omega_0 t)},$$

slowly varying terms in Eq. (8.93)satisfies

$$i\omega_{pe}\frac{\partial E_0}{\partial t} + \frac{3}{2}v_{Te}^2\frac{\partial^2 E_0}{\partial x^2} = \frac{1}{2}\omega_{pe}^2\frac{n_s}{n_0}E_0. \tag{8.95}$$

The slow density modulation $n_s(x,t)$ can be found by taking time average of the electron equation of motion,

$$m\left\langle v_e\frac{\partial v_e}{\partial x}\right\rangle = -eE_s - \frac{T_e}{n_0 + n_s}\frac{\partial n_s}{\partial x}. \tag{8.96}$$

$\langle \cdots \rangle$ indicates time averaging to eliminate fast oscillation at the Langmuir frequency and $E_s(x,t)$ is the electric field associated with slowly varying perturbation. (It will become clear shortly that

E_s is ion acoustic perturbation.) Note that the adiabaticity coefficient $\gamma_e = 1$ is used for slow, nearly isothermal perturbation. Since

$$v_e \simeq \frac{ie}{m\omega_{pe}} E_0,$$

we find

$$\left\langle v_e \frac{\partial v_e}{\partial x} \right\rangle = \frac{1}{2} \left\langle \frac{\partial}{\partial x} v_e^2 \right\rangle = \frac{1}{4} \left(\frac{e}{m}\right)^2 \frac{1}{\omega_{pe}^2} \frac{\partial}{\partial x} |E_0|^2,$$

and Eq. (8.96) becomes

$$\frac{\partial}{\partial x} \left[\frac{1}{4} \left(\frac{e}{m}\right)^2 \frac{1}{\omega_{pe}^2} |E_0|^2 - \frac{e}{m} \phi_s \right] = -\frac{T_e/m}{n_0 + n_s} \frac{\partial n_s}{\partial x}, \tag{8.97}$$

where ϕ_s is the potential associated with the slow electric field,

$$E_s = -\frac{\partial \phi_s}{\partial x}.$$

Eq.(**??**) can be readily integrated with the result

$$n_s(x,t) = n_0 \exp\left[\frac{1}{T_e} \left(e\phi_s - \frac{1}{4} \frac{e^2}{m} \frac{1}{\omega_{pe}^2} |E_0|^2 \right) \right] - n_0. \tag{8.98}$$

Substitution into Eq. (8.95) yields

$$i\omega_{pe} \frac{\partial E_0}{\partial t} + \frac{3}{2} v_{Te}^2 \frac{\partial^2 E_0}{\partial x^2} = \frac{1}{2} \left[\exp\left\{ \frac{1}{T_e} \left(e\phi_s - \frac{1}{4} \frac{e^2}{m} \frac{1}{\omega_{pe}^2} |E_0|^2 \right) \right\} - 1 \right] \omega_{pe}^2 E_0. \tag{8.99}$$

This equation relates the modulated amplitude of the Langmuir wave $E_0(x,t)$ to the slowly varying (ion acoustic) potential $\phi_s(x,t)$. If the ions were infinitely massive, there is no ion acoustic perturbation, and for small amplitude Langmuir mode

$$\frac{1}{4} \frac{e^2}{mT_e} \frac{1}{\omega_{pe}^2} |E_0|^2 \ll 1,$$

Eq. (8.99) reduces to

$$i\omega_{pe} \frac{\partial E_0}{\partial t} + \frac{3}{2} v_{Te}^2 \frac{\partial^2 E_0}{\partial x^2} + \frac{e^2}{4mT_e} |E_0|^2 E_0 = 0. \tag{8.100}$$

This is known by nonlinear Schrödinger equation.

Returning to the problem of modulational instability of Langmuir wave, another relationship between E_0 and ϕ_s can be found from ion dynamics. For simplicity, we assume cold ions. The equation of motion

$$\left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x} \right) v_i = -\frac{e}{M} \frac{\partial \phi_s}{\partial x}, \tag{8.101}$$

and continuity equation

$$\left(\frac{\partial}{\partial t}+v_i\frac{\partial}{\partial x}\right)n_i+n_i\frac{\partial v_i}{\partial x}=0, \tag{8.102}$$

can be solved by the method of self-similar variable,

$$\xi = x - V_0 t \tag{8.103}$$

where V_0 is the propagation velocity of the slowly varying quantities, E_s, ϕ_s, n_i and v_i. We anticipate that V_0 should be of the order of the ion acoustic speed, $V_0 \simeq c_s = \sqrt{T_e/M}$. In terms of the self-similar variable ξ, Eqs. (8.101) and (8.102) reduce to ordinary differential equations,

$$(v_i - V_0)\frac{dv_i}{d\xi}+\frac{e}{M}\frac{d\phi_s}{d\xi} = 0, \tag{8.104}$$

$$(v_i - V_0)\frac{dn_i}{d\xi}+n_i\frac{dv_i}{d\xi} = 0. \tag{8.105}$$

Integrating over ξ, we find

$$\frac{1}{2}M(v_i - V_0)^2 + e\phi_s = \text{const.} \tag{8.106}$$

$$n_i(v_i - V_0) = \text{const.} \tag{8.107}$$

Therefore, the ion density n_i is given by

$$n_i = n_0\frac{1}{\sqrt{1-\dfrac{2e\phi_s}{MV_0^2}}}. \tag{8.108}$$

Finally, Poisson's equation can be used to relate the ion density n_i to the electron density in Eq. (8.98),

$$\begin{aligned}\frac{\partial^2\phi_s}{\partial x^2} &= -4\pi e(n_i - n_e)\\ &= -4\pi n_0 e\frac{1}{\sqrt{1-\dfrac{2e\phi_s}{MV_0^2}}}+4\pi n_0 e\exp\left[\frac{1}{T_e}\left(e\phi_s-\frac{1}{4}\frac{e^2}{m}\frac{1}{\omega_{pe}^2}|E_0|^2\right)\right].\end{aligned} \tag{8.109}$$

Eqs. (8.99) and (8.109) constitute simultaneous equations for the two functions, E_0 and ϕ_s.

In long wavelength limit $\partial/\partial x \ll k_{De}$, charge neutrality is maintained to high degree. In this case, $n_e = n_i$ yields

$$\frac{1}{\sqrt{1-\dfrac{2e\phi_s}{MV_0^2}}} = \exp\left[\frac{1}{T_e}\left(e\phi_s-\frac{1}{4}\frac{e^2}{m}\frac{1}{\omega_{pe}^2}|E_0|^2\right)\right]. \tag{8.110}$$

If the perturbations are small, both sides can be expanded and we obtain

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial x^2}\right) n_s = \frac{c_s^2}{16\pi n_0 T_e} \frac{\partial^2}{\partial x^2} |E_0|^2. \tag{8.111}$$

With the following normalizations,

$$\tau = \frac{m}{M}\omega_{pe} t,\ \tilde{n} = \frac{M}{m}\frac{n_s}{n_0},\ X = \sqrt{\frac{m}{M}} k_{De} x,\ \tilde{E} = \sqrt{\frac{M}{m}} \frac{E_0}{\sqrt{16\pi n_0 T_e}}, \tag{8.112}$$

Eqs. (8.95) and (8.111) can be cast into more compact forms,

$$\left(i\frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial X^2}\right) \tilde{E} = \tilde{n}\tilde{E}, \tag{8.113}$$

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial X^2}\right) \tilde{n} = \frac{\partial^2}{\partial X^2} |\tilde{E}|^2. \tag{8.114}$$

A particular set of solutions for these simultaneous equations has been found by Zakharov,

$$\tilde{E}(X,\tau) = \lambda\sqrt{2(1-s^2)}\operatorname{sech}[\lambda(X - s\tau)] \exp\left[i\frac{s}{2}X + i\left(\lambda^2 - \frac{s^2}{4}\right)\tau\right], \tag{8.115}$$

$$\tilde{n}(X,\tau) = -2\lambda^2 \operatorname{sech}^2[\lambda(X - s\tau)], \tag{8.116}$$

where s and λ are constants related through

$$\lambda\sqrt{2(1-s^2)} = 1. \tag{8.117}$$

Note that the density perturbation is negative while the field intensity of the Langmuir mode becomes maximum in the density hole. An intense Langmuir mode digs a density hole in which the wave energy is trapped. The parameter λ determines the depth and width of the solitary structure and s is the soliton velocity normalized by the ion acoustic speed. The Langmuir soliton is subsonic ($s < 1$). Experimental confirmation of Langmuir soliton has been made by Wong in a series of microwave-plasma interaction experiments.

The analysis presented above is, strictly speaking, one dimensional and whether Langmuir soliton still persists in multi dimensional cases should be examined by analyzing the generalized equations,

$$\left(i\frac{\partial}{\partial \tau} + \nabla^2\right) \tilde{E} = \tilde{n}\tilde{E}, \tag{8.118}$$

$$\left(\frac{\partial^2}{\partial \tau^2} - \nabla^2\right) \tilde{n} = \frac{\partial^2}{\partial X^2} |\tilde{E}|^2. \tag{8.119}$$

In two and three dimensions, stable Langmuir solitons cannot be maintained because of the onset of so-called filamentation instability in 2D and collapse in 3D.

8.7 Ion Acoustic Solitary Waves

In a nonisothermal plasma with an electron temperature far exceeding the ion temperature $T_e \gg T_i$, the ion acoustic mode exists as a weakly damped natural mode. The dispersion relation is

$$\omega^2 = \frac{k^2 c_s^2}{1 + (k/k_{De})^2}. \tag{8.120}$$

In long wavelength region $k^2 \ll k_{De}^2$, this reduces to

$$\omega \simeq k c_s \left(1 - \frac{1}{2}\left(\frac{k}{k_{De}}\right)^2\right). \tag{8.121}$$

The dispersion relation is similar to that of water waves,

$$\omega^2 = \left(gk + \frac{T_s}{\rho}k^3\right)\tanh(hk), \tag{8.122}$$

provided the water depth h is much shallower that the wavelength, $hk \ll 1$. Here, g is the gravitational acceleration, $T_s \simeq 0.07$ N/m is the surface tension of water and $\rho = 10^3$ kg/m^3 is the water mass density. In this case, the surface tension can be ignored and Eq. (8.122) may be approximated by

$$\omega \simeq k\sqrt{gh}\left(1 - \frac{1}{12}(hk)^2\right). \tag{8.123}$$

In fact, an equation to describe nonlinear shallow water wave had been derived by Korteweg and de Vries (known as KdV equation) in 1895. The same equation emerges for nonlinear ion acoustic wave because of the similarity of the dispersion relations.

The basic equation for nonlinear ion acoustic wave can be obtained from Eq. (8.109) if we ignore the Langmuir mode, $E_0 = 0$,

$$\frac{d^2\phi}{d\xi^2} = -4\pi n_0 e \frac{1}{\sqrt{1 - \dfrac{2e\phi}{MV_0^2}}} + 4\pi n_0 e \exp\left(\frac{e\phi}{T_e}\right), \tag{8.124}$$

where V_0 is the self-similar velocity having a meaning

$$V_0 = \frac{\partial/\partial t}{\partial/\partial x}.$$

Assuming a small potential ϕ, we may expand Eq. (8.124) to second order terms of ϕ,

$$\frac{d^2\phi}{d\xi^2} = k_{De}^2\left[\left(1 - \frac{c_s^2}{V_0^2}\right)\phi + \left(\frac{1}{2} - \frac{3c_s^4}{2V_0^4}\right)\frac{e\phi^2}{T_e}\right]. \tag{8.125}$$

With the following normalization

$$X = k_{De}\xi,\ \Phi = e\phi/T_e,\ \mathcal{M} = V_0/c_s \text{ (Mach number)},$$

Eq. (8.125) can be reduced to

$$\frac{d^2\Phi}{dX^2} = \left(1 - \frac{1}{\mathcal{M}^2}\right)\Phi + \left(\frac{1}{2} - \frac{3}{2\mathcal{M}^4}\right)\Phi^2 = A\Phi - B\Phi^2, \tag{8.126}$$

where

$$A = 1 - \frac{1}{\mathcal{M}^2}, \quad B = \frac{3}{2\mathcal{M}^4} - \frac{1}{2}. \tag{8.127}$$

Eq. (8.126) has an integral

$$\left(\frac{d\Phi}{dX}\right)^2 = A\Phi^2 - \frac{2}{3}B\Phi^3, \tag{8.128}$$

or

$$\int^{\Phi} \frac{d\Phi}{\Phi\sqrt{A - \frac{2}{3}B\Phi}} = \pm X. \tag{8.129}$$

Using the integral

$$\int \frac{dx}{x\sqrt{a-bx}} = \frac{1}{a}\ln\left|\frac{\sqrt{a-bx}-\sqrt{a}}{\sqrt{a-bx}+\sqrt{a}}\right|,$$

we readily find the following solution for $\Phi(X)$,

$$\Phi(X) = \frac{3A}{2B}\text{sech}^2\left(\frac{\sqrt{A}}{2}X\right), \tag{8.130}$$

or in terms of the physical coordinate x and time t,

$$\frac{e\phi(x,t)}{T_e} = \frac{\frac{3}{2}\left(1 - \frac{1}{\mathcal{M}^2}\right)}{\frac{3}{2\mathcal{M}^4} - \frac{1}{2}}\text{sech}^2\left(\frac{\sqrt{1-\frac{1}{\mathcal{M}^2}}}{2}k_{De}(x - \mathcal{M}c_s t)\right). \tag{8.131}$$

The amplitude of the soliton is

$$\frac{e\phi_{\max}}{T_e} = \frac{3A}{2B} = \frac{\frac{3}{2}\left(1 - \frac{1}{\mathcal{M}^2}\right)}{\frac{3}{2\mathcal{M}^4} - \frac{1}{2}}, \tag{8.132}$$

and the velocity of the soliton is given by

$$V_0 = \mathcal{M}c_s = \left(1 + \frac{1}{3}\frac{e\phi_{\max}}{T_e}\right)c_s. \tag{8.133}$$

The maximum Mach number allowed within the present approximation for *bright* ($\phi > 0$) solitons is

$$\mathcal{M} < 3^{1/4} \simeq 1.32. \tag{8.134}$$

For $M > 1.32$, ϕ becomes negative suggesting a *dark* soliton. Only large amplitude dark solitons can exist and it is not easy to excite dark solitons. Dark solitons have only recently been observed in liquids. However, in a plasma, there exist no dark solitons as we will see below.

The ion acoustic soliton is supersonic ($\mathcal{M} > 1$) and its velocity increases with the amplitude. The width of the soliton decreases with the amplitude,

$$\Delta x = \frac{2}{\sqrt{A}} \frac{1}{k_{De}} = \frac{2}{\sqrt{1 - \dfrac{1}{\mathcal{M}^2}}} \frac{1}{k_{De}} = \frac{\sqrt{3}}{\sqrt{e\phi_{\max}/T_e}} \frac{1}{k_{De}}. \tag{8.135}$$

Fig. 8.3 shows a numerical solution to the exact differential equation

$$\frac{d^2\phi}{d\xi^2} = -4\pi n_0 e \frac{1}{\sqrt{1 - \dfrac{2e\phi}{MV_0^2}}} + 4\pi n_0 e \exp\left(\frac{e\phi}{T_e}\right), \tag{8.136}$$

or its normalized form

$$\frac{d^2\Phi}{dX^2} = -\frac{1}{\sqrt{1 - \dfrac{2\Phi}{\mathcal{M}^2}}} + \exp(\Phi), \tag{8.137}$$

when $\mathcal{M} = 1 + 0.3/3 = 1.1$. According to the KdV solution, the soliton amplitude is 0.3 while the amplitude numerically found is approximately 0.28. The discrepancy becomes worse as the Mach number increases. Fig. (8.4) shows the dependence of soliton amplitude on the Mach number. The KdV equation, Eq. (8.126), predicts bright soliton in the regime $1 < \mathcal{M} < 4^{1/4} = 1.32$ and dark soliton in the region $\mathcal{M} > 1.32$. The exact equation, Eq. (8.137), allows only bright solitons in the region $1 < \mathcal{M} < 1.6$. There exists no dark ion acoustic solitons in a plasma.

If the ion temperature T_i is finite, Eq. (8.137) is modified as

$$\frac{d^2\Phi}{dX^2} = -\frac{\sqrt{2}}{\sqrt{1 - \dfrac{2\Phi - 3\tau}{\mathcal{M}^2} + \sqrt{\left(1 - \dfrac{2\Phi - 3\tau}{\mathcal{M}^2}\right)^2 - \dfrac{12\tau}{\mathcal{M}^2}}}} + \exp(\Phi), \tag{8.138}$$

where $\tau = T_i/T_e$ and γ_i (the ratio of specific heats for ions) = 3 is assumed. The equation can be solved only numerically. The range of $\Phi_{\max}$ and Mach number $\mathcal{M}$ sensitively depends on τ. For example, when $\tau = T_i/T_e = 0.1$, soliton solution exists in the range $0 < \Phi_{\max} < 0.27$ and corresponding range in $\mathcal{M}$ is $1.14 < \mathcal{M} < 1.27$.

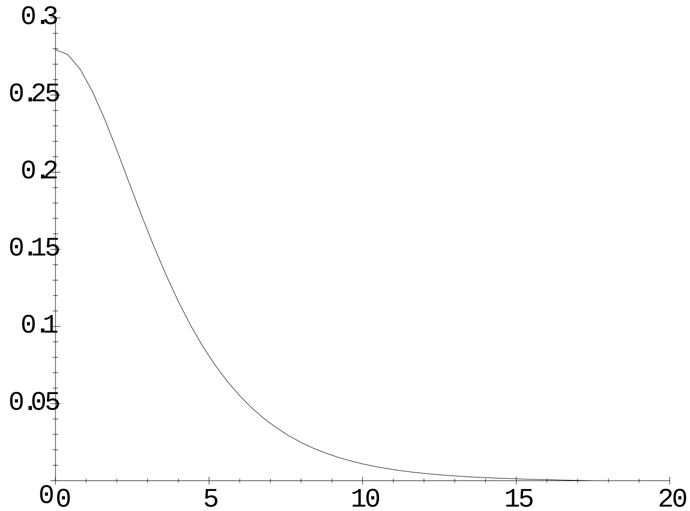

Figure 8.3: Ion acoustic soliton when $\mathcal{M} = 1.1$.

8.8 Diffusion of Particles in Random Electric Fields

The quasilinear equation used in Section 9.2

$$\frac{\partial f}{\partial t} = \left(\frac{e}{m}\right)^2 \sum_k |E_k|^2 \,\mathrm{Re}\, \frac{\partial}{\partial v}\left(\frac{i}{\omega_k - kv}\frac{\partial f}{\partial v}\right),$$

is formally a diffusion equation and thus describes how particles diffuse in the velocity space. Even if the electric field is stationary, the function

$$\frac{1}{\omega_k - kv} = \mathrm{P}\frac{1}{\omega_k - kv} - i\pi\delta(\omega_k - kv),$$

has an imaginary part due to resonant particles satisfying $kv = \omega_k$, and a well defined quasilinear diffusion coefficient

$$D_{QL} = \pi \left(\frac{e}{m}\right)^2 \sum_k |E_k|^2 \,\delta(\omega_k - kv),$$

emerges. However, the validity of such formulation should be limited to weak turbulence because in the quasilinear theory, the particle trajectory is assumed to be unaffected by the electric field. As the turbulence level increases, the motion of resonant particles is expected to deviate from the simple drift motion which will in turn modify the resonance condition.

Dupree was the first to generalize the quasilinear theory for strong turbulence and let us first review his approach. By definition, the diffusion coefficient D in the velocity space is given by the

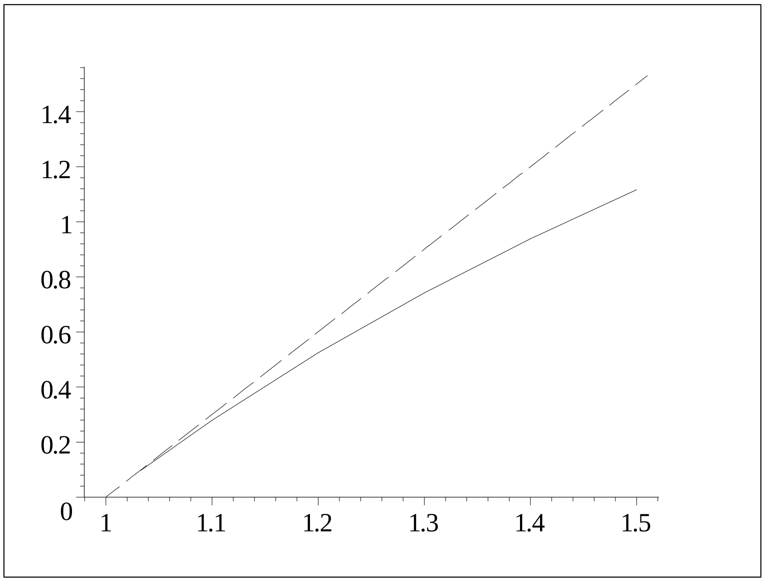

Figure 8.4: Soliton amplitude vs. Mach number. Solid line: exact solution. Dashed line: solution based on KdV equation. Note that KdV equation has an upper limit of Mach number, $\mathcal{M} \leq 3^{1/4} = 1.32$ while in Eq. (8.137), there is no such limitation.

time derivative of the velocity variance,

$$D = \frac{1}{2}\frac{d}{dt}\left\langle [\Delta v(t)]^2 \right\rangle,$$

where $[\cdots]$ indicates ensemble average and $\Delta v(t)$ is deviation of particle velocity from simple drift motion,

$$\Delta v(t) = \frac{e}{m}\int^t E[x(t'), t']dt'.$$

The electric field in this equation is the field experienced by the particle. The particle trajectory also deviates from that in simple drift, and is given by

$$x(t) = x + vt + \Delta x(t),$$

where x is the initial coordinate and $\Delta x(t)$ is

$$\begin{aligned}\Delta x(t) &= \int^t \Delta v(t')dt' \\ &= \frac{e}{m}\int^t dt' \int^{t'} dt'' E[x(t''), t''] \\ &= \frac{e}{m}\int^t sE[x(t-s), t-s]ds.\end{aligned}$$

The electric field may be decomposed into Fourier components,

$$E[x(t), t] = \sum_k E_k e^{i(kx(t)-\omega_k t)}.$$

Then, the velocity variance becomes

$$\left\langle [\Delta v(t)]^2 \right\rangle = \left(\frac{e}{m}\right)^2 \sum_k |E_k|^2 \int^t ds \int^s ds' e^{i(kv-\omega_k)s'} \left\langle e^{ik[\Delta x(s)-\Delta x(s-s')]} \right\rangle.$$

If $\Delta x = 0$, the double time integral reduces to

$$2\pi\delta(\omega_k - kv)t,$$

and we recover the quasilinear diffusivity,

$$D_{QL} = \pi \left(\frac{e}{m}\right)^2 \sum_k |E_k|^2 \, \delta(\omega_k - kv).$$

The ensemble average of the exponential function may be written as

$$\left\langle e^{ik[\Delta x(s)-\Delta x(s-s')]} \right\rangle = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n!} (ik)^n \left\langle [\Delta x(s, s')]^n \right\rangle\right)$$

where

$$\Delta x(s, s') \equiv \Delta x(s) - \Delta x(s - s').$$

$\langle \Delta x(s, s') \rangle$ is not essential for diffusion because it only produces a turbulence induced drift. Cumulants of orders higher than $n = 2$ may be ignored if statistics of the turbulent electric field is Gaussian. Thus, in the lowest order,

$$\left\langle e^{ik[\Delta x(s)-\Delta x(s-s')]} \right\rangle \simeq \exp\left(-\frac{k^2}{2} \left\langle [\Delta x(s, s')]^2 \right\rangle\right).$$

It is customary to assume that the turbulence is Markovian, that is, the cumulant

$$\left\langle [\Delta x(s) - \Delta x(s - s')]^2 \right\rangle, \tag{8.139}$$

depends on the relative time s' only. In this case, the following equation for the velocity diffusivity emerges,

$$\begin{aligned} D &= \frac{1}{2}\frac{d}{dt} \left\langle [\Delta v(t)]^2 \right\rangle \\ &= \frac{1}{2}\left(\frac{e}{m}\right)^2 \sum_k |E_k|^2 \int_0^{\infty} \exp\left[i(kv - \omega)t - \frac{1}{3}k^2 D t^3\right] dt. \end{aligned}$$

For resonant particles with $kv \simeq \omega$, the diffusivity is approximately given by

$$D \simeq \left(\frac{\Gamma(\frac{1}{3})}{2 \times 3^{2/3}} \left(\frac{e}{m} \right)^2 \right)^{3/4} \left(\sum_k \frac{|E_k|^2}{k^{2/3}} \right)^{3/4},$$

which is proportional to $|E_k|^{3/2}$ in contrast to the quasilinear diffusivity, $D_{QL} \propto |E_k|^2$. Ishihara et al. have removed the Markovian ansatz and found that the diffusivity becomes time-dependent and the velocity variance is subdiffusive,

$$\left\langle [\Delta v(t)]^2 \right\rangle \propto t^{2/3}. \tag{8.140}$$