Chapter 7

Electrostatic Waves and Instabilities

7.1 Introduction

In Chapter 6, the dispersion relation for general waves, both electromagnetic and electorstatic, in a uniform plasma has been formulated. The reader may have noticed that no explicit use has been made of the Gauss' law for longitudinal electric field,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{7.1}$$

where ρ is the charge density. The electric field in the Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{7.2}$$

is evidently transverse, since the longitudinal component satisfies, by definition,

$$\nabla \times \mathbf{E}_L = 0. \tag{7.3}$$

In the expansion of

$$\nabla \times \nabla \times \mathbf{E} = \nabla\nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E}, \tag{7.4}$$

the longitudinal component indeed vanishes identically. Therefore, longitudinal waves, or often called electrostatic waves, should be treated separately using the Gauss' law in Eq. (7.1).

Since

$$\nabla\cdot(\nabla\times\mathbf{B}) = \frac{1}{c}\nabla\cdot\mathbf{D} = \frac{1}{c}\nabla\cdot\left(\overleftrightarrow{\boldsymbol{\varepsilon}}\cdot\mathbf{E}\right) = 0, \tag{7.5}$$

the dispersion relation of electrostatic waves characterized by $\mathbf{E} = -\nabla\phi = -i\mathbf{k}\phi$, where ϕ is the scalar potential, can readily be found as

$$\mathbf{k}\cdot\overleftrightarrow{\boldsymbol{\epsilon}}\cdot\mathbf{k} = \sum_{ij} k_i k_j \varepsilon_{ij} = 0, \tag{7.6}$$

where ε_{ij} is the component of the dielectric tensor calculated in Chapter 6. In this Chapter, electrostatic waves and instabilities in a uniform plasma will be discussed. Low frequency drift type modes in a nonuniform plasma have already been analyzed in Chapter 3.

7.2 Dispersion Relation

For electrostatic modes, it is not necessary to use the whole Maxwell's equations, for the magnetic perturbation is assumed to be negligible. The electric field associated with electrostatic modes is curl free and description in terms of the scalar potential

$$\mathbf{E} = -\nabla\phi, \tag{7.7}$$

will suffice. The linearized Vlasov equation with ignorable magnetic perturbation is

$$\frac{df}{dt} - \frac{e}{m}\nabla\phi\cdot\frac{\partial f_0}{\partial\mathbf{v}} = 0, \tag{7.8}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e}{m}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial}{\partial\mathbf{v}}. \tag{7.9}$$

We assume a uniform plasma confined by a uniform magnetic field in the $z-$direction. Then, Eq. (7.8) can be readily integrated as shown in Chapter 6,

$$f = -\frac{e}{m}\sum_{n=-\infty}^{\infty}\frac{J_n^2(\Lambda)}{\omega - k_\parallel v_\parallel - n\Omega}\left(\frac{n\Omega}{v_\perp}\frac{\partial f_0}{\partial v_\perp} + k_\parallel\frac{\partial f_0}{\partial v_\parallel}\right)\phi. \tag{7.10}$$

Substituting this into the Poisson's equation

$$\nabla^2\phi = -4\pi e \int (f_i - f_e) d\mathbf{v}, \tag{7.11}$$

we obtain the following dispersion relation,

$$k^2 + \sum_s \omega_{ps}^2 \sum_n \int \frac{J_n^2(\Lambda_s)}{\omega - k_\parallel v_\parallel - n\Omega_s} \left(\frac{n\Omega}{v_\perp} \frac{\partial f_0}{\partial v_\perp} + k_\parallel \frac{\partial f_0}{\partial v_\parallel} \right) d\mathbf{v} = 0. \tag{7.12}$$

For isotropic Maxwellian distribution, $f_0 = f_M(v^2)$, the perturbed distribution function reduces to

$$f = -\frac{e\phi}{T} f_M + \sum_{n=-\infty}^{\infty} \frac{\omega}{\omega - k_\parallel v_\parallel - n\Omega} J_n^2(\Lambda) \frac{e\phi}{T} f_M, \tag{7.13}$$

where use has been made of the following identity,

$$\sum_{n=-\infty}^{\infty} J_n^2(\Lambda) = 1.$$

In this case, the dispersion relation becomes

$$k^2 + \sum_{s=e,i} k_{Ds}^2 \left[1 + \zeta_{s0} \sum_n \Gamma_n(b_s) Z(\zeta_{sn}) \right] = 0, \tag{7.14}$$

where

$$k_{Ds}^2 = \frac{4\pi n_0 e^2}{T_s},\ \Gamma_n(b) = e^{-b} I_n(b),\ \zeta_{sn} = \frac{\omega - n\Omega_s}{k_\parallel v_{Ts}}, \tag{7.15}$$

and $Z(\zeta)$ is the plasma dispersion function.

It is noted that the dispersion relation can alternatively be found from

$$\mathbf{k} \cdot \overleftrightarrow{\varepsilon} \cdot \mathbf{k} = k_i k_j \epsilon_{ij} = 0, \tag{7.16}$$

as explained in the Introduction. For the geometry assumed in Fig. 7.1, $\mathbf{k} = k_\perp \mathbf{e}_x + k_\parallel \mathbf{e}_z$. Therefore, $\mathbf{k} \cdot \overleftrightarrow{\varepsilon} \cdot \mathbf{k} = k_i k_j \epsilon_{ij} = 0$ reduces to

$$k_\perp^2 \epsilon_{xx} + k_\perp k_\parallel (\epsilon_{xz} + \epsilon_{zx}) + k_\parallel^2 \epsilon_{zz} = 0, \tag{7.17}$$

where

$$\epsilon_{xx} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_n \int \frac{v_\perp (n/\Lambda_s)^2 J_n^2(\Lambda_s)}{\omega - k_\parallel v_\parallel - n\Omega} U d\mathbf{v}, \tag{7.18}$$

$$\epsilon_{xz} = \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_n \int \frac{v_\perp (n/\Lambda_s) J_n^2(\Lambda_s)}{\omega - k_\parallel v_\parallel - n\Omega} W d\mathbf{v}, \tag{7.19}$$

$$\epsilon_{zx} = \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_n \int \frac{v_\parallel (n/\Lambda_s) J_n^2(\Lambda_s)}{\omega - k_\parallel v_\parallel - n\Omega} U d\mathbf{v}, \tag{7.20}$$

$$\epsilon_{zz} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_n \int \frac{v_\parallel J_n^2(\Lambda_s)}{\omega - k_\parallel v_\parallel - n\Omega} W d\mathbf{v}, \tag{7.21}$$

with

$$U = (\omega - k_\parallel v_\parallel) \frac{\partial f_{0s}}{\partial v_\perp} + v_\perp k_\parallel \frac{\partial f_{0s}}{\partial v_\parallel}, \tag{7.22}$$

$$W = (\omega - n\Omega_s) \frac{\partial f_{0s}}{\partial v_\parallel} + \frac{n\Omega_s v_\parallel}{v_\perp} \frac{\partial f_{0s}}{\partial v_\perp}. \tag{7.23}$$

We see that Eq. (7.17) is indeed identical to Eq. (7.12).

For electrostatic modes in an unmagnetized plasma ($\mathbf{B}_0 = 0$), the original Vlasov equation is

$$\frac{df}{dt} - \frac{e}{m} \nabla\phi \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{7.24}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla. \tag{7.25}$$

If f_0 is isotropic Maxwellian, the dispersion relation reduces to

$$k^2 + \sum_{s=e,i} k_{Ds}^2 \left[1 + \zeta_s Z(\zeta_s)\right] = 0, \tag{7.26}$$

where

$$\zeta_s = \frac{\omega}{k v_{Ts}}. \tag{7.27}$$

7.3 Electron Plasma Mode

The electron plasma mode (or often called Langmuir mode) can easily be excited by a tenuous electron beam through beam-plasma interaction. The mode has a frequency close to the electron plasma frequency $\omega \simeq \omega_{pe}$. If the electron distribution function is Maxwellian without any beam components, the mode is Landau damped by electrons through resonant wave-particle interaction. The dispersion relation can be found from Eq. (7.26) by ignoring the ion term,

$$2k^2 = k_{De}^2 Z'(\zeta_e), \tag{7.28}$$

where $Z'(\zeta)$ is the derivative of the plasma dispersion function which satisfies

$$Z' + 2[1 + \zeta Z(\zeta)] = 0. \tag{7.29}$$

Assuming $\zeta_e \gg 1$ and using the asymptotic form of Z',

$$Z'(\zeta) \simeq \frac{1}{\zeta^2} + \frac{3}{2\zeta^4} + \cdots - 2i\sqrt{\pi}\zeta e^{-\zeta^2}, \ \zeta \gg 1, \tag{7.30}$$

we find the mode frequency and damping rate which are valid in long wavelength regime, $k^2 \ll k_{De}^2$,

$$\omega_r^2 \simeq \omega_{pe}^2 + \frac{3}{2}(kv_{Te})^2 = \omega_{pe}^2 + 3k^2\frac{T_e}{m}, \tag{7.31}$$

$$\frac{\gamma}{\omega_r} \simeq -\sqrt{\pi}\left(\frac{\omega_{pe}}{|k|v_{Te}}\right)^3 \exp\left[-\left(\frac{\omega_{pe}}{kv_{Te}}\right)^2\right]. \tag{7.32}$$

This collisionless damping of the electron plasma mode was first predicted by Landau (1945). Experimental confirmation of Landau damping was made in the early 1960's.

Intuitively, the physical mechanism of Landau damping can be understood from the unbalance in the energy exchange between electrons and electric field. For this purpose, we rewrite the dispersion relation in the form

$$k^2 = \omega_{pe}^2 \int \frac{k\dfrac{df_M}{dv}}{kv - \omega} dv = \omega_{pe}^2 k \left(\mathrm{P}\int \frac{\dfrac{df_M}{dv}}{kv-\omega} dv + i\pi \int \frac{df_M}{dv}\delta(kv - \omega) dv \right), \tag{7.33}$$

where P indicates the principal part of the integral,

$$\mathrm{P}\int_{-\infty}^{\infty} dv = \int_{-\infty}^{\omega/k-0} dv + \int_{\omega/k+0}^{\infty} dv.$$

The principal part in the limit $\omega/k \gg v$ reduces to

$$\mathrm{P}\int \frac{\frac{df_M}{dv}}{kv-\omega}dv \simeq \frac{k}{\omega^2}.$$

Therefore, the dispersion relation is

$$\omega^2 \simeq \omega_{pe}^2 + i\pi\frac{\omega_{pe}^4}{k^2}\left.\frac{df_M}{dv}\right|_{v=\omega/k}.$$

Assuming $\omega = \omega_{pe} + i\gamma$, $|\gamma| \ll \omega_{pe}$, we obtain

$$\gamma = \frac{\pi}{2}\frac{\omega_{pe}^3}{k^2}\left.\frac{df_M}{dv}\right|_{v=\omega/k} < 0. \tag{7.34}$$

The damping factor γ is proportional to the derivative of the distribution function df_M/dv at the resonance, $v = \omega/k$. Electrons having a velocity close the phase velocity $v \simeq \omega/k$ strongly interact (resonate) with the electric field because they essentially experience a stationary (dc) field. Electrons having a velocity smaller than the phase velocity are continuously accelerated (they are pushed by the wave) while those with larger velocity are decelerated (they push the wave). If the electron distribution is Maxwellian, there are more electrons travelling slower than the wave as illustrated in Fig. 8.1 and the net result is the electrons as a whole gain energy from the wave. Thus the wave loses energy to the electrons and its amplitude decreases with time. The story remains unchanged even if the direction of wave propagation is reversed, $k < 0$. In this case, the resonance occurs for electrons travelling with a negative velocity close to $v = \omega/k < 0$. However, the sign of function

$$k\frac{df_M}{dv} = -\frac{m}{T_e}kvf_M < 0,$$

in the numerator remains unchanged. What matters in the argument of energy unbalance is the magnitude of the electron velocity and that of the wave phase velocity. Physically, a Maxwellian distribution has no free energy to excite waves and thus no instabilities (growing modes) are expected. Any waves in a *uniform* plasma with Maxwellian velocity distributions of ions and electrons are damped regardless of the direction of wave propagation. In a *nonuniform* plasma, the pressure gradient provides free energy for low frequency modes even if particle distribution is Maxwellian as we saw in Chapter 3.

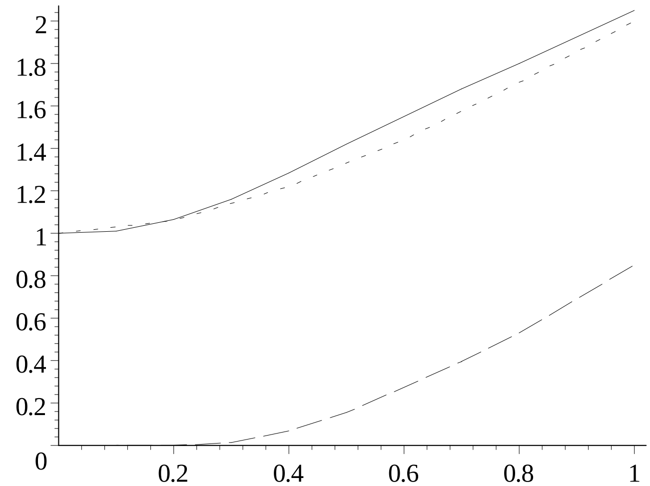

Figure 7-1: Dispersion relation of the electron plasma mode. Solid line shows ω_r/ω_{pe} and dashed line $-\gamma/\omega_{pe}$ as functions of k/k_{De}. $\gamma\,(<0)$ is the Landau damping rate. The dotted line shows the frequency based on the approximation in 7.31.

A more detailed explanation for physical mechanism of Landau damping was given by Dawson who actually evaluated the amount of energy gained (or lost) by the resonant electrons. Let us consider an electron beam placed in a propagating electric field described by

$$E(x,t) = E_0 e^{\gamma t}\sin(kx-\omega t),$$

where γ is the yet unknown damping rate which we seek. The beam density is denoted by N and beam velocity by V. A collection of many such beams constitutes a velocity distribution function of electrons. From the linearized equation of motion,

$$\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)v = -\frac{e}{m}E_0 e^{\gamma t}\sin(kx-\omega t),$$

with the initial condition $v(t=0)=0$,

$$\begin{aligned} v(x,t) &= \frac{\frac{e}{m}E_0}{(kV-\omega)^2+\gamma^2}\left\{(kV-\omega)\left[e^{\gamma t}\cos(kx-\omega t)-\cos[k(x-Vt)]\right]\right. \\ &\quad \left. -\,\gamma\left[e^{\gamma t}\sin(kx-\omega t)-\sin[k(x-Vt)]\right]\right\}. \end{aligned} \tag{7.35}$$

The density perturbation can then be found from the linearized continuity equation,

$$\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)n=-N(V)\frac{\partial v}{\partial x}. \tag{7.36}$$

If $n=0$ at $t=0$, this yields

$$\begin{aligned} n(x,t) &= N(V)\frac{eE_0}{m}\left\{\frac{2k\gamma(kV-\omega)}{[(kV-\omega)^2+\gamma^2]^2}\left[e^{\gamma t}\sin(kx-\omega t)-\sin[k(x-Vt)]\right]\right. \\ &\quad +\frac{k[\gamma^2-(kV-\omega)^2]}{[(kV-\omega)^2+\gamma^2]^2}\left[e^{\gamma t}\cos(kx-\omega t)-\cos[k(x-Vt)]\right] \\ &\quad \left.-\frac{kt}{(kV-\omega)^2+\gamma^2}\left[\gamma\cos[k(x-Vt)]+(kV-\omega)\sin[k(x-Vt)]\right]\right\}. \end{aligned} \tag{7.37}$$

The kinetic energy density associated with the beam is by definition

$$U_b=\frac{1}{2}m(N+n)(V+v)^2\simeq\frac{1}{2}mNV^2+\frac{1}{2}mNv^2+mVvn. \tag{7.38}$$

Therefore the change in the beam energy due to the electric field is

$$\frac{1}{2}mNv^2+mVvn. \tag{7.39}$$

Substituting v and n, and taking the spatial average, we find the average energy density,

$$\overline{U_b}=\frac{e^2}{2m}E_0^2N(V)\left\{\frac{A(V,t)}{(kV-\omega)^2+\gamma^2}-\frac{2kV(kV-\omega)A(V,t)}{[(kV-\omega)^2+\gamma^2]^2}+\frac{kVt}{(kV-\omega)^2+\gamma^2}e^{\gamma t}\sin[(kV-\omega)t]\right\}, \tag{7.40}$$

where the function $A(V,t)$ is defined by

$$A(V,t)=\frac{e^{\gamma t}+1}{2}-e^{\gamma t}\cos[(kV-\omega)t].$$

Integrating Eq. (7.40) over the velocity V yields the total energy,

$$\begin{aligned} W &= \int\overline{U_b}dV \\ &= -\frac{e^2}{4m}E_0^2\int\frac{V\dfrac{dN}{dV}}{(kV-\omega)^2+\gamma^2}\left[(e^{\tau t}-1)^2+4e^{\tau t}\sin^2\left(\frac{(kV-\omega)t}{2}\right)\right]dV, \end{aligned} \tag{7.41}$$

where it is noted that

$$\frac{d}{dV}\frac{1}{(kV-\omega)^2+\gamma^2} = -2\frac{k(kV-\omega)}{[(kV-\omega)^2+\gamma^2]^2},$$

and use has been made of integration by parts. The function

$$\frac{1}{(kV-\omega)^2+\gamma^2}\left[(e^{\tau t}-1)^2+4e^{\tau t}\sin^2\left(\frac{(kV-\omega)t}{2}\right)\right], \tag{7.42}$$

is sharply peaked at the resonance $V=\omega/k$. Therefore, the integration limits can be extended from $-\infty$ to ∞ and the relatively slowly varying function $V(dN/dV)$ can be taken out of the integrand. The result is

$$W \simeq \frac{e^2}{4m}E_0^2 \left.\frac{dN}{dV}\right|_{V=\omega/k}\frac{\omega}{k}\frac{\omega}{\gamma k}(e^{2\gamma t}-1). \tag{7.43}$$

Finally, the damping factor $-\gamma$ can be found from energy conservation principle,

$$\frac{d}{dt}(W+W_w)=0, \tag{7.44}$$

where

$$W_w = \frac{E_0^2 e^{2\gamma t}}{16\pi}\left.\omega\frac{\partial}{\partial\omega}\left(1-\frac{\omega_{pe}^2}{\omega^2}\right)\right|_{\omega=\omega_{pe}} = \frac{E_0^2 e^{2\gamma t}}{8\pi}, \tag{7.45}$$

is the wave energy density averaged over one wavelength. Eq. (7.44) yields

$$\gamma = \frac{\pi}{2}\frac{4\pi e^2}{m}\frac{\omega_{pe}}{k^2}\left.\frac{dN}{dV}\right|_{V=\omega/k}, \tag{7.46}$$

which agrees with Eq. (7.34).

7.4 Excitation of the Langmuir Mode by a Tenuous Electron Beam

The electron plasma mode can be easily excited by an electron beam through beam-plasma interaction. Let us consider a tenuous electron beam with density n_b and velocity V injected into a plasma. The relevant dispersion relation is

$$2k^2 = k_{De}^2 Z'(\zeta_e) + k_b^2 Z'(\zeta_b), \tag{7.47}$$

where

$$k_b^2 = \frac{4\pi n_b e^2}{T_b}, \quad \zeta_b = \frac{\omega - kV}{v_{Tb}}, \quad v_{Tb} = \sqrt{2T_b/m}.$$

If the thermal spread of the beam is small and the phase velocity ω/k is much larger than the electron thermal speed, the arguments ζ_e and ζ_b are large and the following dispersion relation emerges,

$$1 = \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_b^2}{(\omega - kV)^2}, \tag{7.48}$$

where $\omega_b^2 = 4\pi n_b e^2/m$ is the plasma frequency of the beam electrons. We assume that the beam electron density n_b is much smaller than the background electron density n_0, $n_b \ll n_0$. Then, the beam term in Eq. (7.48) can be handled perturbatively. Assuming $\omega = \omega_{pe} + \delta \simeq kV$, $|\delta| \ll \omega_{pe}$, we obtain

$$\delta^3 = \frac{1}{2}\omega_{pe}\omega_b^2. \tag{7.49}$$

This has an unstable solution,

$$\delta = \frac{-1 + i\sqrt{3}}{2}\omega_0, \text{ at } kV \simeq \omega_{pe} \tag{7.50}$$

where

$$\omega_0 = \left(\frac{\omega_{pe}\omega_b^2}{2}\right)^{1/3} = \left(\frac{n_b}{2n_0}\right)^{1/3} \omega_{pe}. \tag{7.51}$$

The strongest interaction between the beam and background electrons occurs at the resonance $kV = \omega_{pe}$ at which the beam electrons essentially experience a retarding dc electric field because of the Doppler shift. The beam electrons give away energy to the electron plasma wave which in turn grows exponentially in time with the growth rate

$$\gamma_{\max} = \frac{\sqrt{3}}{2}\omega_0 = \frac{\sqrt{3}}{2}\left(\frac{n_b}{2n_0}\right)^{1/3} \omega_{pe}, \quad kV \simeq \omega_{pe}. \tag{7.52}$$

The growth rate can be a large fraction of the electron plasma frequency even for a low density beam. The growth rate at arbitrary value of k is qualitatively shown in Fig. 7.2 for the case $n_b/n_0 = 0.01$. The half width of the growth rate profile at the resonance is approximately

$$\Delta k \simeq \left(\frac{n_b}{n_0}\right)^{1/3} \frac{\omega_{pe}}{V}.$$

Outside this range, the growth rate is much smaller than the maximum growth rate.

The instability is a typical example of nonresonant (or hydrodynamic) excitation through the interaction between negative energy wave carried by the beam and positive energy wave (electron plasma mode). In the absence of background electrons, the dielectric constant associated with the electron beam is

$$\epsilon_b = 1 - \frac{\omega_b^2}{(\omega - kV)^2}. \tag{7.53}$$

The wave energy density in a dispersive medium is

$$U_b = \frac{E^2}{8\pi}\frac{\partial}{\partial\omega}(\omega\epsilon_b) = \frac{E^2}{8\pi}\frac{2\omega\omega_b^2}{(\omega - kV)^3}. \tag{7.54}$$

This can be negative if $\omega < kV$, that is, if the beam velocity is faster than the wave phase velocity. This is the familiar Cerenkov condition for emission of electromagnetic waves in a dielectric medium.

The energy associated with the electron plasma mode, $\epsilon_e = 1 - (\omega_{pe}/\omega)^2$, is positive definite,

$$U_e = \frac{E^2}{8\pi}\frac{\partial}{\partial\omega}(\omega\epsilon_e) = \frac{E^2}{8\pi} \times (1 + 1), \tag{7.55}$$

where one part resides in the electric field and another one part in the electron kinetic energy.

As the wave amplitude grows, the beam loses its kinetic energy to the electron plasma mode. One may prematurely conclude that the nonlinear saturation of the instability should occur when the original beam energy is totally depleted,

$$\frac{E_{\text{sat}}^2}{4\pi} \simeq \frac{1}{2}n_b m V^2. \tag{7.56}$$

However, the wave growth ceases well before the depletion because the resonance condition $kV = \omega_{pe}$ is violated for a small reduction in the beam velocity, $\Delta V \simeq -(n_b/n_0)^{1/3}V$. Therefore, the wave energy density at saturation of the beam instability is approximately given by

$$\frac{E_{\text{sat}}^2}{8\pi} \simeq \left(\frac{n_b}{n_0}\right)^{1/3} n_b m V^2. \tag{7.57}$$

A more accurate analysis based on beam electron trapping condition yields the same result.

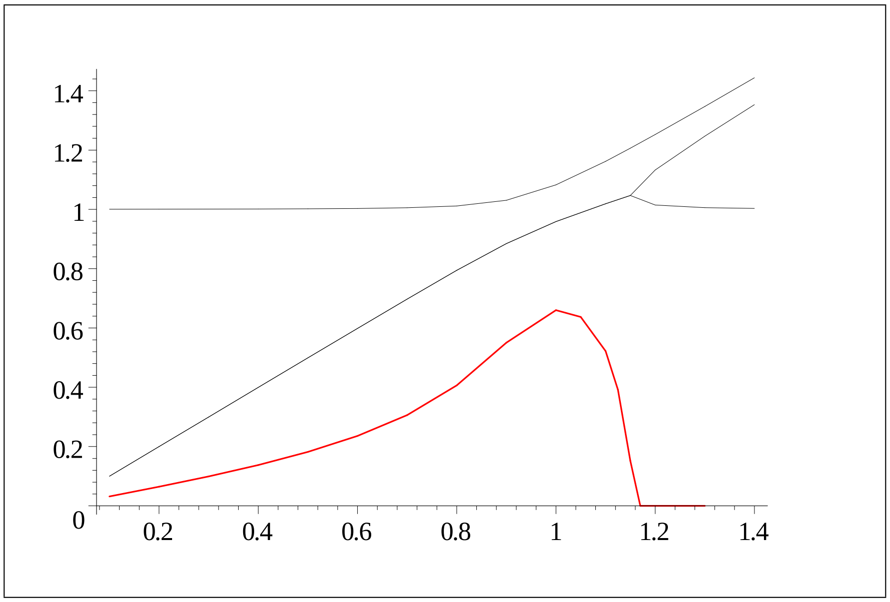

Figure 7-2: Solution of the disperion relation in Eq. (7.48) when $n_b/n_0 = 0.001$. $y = (\omega_r + i\gamma)/\omega_{pe}$ vs. $x = kV/\omega_{pe}$. The growth rate $10\gamma/\omega_{pe}$ is shown in red. Black lines indicate ω_r/ω_{pe}.

7.5 Ion Acoustic Mode

The ion acoustic mode is a low frequency electrostatic mode in which both electrons and ions participate. In a plasma with dominant electron temperature, $T_e \gg T_i$, ion Landau damping is small and the ion acoustic mode can be excited by a modest electron current. In nonlinear dynamics of electron plasma mode, the ion acoustic mode plays an important role as well, as briefly touched upon in the preceding Section.

The dispersion relation of the ion acoustic mode is given by

$$2k^2 = k_{De}^2 Z'(\zeta_e) + k_{Di}^2 Z'(\zeta_i), \tag{7.58}$$

where

$$\zeta_e = \frac{\omega - kV_e}{kv_{Te}},\ \zeta_i = \frac{\omega}{kv_{Ti}}, \tag{7.59}$$

with V_e the average electron drift velocity relative to the ions. If $T_e > T_i$, and $|\zeta_e| < 1$, $\zeta_i > 1$, the following approximate solution emerges,

$$\omega_r \simeq \left(\frac{\omega_{pi}^2}{1 + (k_{De}/k)^2} + \frac{3T_i}{M} k^2 \right)^{1/2}, \tag{7.60}$$

$$\frac{\gamma}{\omega_r} = \sqrt{\pi} \left(\frac{\omega_r}{k v_{Ti}} \right)^2 \left(\frac{T_i}{T_e} \frac{kV_e - \omega_r}{k v_{Te}} - \frac{\omega_r}{k v_{Ti}} e^{-(\omega_r/k v_{Ti})^2} \right). \tag{7.61}$$

The necessary condition for instability is $kV_e > \omega_r$ (Cerenkov condition). The second exponential term is due to ion Landau damping which rapidly increases with the ion temperature. When $T_i \simeq T_e$, the critical velocity for the instability found from the dispersion relation, Eq. (7.58), is of the order of the electron thermal speed as depicted in Fig. 7.3 for the mode $(k/k_{De})^2 = 0.1$.

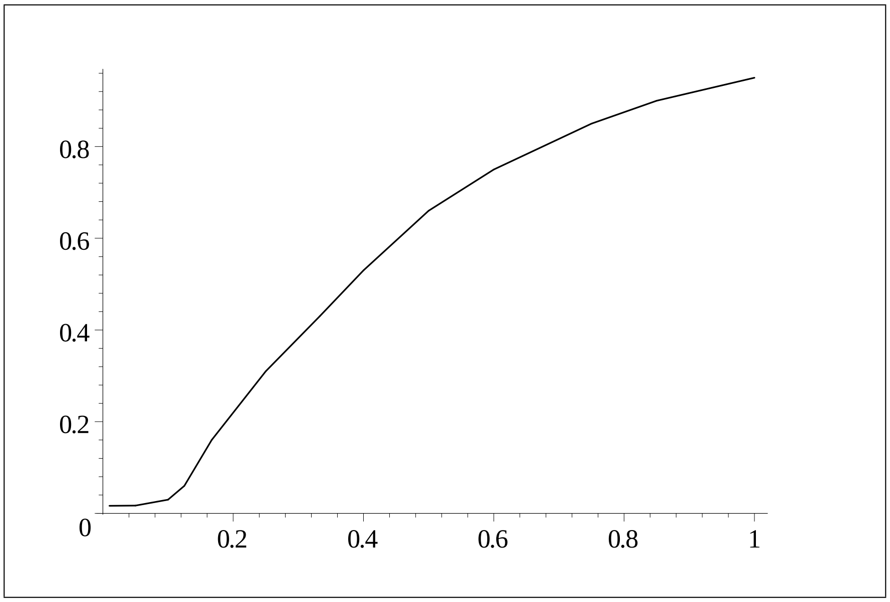

Figure 7-3: The critical electron drift velocity V_{cr}/v_{Te} ($v_{Te} = \sqrt{2T_e/m_e}$) for the ion acoustic instability in a hydrogen plasma as a function of T_i/T_e. For $T_i \ll T_e$, the critical velocity is approximately equal to the ion acosutic velocity. For $T_i \simeq T_e$, it is of the order of the electon thermal velocity.

When $T_e \gg T_i$, the dispersion relation in the long wavelength regime ($k < k_{De}$) becomes

$$\omega \simeq k\sqrt{\frac{T_e}{M}}, \ \ k < k_{De}. \tag{7.62}$$

In this mode, the electron pressure provides the restoring force while ions provide inertia as in the ordinary sound wave. The wave energy density associated with the ion acoustic mode can be calculated as

$$U = \frac{E^2}{8\pi}\omega\frac{\partial\epsilon}{\partial\omega} = \frac{E^2}{8\pi}\left[1 + \left(\frac{\omega_{pi}}{\omega}\right)^2 + \left(\frac{k_{De}}{k}\right)^2\right], \tag{7.63}$$

where the dielectric function ϵ is

$$\epsilon = 1 - \left(\frac{\omega_{pi}}{\omega}\right)^2 + \left(\frac{k_{De}}{k}\right)^2. \tag{7.64}$$

In Eq. (7.63), the energy is shared by the electric field (1), ion kinetic motion $(\omega_{pi}/\omega)^2$, and electron thermal potential $(k_{De}/k)^2$.

The dispersion relation of the ion acoustic wave can be derived from hydrodynamic equations as follows. Since the phase velocity of ion acoustic wave is much smaller than the electron thermal velocity, electrons obey the Boltzmann distribution,

$$n_0 + n_e = n_0 \exp\left(\frac{e\phi}{T_e}\right), \tag{7.65}$$

where n_e is the perturbation. For $e\phi \ll T_e$, we find

$$n_e = \frac{e\phi}{T_e}n_0. \tag{7.66}$$

The equation of motion for ions is

$$n_0 M\frac{\partial v_i}{\partial t} = -en_0\frac{\partial\phi}{\partial x} - \frac{\partial p_i}{\partial x}, \tag{7.67}$$

where p_i is ion pressure perturbation which consists of density and ion temperature perturbations,

$$p_i = T_i n_i + n_0 T_i'. \tag{7.68}$$

If the ion temperature is lower than the electron temperature, $T_i < T_e$, the ion pressure perturbation may be approximated by

$$p_i \simeq \gamma_i T_i n_i, \tag{7.69}$$

where the adiabaticity coefficient is $\gamma_i \simeq 3$. Then,

$$n_0 M\frac{\partial v_i}{\partial t} \simeq -en_0\frac{\partial\phi}{\partial x} - 3T_i\frac{\partial n_i}{\partial x}, \tag{7.70}$$

which yields

$$v_i = \frac{k}{\omega n_0 M}\left(en_0\phi + 3T_i n_i\right). \tag{7.71}$$

Finally, the ion continuity equation is

$$\frac{\partial n_i}{\partial t} + n_0\frac{\partial v_i}{\partial x} = 0, \tag{7.72}$$

and the ion density perturbation thus becomes

$$n_i = \frac{ek^2\phi}{M\left(\omega^2 - 3\frac{T_i}{M}k^2\right)}n_0. \tag{7.73}$$

Substituting n_e and n_i into the Poisson's equation,

$$\nabla^2\phi = -4\pi e\left(n_i - n_e\right), \tag{7.74}$$

we find

$$\omega^2 = \frac{k^2\omega_{pi}^2}{k^2 + k_{De}^2} + \frac{3T_i}{M}k^2. \tag{7.75}$$

This is in agreement with the kinetic result in Eq. (7.60).

When the electron drift velocity is larger than the electron thermal velocity, $V > v_{Te}$, a strong electron-ion instability known as the Buneman instability develops. The dispersion relation is similar to the beam-plasma mode discussed in the preceding Section,

$$1 = \left(\frac{\omega_{pi}}{\omega}\right)^2 + \frac{\omega_{pe}^2}{(\omega - kV)^2}. \tag{7.76}$$

The maximum growth rate occurs at the resonance $kV \simeq \omega_{pe}$ and the solution for ω is given by

$$\omega = \frac{1 + i\sqrt{3}}{2}\left(\frac{m}{2M}\right)^{1/3}\omega_{pe},\ kV \simeq \omega_{pe}. \tag{7.77}$$

The growth rate is quite comparable with the mode frequency in contrast to the case of the beam-plasma instability in which $\omega_r \simeq \omega_{pe} \gg \gamma_{\max}$. The nonlinear development of the Buneman instability is therefore expected to be markedly different from the case of electron-beam instability. According to the analysis by Ishihara and Hirose, deviation from the linear stage occurs when the field energy becomes of the order of

$$\frac{E^2}{8\pi} \simeq \left(\frac{m}{M}\right)^{1/3} n_0 m V_e^2.$$

The instability continues to grow algebraically, however, and eventual saturation due to electron trapping occurs at a turbulence level of

$$\frac{E_{\text{sat}}^2}{8\pi} \simeq 0.1\, n_0 m V_e^2,$$

relatively independent of the ion mass. Large amplitude ion oscillation at the ion plasma frequency ω_{pi} has also been predicted in the final nonlinear stage of the instability. The kinetic energy acquired by the ions is comparable with the field energy and efficient ion heating as well as electron heating should take place.

In the ion acoustic, Buneman and lower hybrid instabilities, electrons are effectively scattered (heated) by the turbulent electric field and this collisionless heating process is called turbulent heating. Plasma temperatures in keV range can easily be achieved by drawing a large current in a plasma. The effective plasma resistivity can be enhanced by orders of magnitude over the classical Spitzer resistivity. For application of turbulent heating in toroidal plasmas, the reader is referred to a review paper by de Kluiver, Perepelkin and Hirose (1991).

7.6 Electrostatic Waves and Instabilities in a Magnetized Plasma

In a plasma confined by a magnetic field, the cyclotron frequencies of electrons and ions enter as characteristic frequencies and significantly modify the dispersion relations of electron and ion modes. In addition, some new modes appear.

In a cold plasma, the dispersion relation

$$\mathbf{k} \cdot \overleftrightarrow{\varepsilon} \cdot \mathbf{k} = 0, \tag{7.78}$$

allows simple solutions. Recalling we have assumed $k = k_\perp e_x + k_\parallel e_z$ and

$$\varepsilon_{xx} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}, \quad \varepsilon_{zz} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}, \tag{7.79}$$

we obtain

$$k_\perp^2\left(1-\sum_s \frac{\omega_{ps}^2}{\omega^2-\Omega_s^2}\right)+k_\parallel^2\left(1-\sum_s \frac{\omega_{ps}^2}{\omega^2}\right)=0. \tag{7.80}$$

If the electrons are strongly magnetized such that $\Omega_e \gg \omega_{pe}$, a high frequency solution to the dispersion relation is

$$\omega \simeq \frac{k_\parallel}{k}\omega_{pe} = \omega_{pe}\cos\theta,$$

where

$$\theta = \arccos\left(\frac{k_\parallel}{k}\right),$$

is the propagation angle relative to the magnetic field. The mode is called the magnetized electron plasma mode. The group velocity is perpendicular to the phase velocity,

$$\frac{d\omega}{dk} = -\frac{\omega_{pe}}{k}\sin\theta\ e_\theta.$$

For perpendicular propagation ($k_\parallel = 0$), we recover the upper and lower hybrid modes,

$$\omega \simeq \omega_{UH} = \sqrt{\omega_{pe}^2+\Omega_e^2}, \quad \omega \simeq \omega_{LH} = \frac{\omega_{pi}}{\sqrt{1+(\omega_{pe}/\Omega_e)^2}}.$$

We have encountered these modes as the resonance frequencies of electromagnetic modes propagating perpendicular to the magnetic field. Resonance is characterized by short wavelengths ($k_\perp \to \infty$) which lower the phase velocity well below those characteristic of electromagnetic modes.

Bernstein Modes

In cold plasma approximation, the Larmor radii of electrons and ions are assumed negligibly small compared with the cross field wavelength. If this assumption is removed, harmonics of the cyclotron frequency appear. Physically, the appearance of cyclotron harmonics is due to deformation of the circular Larmor orbit caused by the waves. Bernstein modes are characterized by pure perpendicular propagation ($k_\parallel = 0$). In this case, both Landau and cyclotron damping disappear and deviation from Maxwellian velocity distribution may drive such weakly damped modes unstable.

We first consider the electron Bernstein mode in which ion dynamics can be ignored ($\omega \gg \omega_{pi}$).

If we assume $k_\parallel = 0$ in Eq. (7.26), the plasma dispersion function may be approximated by

$$Z(\zeta_{en}) = -\frac{1}{\zeta_{en}} = -\frac{k_\parallel v_{Te}}{\omega - n\Omega_e},$$

and we obtain

$$k_\perp^2 + k_{De}^2 \left(1 - \sum_{n=-\infty}^{\infty} e^{-\lambda_e} I_n(\lambda_e) \frac{\omega}{\omega - n\Omega_e}\right) = 0. \tag{7.81}$$

Noting the identity

$$e^{-\lambda_e} \sum_{n=-\infty}^{\infty} I_n(\lambda_e) = 1, \tag{7.82}$$

we can simplify the dispersion relation as

$$\frac{\omega_{pe}^2}{\lambda_e} \sum_{n=1}^{\infty} e^{-\lambda_e} I_n(\lambda_e) \frac{2n^2}{\omega^2 - n^2\Omega_e^2} = 1. \tag{7.83}$$

In the long wavelength limit $\lambda_e \ll 1$, the modified Bessel function $I_n(\lambda_e)$ may be approximated by

$$I_n(\lambda_e) \simeq \frac{1}{n!}\left(\frac{\lambda_e}{2}\right)^n, \tag{7.84}$$

and Eq. (7.81) becomes

$$\omega_{pe}^2 \left(\frac{1}{\omega^2 - \Omega_e^2} + \frac{\lambda_e}{\omega^2 - (2\Omega_e)^2} + \frac{3}{8}\frac{\lambda_e^2}{\omega^2 - (3\Omega_e)^2} + \cdots\right) = 1. \tag{7.85}$$

Approximate solutions are

$$\omega \simeq \sqrt{\omega_{pe}^2 + \Omega_e^2}, \quad 2\Omega_e, \quad 3\Omega_e. \cdots. \tag{7.86}$$

Note that the upper hybrid mode encountered in cold plasma approximation is recovered. In addition, the cyclotron harmonics appear as undamped modes.

In the opposite limit $\lambda_e \gg 1$, the asymptotic form of the modified Bessel function is

$$e^{-\lambda_e} I_n(\lambda_e) \simeq \frac{1}{\sqrt{2\pi\lambda_e}} e^{-n^2/2\lambda_e}, \tag{7.87}$$

and the dispersion relation becomes

$$\frac{\omega_{pe}^2}{\sqrt{2\pi}\lambda_e^{3/2}} \sum_{n=1}^{\infty} \frac{2n^2 e^{-n^2/2\lambda_e}}{\omega^2 - n^2\Omega_e^2} = 1. \tag{7.88}$$

Solutions are

$$\omega \simeq n\Omega_e, \quad n = 1, 2, 3, \cdots$$

For arbitrary value of λ_e, the dispersion relation can only be solved numerically. Fig. 7.4 shows the mode frequency against the electron finite Larmor radius parameter $\sqrt{\lambda_e} = k_\perp \sqrt{T_e/m_e}/\Omega_e$ for the case $\omega_{pe}^2 = 4\Omega_e^2$. The upper hybrid frequency is $\omega_{UH} = \sqrt{5}\Omega_e$. The mode frequency below ω_{UH} starts off at the cyclotron harmonics and monotonically decreases with $k_\perp$. Above ω_{UH}, the frequency remains close to the cyclotron harmonic frequencies and the maximum deviation from the $n-$th harmonic is approximately given by

$$\Delta\omega \simeq \frac{\omega_{pe}^2}{\sqrt{2\pi} n^2 \Omega_e}.$$

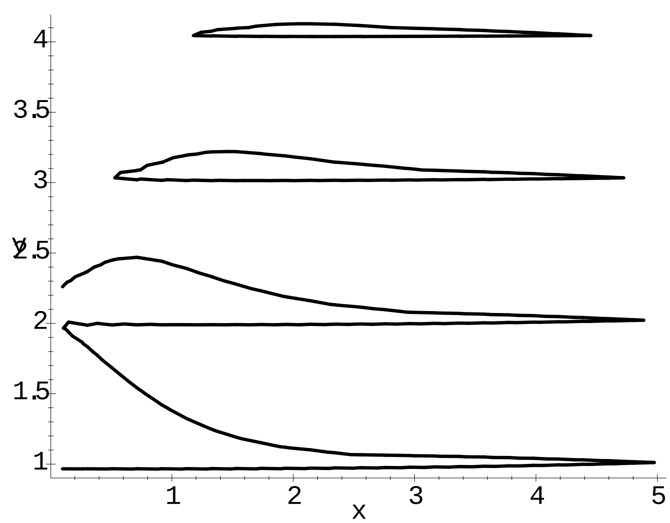

Figure 7-4: Electron Bernstein mode when $\omega_{pe} = 2\Omega_e$. $y = \omega/\Omega_e$, $x = k_\perp \rho_e$.

The ion Bernstein mode can be analyzed in a similar manner. In the frequency regime $\omega \ll \omega_{pe}, \Omega_e$, the dispersion relation becomes

$$\frac{\omega_{pi}^2}{\lambda_i} \sum_{n=1}^{\infty} e^{-\lambda_i} I_n(\lambda_i) \frac{2n^2}{\omega^2 - n^2\Omega_i^2} = 1 + \left(\frac{\omega_{pe}}{\Omega_e}\right)^2, \tag{7.89}$$

or

$$\frac{\omega_{LH}^2}{\lambda_i}\sum_{n=1}^{\infty}e^{-\lambda_i}I_n(\lambda_i)\frac{2n^2}{\omega^2-n^2\Omega_i^2}=1. \tag{7.90}$$

Since in usual laboratory plasmas the lower hybrid frequency is close to the ion plasma frequency which in turn is much higher than the ion cyclotron frequency, the frequency is a monotonically decreasing function of $\sqrt{\lambda_i}=k_\perp\sqrt{T_i/m_i}/\Omega_i$ in most regions. Fig. 7.5 shows the dispersion relation of the ion Bernstein mode when $\omega_{LH}=10\Omega_i$ which corresponds to a somewhat low density plasma.

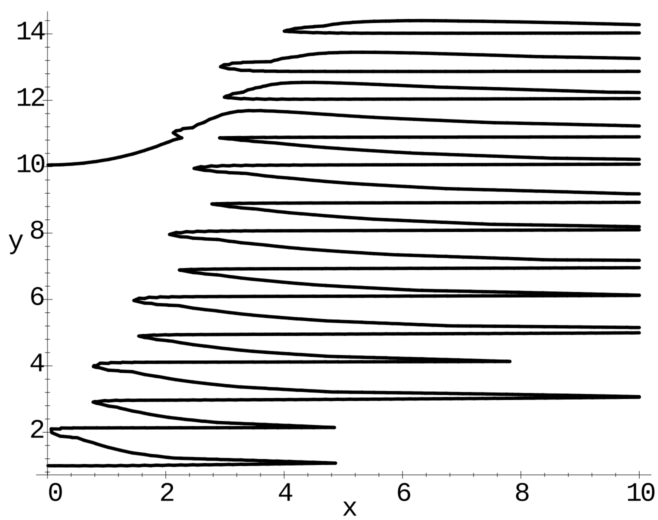

Figure 7-5: Ion Bernstein modes. $y=\omega/\Omega_i$, $x=k_\perp\rho_i$.

Since the frequency domain of the ion Bernstein mode is of order of the ion cyclotron frequency, slight deviation from pure perpendicular propagation can violate the condition

$$\omega\ll k_\parallel v_{Te},$$

where $k_\parallel v_{Te}$ is the electron transit frequency. In the intermediate region such that

$$k_\parallel v_{Ti}\ll\omega\ll k_\parallel v_{Te},$$

electron response becomes adiabatic and the dispersion relation is modified as

$$\frac{\omega_{pi}^2}{\lambda_i}\sum_{n=1}^{\infty}e^{-\lambda_i}I_n(\lambda_i)\frac{2n^2}{\omega^2-n^2\Omega_i^2}=1+\left(\frac{k_{De}}{k}\right)^2\simeq 1+\left(\frac{k_{De}}{k_\perp}\right)^2, \quad (k_\perp \gg k_\parallel) \tag{7.91}$$

In this case, the ion cyclotron harmonics are coupled to the ion acoustic mode. Noting

$$\left(\frac{k_{De}}{k_\perp}\right)^2=\frac{T_i}{T_e}\frac{1}{\lambda_i}\left(\frac{\omega_{pi}}{\Omega_i}\right)^2,$$

we rewrite Eq. (7.91) as

$$\frac{\omega_{pi}^2}{\lambda_i}\sum_{n=1}^{\infty}e^{-\lambda_i}I_n(\lambda_i)\frac{2n^2}{\omega^2-n^2\Omega_i^2}=1+\frac{T_i}{T_e}\frac{1}{\lambda_i}\left(\frac{\omega_{pi}}{\Omega_i}\right)^2. \tag{7.92}$$

Fig. 7.6 shows solutions to this dispersion relation when $T_e = 10T_i$ and $\omega_{pi} = 10\Omega_i$. Note that the frequency starts off at the ion cyclotron frequency Ω_i. Near Ω_i, the dispersion relation is well described by

$$\omega^2 \simeq \Omega_i^2 + k_\perp^2 c_s^2,$$

where $c_s = \sqrt{T_e/M_i}$ is the ion acoustic speed. This mode is called the electrostatic ion cyclotron mode and its dispersion relation has been experimentally observed by D'Angelo and Hirose et al.

Loss-Cone Instability in Mirror Magnetic Field

The Bernstein modes are weakly damped because of absence of Landau and cyclotron damping and thus can become unstable if the velocity distribution function is not Maxwellian. In a mirror magnetic field, particle confinement exploits two constants of motion,

$$\text{magnetic dipole moment } \mu = \frac{\frac{1}{2}mv_\perp^2(z)}{B(z)} = \text{ const.}$$

and

$$\text{energy } E = \frac{1}{2}m\left[v_\perp^2(z)+v_\parallel^2(z)\right] = \text{ const.}$$

where z indicates the coordinate along the magnetic field. It can be easily seen that only those particles satisfying

$$\frac{v_\parallel^2}{v_\perp^2} < \frac{B_{\max}-B_{\min}}{B_{\min}}$$

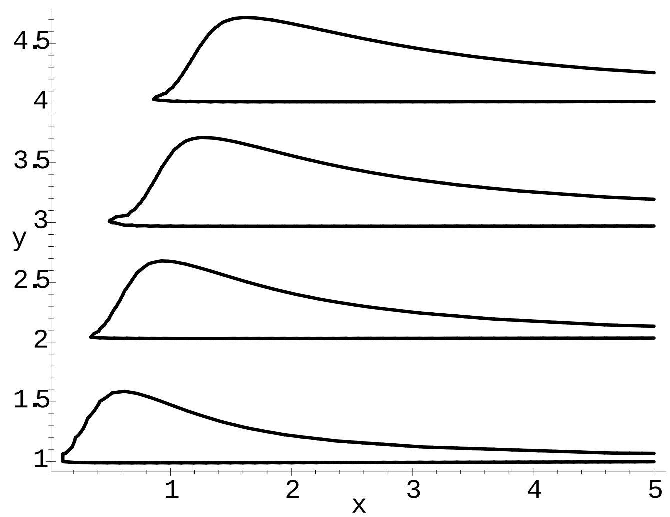

Figure 7-6: Coupling between the ion cyclotron harmonic modes and ion acoustic mode. Near the ion cyclotron frequency Ω_i, the dispersion relation is $\omega^2 \simeq \Omega_i^2 + k_\perp^2 c_s^2$ where $c_s = \sqrt{T_e/M_i}$, $(T_e > T_i)$, is the ion acoustic speed. $y = \omega/\Omega_i, x = k_\perp \rho_s$ where $\rho_s = c_s/\Omega_i$ is the ion acoustic Larmor radius.

can be confined and those particles with

$$\frac{v_\parallel^2}{v_\perp^2} > \frac{B_{\max} - B_{\min}}{B_{\min}}$$

are lost through the throats. Therefore, velocity distribution of plasma particles confined in a mirror device is inherently non-Maxwellian which may cause plasma instabilities and degrade confinement. The loss-cone distribution is characterized by particle deficiency in small perpendicular velocity region and may be modelled by

$$g(v_\perp) = v_\perp^m f_M(v_\perp^2)$$

where m is an integer and $f_M(v_\perp^2)$ is a Maxwellian distribution. When $m \neq 0$, the derivative $dg(v_\perp)/dv_\perp$ can become positive in a region of small velocity which significantly modifies the dispersion relation of the Bernstein mode. The loss-cone mode had been studied extensively in the past when fusion research based mirror devices was active.

A simple case which allows analytic treatment is the instability of the magnetized electron plasma oscillation driven by an ion loss-cone distribution. The mode frequency is assumed to be much higher than the ion cyclotron frequency and the cross-field wavelength much shorter than the ion Larmor radius. In this case, the ions are essentially unmagnetized and for nearly perpendicular propagation in the $x-$direction, the dispersion relation may be approximated by

$$1+\left(\frac{\omega_{pe}}{\Omega_e}\right)^2=\frac{\omega_{pe}^2}{\omega^2}\left(\frac{k_\parallel}{k_\perp}\right)^2+\frac{\omega_{pi}^2}{k_\perp^2}\int_{-\infty}^{\infty}\frac{1}{v_x-\frac{\omega}{k_\perp}}\frac{d}{dv_x}F(v_x)dv_x,$$

where

$$F(v_x)=\int_{-\infty}^{\infty}g(v_\perp)dv_y.$$

As an example, let us consider the following velocity distribution,

$$g(v_\perp)=\frac{1}{\pi v_{Ti}^4}v_\perp^2\exp\left(-\frac{v_\perp^2}{v_{Ti}^2}\right).$$

Then,

$$F(v_x)=\frac{1}{\sqrt{\pi}v_{Ti}}\left(\frac{v_x^2}{v_{Ti}^2}+\frac{1}{2}\right)\exp\left(-\frac{v_x^2}{v_{Ti}^2}\right),$$

and the dispersion relation becomes

$$1+\left(\frac{\omega_{pe}}{\Omega_e}\right)^2=\frac{\omega_{pe}^2}{\omega^2}\left(\frac{k_\parallel}{k_\perp}\right)^2+\left(\frac{k_{Di}}{k_\perp}\right)^2\zeta_i\left[(1-2\zeta_i^2)Z(\zeta_i)-2\zeta_i\right],$$

where $\zeta_i=\omega/k_\perp v_{Ti}$ is the argument of the plasma dispersion function $Z(\zeta_i)$. An unstable solution emerges if $\zeta_i\ll 1$,

$$\omega=\omega_r+i\gamma,$$

where

$$\omega_r\simeq\frac{k_\parallel}{k_\perp}\frac{\omega_{pe}}{\sqrt{1+(\omega_{pe}/\Omega_e)^2}},$$

$$\gamma\simeq\frac{\sqrt{\pi}}{2}\frac{(k_{Di}/k_\perp)^2}{1+(\omega_{pe}/\Omega_e)^2}\frac{\omega_r^2}{k_\perp v_{Ti}}.$$

Note that the distribution $F(v_x)$ is double-humped and the mechanism of the instability is similar to that in two-stream instability.

Lowerhybrid Instabilities

If a large current flows across an external magnetic field, the dispersion relation of the Buneman instability is modified as

$$1 = \left(\frac{\omega_{pi}}{\omega}\right)^2 + \frac{\omega_{pe}^2}{(\omega - kV\sin\theta)^2}\cos^2\theta - \frac{\omega_{pe}^2}{\Omega_e^2}\sin^2\theta, \tag{7.93}$$

where ω_{LH} is the lower hybrid frequency, θ is the angle between k and B_0 and the frequency ω is assumed to be in the range $\Omega_i \ll \omega \ll \Omega_e$ (that is, unmagnetized ions and strongly magnetized electrons). For nearly perpendicular propagation, we have

$$1 = \left(\frac{\omega_{LH}}{\omega}\right)^2 + \frac{M}{m}\cos^2\theta\frac{\omega_{LH}^2}{(\omega - kV)^2}, \tag{7.94}$$

where m/M is the electron/ion mass ratio. The maximum growth rate occurs when $\cos\theta \simeq \sqrt{m/M}$ and is given by

$$\gamma_{\max} \simeq 0.21\omega_{LH}. \tag{7.95}$$

Another lower hybrid instability occurs in the presence of density gradient,

$$1 + \frac{\omega_{pe}^2}{\Omega_e^2} = \left(\frac{\omega_{pi}}{\omega}\right)^2 - \frac{\omega_{pe}^2\kappa_n}{|\Omega_e|\,(\omega - k_\perp V\sin\theta)k_\perp}, \tag{7.96}$$

where κ_n is the density gradient across the magnetic field. The instability is known as the lower hybrid drift mode originally discovered by Mikhailovskii and Tsypin. Kinetic lowerhybrid drift instability was analyzed by Hirose and Alexeff.