Chapter 6

Waves in a Uniform Plasma

6.1 Introduction

Although we seldom encounter uniform unbounded plasmas in practice, studying wave phenomena in such an idealized case reveals numerous fundamental waves that can be excited in a plasma. Also, when the characteristic lengths of plasma nonuniformities are much longer than wavelengths of concerned waves, wave propagation can still be treated by the method essentially identical to those developed for uniform plasmas (*e.g.* WKB approximation).

A magnetized plasma is a typical anisotropic medium for electromagnetic waves and can support various kinds of waves. Since a plasma consists of light electrons and heavy ions, characteristic frequencies range from low frequency ion cyclotron frequency to high frequency electron cyclotron frequency. In general, particle dynamics of both electrons and ions must be incorporated in rigorous analysis of plasma waves.

Another characteristic feature of waves in a plasma is that they are subject to damping even in the absence of particle collisions. Well known examples are Landau and cyclotron damping. The collisionless wave damping plays important roles in plasma heating (and current drive) which can be effectively used in further raising the temperature of a plasma already having a temperature high enough so that collisional Joule heating is ineffective.

In this Chapter, a general kinetic method based on particle velocity distribution function will be presented for waves in a collisionless, magnetized, uniform plasma. Contribution of both electrons and ions to the conduction current will be found first. The resultant current

$$\mathbf{J} = \overleftrightarrow{\boldsymbol{\sigma}} \cdot \mathbf{E}$$

with $\overleftrightarrow{\boldsymbol{\sigma}}$ being the conductivity tensor will then be substituted into Maxwell's equations which yield a general dispersion relation.

6.2 Perturbed Particle Distribution Function

Although fluid or hydrodynamic approximation is much less involved than kinetic description and provides more transparent physical pictures, it overlooks collisionless damping such as Landau and cyclotron damping. Also, when wave phase velocities approach thermal velocity of electrons or ions, fluid approximation tends to break down. For these reasons, satisfactory description of plasma waves requires kinetic theory.

The starting equation is the collisionless Boltzmann equation or Vlasov equation for a particle species having a charge e and mass m,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{6.1}$$

Both electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$ contain in general external fields $\mathbf{E}_0$ and $\mathbf{B}_0$, and those associated with plasma waves $\mathbf{E}_1$ and $\mathbf{B}_1$. Since the distribution function f implicitly depends on the fields $\mathbf{E}$ and $\mathbf{B}$, Eq. (6.1) is a nonlinear differential equation. Throughout this chapter, we assume that perturbed field quantities are small, so that Eq. (6.1) may be linearized as

$$\frac{df_1}{dt} + \frac{e}{m}\left(\mathbf{E}_1 + \frac{1}{c}\mathbf{v} \times \mathbf{B}_1\right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{6.2}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{e}{mc}(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial}{\partial \mathbf{v}}, \tag{6.3}$$

is the substantive derivative along the particle unperturbed trajectory. For simplicity, we have assumed that there is no external electric field, $\mathbf{E}_0 = 0$. In magnetically confined plasmas, electric fields usually exist. In tokamaks, for example, the toroidal current is driven by an inductive toroidal electric field. This field is much smaller than the runaway

(Dreicer's) field in most tokamaks. Except for a small number of runaway electrons, deviation of the electron velocity distribution from Maxwellian may be ignored. In the direction perpendicular to the magnetic field, an ambipolar electric field tends to develop near the edge of a magnetically confined plasma. In general, a plasma self-consistently induces an ambipolar field to maintain charge neutrality. The assumption $\mathbf{E}_0 = 0$ should therefore be made with some caution. In axisymmetric devices such as tokamaks, the radial electric field E_r causes $E \times B$ plasma rotation in the toroidal direction,

$$V_\phi = -c\frac{E_r}{B_\theta},$$

where B_θ is the poloidal magnetic field. The toroidal rotation velocity may approach the ion thermal speed v_{Ti}. However, transport in tokamaks is fairly insensitive to the toroidal rotation and in the lowest order the radial electric field may be ignored. It should be noted that the radial electric field does not cause plasma rotation in the poloidal direction which is proportional to the ion temperature gradient, dT_i/dr.

There are several known methods to solve Eq. (6.1). The method most frequently used is to integrate it along the particle unperturbed trajectory (the method of characteristic) as briefly outlined in Chapter 3. Here, we directly integrate Eq. (6.1) over the gyro angle α,

$$\tan\alpha = v_y/v_x, \tag{6.4}$$

in the geometry shown in Fig. 6.1. A uniform (thus straight) magnetic field $\mathbf{B}_0$ is assumed in the z direction, and the wavevector $\mathbf{k}$ in the x-z plane without loss of generality in a uniform plasma. Assuming that all perturbed quantities, f_1, $\mathbf{E}_1$ and $\mathbf{B}_1$ are proportional to

$$\exp\left[i(\mathbf{k}\cdot\mathbf{x}-\omega t)\right],$$

and eliminating the perturbed magnetic field $\mathbf{B}_1$ via Faraday's law,

$$\nabla\times\mathbf{E}_1 = -\frac{1}{c}\frac{\partial\mathbf{B}_1}{\partial t} \quad \text{or} \quad \mathbf{k}\times\mathbf{E}_1 = \frac{\omega}{c}\mathbf{B}_1, \tag{6.5}$$

we rewrite Eq. (6.1) as

$$i(\mathbf{k}\cdot\mathbf{v}-\omega)f_1(\mathbf{v}) - \Omega\frac{\partial f_1}{\partial\alpha} = -\frac{e}{m}\left[\left(1-\frac{\mathbf{k}\cdot\mathbf{v}}{\omega}\right)\mathbf{E}_1\cdot\frac{\partial f_0}{\partial\mathbf{v}} + \frac{\mathbf{v}\cdot\mathbf{E}_1}{\omega}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{v}}\right], \tag{6.6}$$

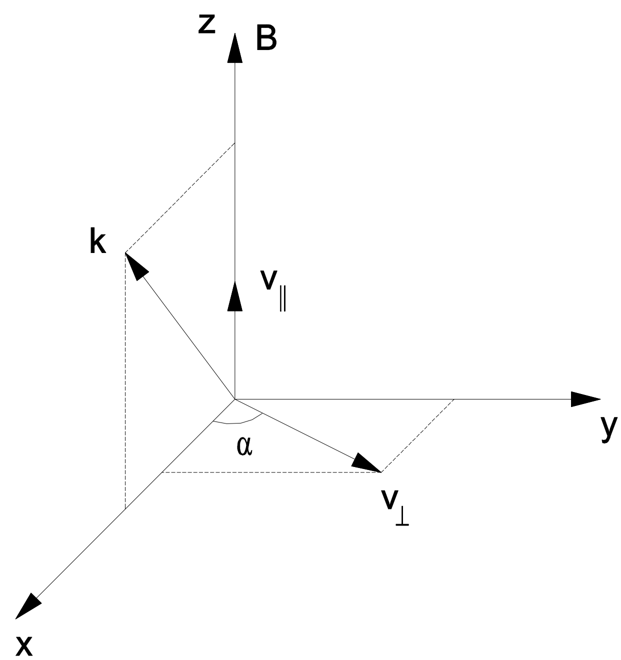

Figure 6.1: Geometry assumed in the analysis. The external magnetic field is in the z direction. The wavevector $\mathbf{k}$ is assumed to be in the $x - z$ plane, $\mathbf{k} = k_\perp \mathbf{e}_x + k_\parallel \mathbf{e}_z$.

where

$$\Omega = \frac{eB_0}{mc}, \tag{6.7}$$

is the cyclotron frequency ($\Omega_e < 0$ for electrons, and $\Omega_i > 0$ for ions). Also, note the identity,

$$\mathbf{v} \times \mathbf{B}_0 \cdot \frac{\partial}{\partial \mathbf{v}} = -B_0 \frac{\partial}{\partial \alpha}. \tag{6.8}$$

Introducing

$$U = \frac{1}{\omega}\left[(\omega - k_\parallel v_\parallel)\frac{\partial f_0}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_0}{\partial v_z}\right], \tag{6.9}$$

$$W = \frac{\partial f_0}{\partial v_\parallel} - \frac{k_\perp \cos\alpha}{\omega}\left(v_\perp \frac{\partial f_0}{\partial v_\parallel} - v_\parallel \frac{\partial f_0}{\partial v_\perp}\right), \tag{6.10}$$

where

$$v_\perp^2 = v_x^2 + v_y^2\,, \quad v_\parallel$$

are constants of motion in a uniform magnetic field ($v_\perp^2 = \text{const.}$ from the constant magnetic moment and $v_\parallel = \text{const.}$ from the assumption $\mathbf{E}_0 = 0$) and characterize the unperturbed distribution function $f_0(\mathbf{v})$,

$$f_0(\mathbf{v}) = f_0(v_\perp^2, v_\parallel), \tag{6.11}$$

we can rewrite Eq. (6.6) in the form

$$\frac{\partial f_1}{\partial \alpha} - i\frac{k_\perp v_\perp \cos\alpha + k_\parallel v_z - \omega}{\Omega} f_1 = \frac{e}{m\Omega}\left[U(E_x\cos\alpha + E_y\sin\alpha) + WE_z\right], \tag{6.12}$$

which can be readily integrated as

$$\begin{aligned} f_1(\mathbf{v},\mathbf{k},\omega) \quad = \quad & \frac{e}{m\Omega}\exp\left[\frac{i}{\Omega}\int_0^\alpha (k_\perp v_\perp \cos\alpha' + k_\parallel v_z - \omega)\ d\alpha'\right] \\ & \times \int_0^\alpha \exp\left[-\frac{i}{\Omega}\int_0^{\alpha'} (k_\perp v_\perp \cos\alpha'' + k_\parallel v_\parallel - \omega)\, d\alpha''\right] \\ & \times \left[U(E_x\cos\alpha' + E_y\sin\alpha') + WE_z\right] d\alpha'. \end{aligned} \tag{6.13}$$

Note that f_1 must be a periodic function of α, $f_1(\alpha + 2\pi n) = f_1(\alpha)$ with n an integer. For this reason, the general solution of Eq. (6.12) has been discarded, and only the particular solution retained.

The integral over α in Eq. (6.13) can be performed if the following expansion is exploited,

$$\exp(ix\sin\alpha) = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\alpha},$$

where $J_n(x)$ is the n-th order Bessel function. The result is

$$f_1(\mathbf{v},\mathbf{k},\omega) = i\frac{e}{m}\sum_m\sum_n J_m(\Lambda)e^{i(m-n)\alpha}\,\frac{\dfrac{n}{\Lambda}J_n(\Lambda)UE_x + iJ_n'(\Lambda)UE_y + J_n(\Lambda)WE_z}{k_\parallel v_\parallel - \omega + n\Omega}, \tag{6.14}$$

where

$$\Lambda = \frac{k_\perp v_\perp}{\Omega}, \quad J_n'(\Lambda) = \frac{dJ_n(\Lambda)}{d\Lambda}, \tag{6.15}$$

W is now modified as

$$W = \frac{\omega - n\Omega}{\omega}\frac{\partial f_0}{\partial v_\parallel} + \frac{n\Omega v_\parallel}{\omega v_\perp}\frac{\partial f_0}{\partial v_\perp}, \tag{6.16}$$

and we have used the following recurrence formulae of the Bessel functions,

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x}J_n(x), \tag{6.17}$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x). \tag{6.18}$$

Equation (6.14) is our desired expression for the perturbed distribution function. Various moments of $f_1(\mathbf{v})$ can be calculated from f_1. For our purpose, it suffices to find the perturbed current density given by

$$\mathbf{J}_1 = \sum_s e_s n_{0s} \int \mathbf{v}\, f_{1s}(\mathbf{v})\, d^3v, \tag{6.19}$$

where s indicates particle species (electrons and ions).

If $f_0(\mathbf{v})$ is isotropic,

$$f_0(\mathbf{v}) = f_0(v^2),$$

U and W are simplified as

$$U = \frac{\partial f_0}{\partial v_\perp},\ W = \frac{\partial f_0}{\partial v_\parallel}. \tag{6.20}$$

In this case, the perturbed magnetic field $\mathbf{B}_1$ disappears from the original Eq. (6.6), since

$$\mathbf{v} \times \mathbf{B}_1 \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{6.21}$$

identically, provided f_0 is isotropic. However, this does not mean that waves are electrostatic, since the electric field is given by

$$\mathbf{E}_1 = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{6.22}$$

where ϕ is the scalar potential and $\mathbf{A}$ is the magnetic vector potential. Electrostatic waves require that the electric field $\mathbf{E}_1$ be written in terms of a scalar potential alone,

$$\mathbf{E}_1 = -\boldsymbol{\nabla}\phi, \tag{6.23}$$

and so far we have not made any such assumptions.

6.3 Dispersion Relation

The perturbed velocity distribution function calculated in the preceding Section yields the perturbed current density through

$$\mathbf{J}_1 = n_0 e \int (f_{1i} - f_{1e})\, \mathbf{v}\, d^3v, \tag{6.24}$$

where for simplicity we assume a plasma composed of electrons and singly ionized ions, both having the same particle density n_0. Since f_1 consists of terms proportional to each component of the electric field $\mathbf{E}$, Eq. (6.24) can be cast into the form

$$\mathbf{J} = \overleftrightarrow{\boldsymbol{\sigma}} \cdot \mathbf{E}, \tag{6.25}$$

where $\overleftrightarrow{\boldsymbol{\sigma}}$ defines the conductivity tensor, which can be evaluated once the unperturbed distribution function $f_0(\mathbf{v})$ is prescribed. (For brevity, we hereafter omit the subscript "1" in the perturbed quantities.) The dispersion relation of electromagnetic waves can then be found from Maxwell's equations,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{6.26}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J}\right) = \frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + 4\pi \overleftrightarrow{\boldsymbol{\sigma}} \cdot \mathbf{E}\right), \tag{6.27}$$

by eliminating the perturbed magnetic field $\mathbf{B}$ (or electric field $\mathbf{E}$) between these equations. It is convenient to introduce a dielectric tensor defined by

$$\overleftrightarrow{\boldsymbol{\varepsilon}} = \mathbf{1} + i\frac{4\pi}{\omega}\overleftrightarrow{\boldsymbol{\sigma}}. \tag{6.28}$$

Eliminating $\mathbf{B}$ between Eqs. (6.26) and (6.27), we obtain

$$k^2\mathbf{E} - (\mathbf{k} \cdot \mathbf{E})\,\mathbf{k} - \frac{\omega^2}{c^2}\overleftrightarrow{\boldsymbol{\varepsilon}} \cdot \mathbf{E} = 0, \tag{6.29}$$

or in the tensorial form

$$\left(k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\epsilon_{ij}\right) E_j = 0, \tag{6.30}$$

where δ_{ij} is the Kronecker's delta. The dispersion relation is thus given by

$$\det\left(k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\epsilon_{ij}\right) = 0. \tag{6.31}$$

The tensor $\overleftrightarrow{\boldsymbol{\varepsilon}}$ has nine components. Only under special circumstances, the tensor becomes Hermitian, $\epsilon_{ij} = \epsilon_{ji}^*$, which is the condition for the absence of wave energy dissipation by the plasma. The expression for ϵ_{ij} is shown below without specifying the velocity distribution function $f_0(\mathbf{v})$ contained in U and W,

$$\boldsymbol{\epsilon} = \mathbf{1} + \sum_s \frac{\omega_{ps}^2}{\omega}\sum_{n=-\infty}^{\infty}\int d^3v\,\frac{\overleftrightarrow{\mathbf{S}}}{\omega - k_\parallel v_\parallel - n\Omega}, \tag{6.32}$$

where the tensor $\overleftrightarrow{\mathbf{S}}$ is given by

$$\overleftrightarrow{\mathbf{S}} = \begin{bmatrix} v_\perp \left(\frac{n}{\Lambda}\right)^2 J_n^2 U & i v_\perp \frac{n}{\Lambda} J_n J_n' U & v_\perp \frac{n}{\Lambda} J_n^2 W \\ -i v_\perp \frac{n}{\Lambda} J_n J_n' U & v_\perp (J_n')^2 U & -i v_\perp J_n J_n' W \\ v_\parallel \frac{n}{\Lambda} J_n^2 U & i v_\parallel J_n J_n' U & v_\parallel J_n^2 W \end{bmatrix}, \tag{6.33}$$

ω_{ps} is the plasma frequency defined by

$$\omega_{ps}^2 = \frac{4\pi n_0 e_s^2}{m_s}, \tag{6.34}$$

and

$$\int d^3 v = 2\pi \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_\parallel .$$

To proceed further, the unperturbed velocity distribution function $f_0(v_\perp^2, v_z)$ must be specified.

If the velocity distribution function $f_0(v_\perp^2, v_z)$ is isotropic Maxwellian

$$f_0(v^2) = \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right), \tag{6.35}$$

with $v^2 = v_\perp^2 + v_\parallel^2$, ϵ_{ij} can be simplified somewhat. For example, ϵ_{xx} becomes

$$\begin{aligned} \epsilon_{xx} &= 1 + \sum_s \frac{\omega_{ps}^2}{\omega} \int \sum_n \frac{v_\perp (n/\Lambda)^2 J_n^2(\Lambda)}{\omega - k_\parallel v_\parallel - n\Omega_s} \left(-\frac{m_s v_\perp}{T_s}\right) \left(\frac{m_s}{2\pi T_s}\right)^{3/2} \exp\left(-\frac{m_s v^2}{2T_s}\right) d^3 v \\ &= 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_n \frac{2n^2}{\lambda_s} \int_0^\infty x e^{-x^2} J_n^2\left(\sqrt{2\lambda_s}\, x\right) dx \times \frac{\omega}{k_\parallel v_{Ts}} \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{e^{-t^2}}{t - \zeta_{sn}} dt, \end{aligned} \tag{6.36}$$

where $x = \nu_\perp / v_{Ts}$, $v_{Ts} = \sqrt{2T_s/m_s}$ is the thermal velocity, $t = \nu_\parallel / v_{Ts}$, $\zeta_{sn} = (\omega - n\Omega_s)/k_\parallel v_{Ts}$, and

$$\lambda_s = \frac{1}{2} \frac{k_\perp^2 v_{Ts}^2}{\Omega^2} = \frac{k_\perp^2 T_s / m_s}{\Omega^2} = (k_\perp \rho_s)^2, \tag{6.37}$$

with

$$\rho_s = \frac{\sqrt{T_s/m_s}}{\Omega_s},$$

being the thermal Larmor radius. The integral over x reduces to

$$\int_0^\infty x e^{-x^2} J_n^2(\sqrt{2\lambda} x) dx = \frac{1}{2} \exp(-\lambda)\, I_n(\lambda), \tag{6.38}$$

where $I_n(\lambda)$ is the modified Bessel function of the first kind. The integral over t can be written in terms of the plasma dispersion function defined by

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-\zeta}\, dt. \tag{6.39}$$

Therefore, Eq. (6.36) becomes

$$\epsilon_{xx} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_n \frac{n^2}{\lambda_s} e^{-\lambda_s} I_n(\lambda_s) \zeta_0 Z(\zeta_{sn}). \tag{6.40}$$

The other components can be calculated in a similar manner:

$$\epsilon_{xy} = -\epsilon_{yx} = -i \sum_s \frac{\omega_p^2}{\omega^2} \sum_n n e^{-\lambda}(I_n - I_n')\zeta_0 Z(\zeta_n), \tag{6.41}$$

$$\epsilon_{xz} = \varepsilon_{zx} = \sum_s \frac{\omega_p^2}{\omega} \sum_n \frac{n\Omega m}{k_\perp k_\| T}\, e^{-\lambda} I_n(\lambda) \zeta_n Z(\zeta_n), \tag{6.42}$$

$$\begin{aligned}\epsilon_{yy} &= 1 + \sum_s \frac{\omega_p^2}{\omega^2} \sum_n e^{-\lambda} \left[\frac{n^2}{\lambda} I_n + 2\lambda (I_n - I_n') \right] \zeta_0 Z(\zeta_n) \\ &= \varepsilon_{xx} + 2 \sum_s \frac{\omega_p^2}{\omega^2} \sum_n \lambda e^{-\lambda} (I_n - I_n') \zeta_0 Z(\zeta_n),\end{aligned} \tag{6.43}$$

$$\varepsilon_{yz} = -\varepsilon_{zy} = i \sum_s \frac{\omega_p^2}{\omega^2} \sum_n \sqrt{\frac{\lambda}{2}} e^{-\lambda} (I_n - I_n') \zeta_0 Z'(\zeta_n) \tag{6.44}$$

$$\epsilon_{zz} = 1 - \sum_s \frac{\omega_p^2}{\omega^2} \sum_n e^{-\lambda} I_n(\lambda) \zeta_0 \zeta_n Z'(\zeta_n). \tag{6.45}$$

The subscript "s" has been omitted for brevity.

In deriving these expressions, use has been made of the following identities:

$$e^{-\lambda} \sum_{n=-\infty}^{\infty} I_n(\lambda) = 1, \tag{6.46}$$

$$\int_0^\infty e^{-x^2} x J_n(ax) J_n(bx) dx = \frac{1}{2} \exp\left(-\frac{a^2+b^2}{4}\right) I_n\left(\frac{ab}{2}\right). \tag{6.47}$$

Differentiating by a yields

$$\int_0^\infty x^2 J_n'(ax) J_n(bx) e^{-x^2} dx = \frac{1}{4}\left(b I_n'\left(\frac{ab}{2}\right) - a I_n\left(\frac{ab}{2}\right)\right) \exp\left(-\frac{a^2+b^2}{4}\right). \tag{6.48}$$

Since $a = b = \sqrt{2\lambda}$, we obtain

$$\int_0^\infty x^2 J_n'(\sqrt{2\lambda}x) J_n(\sqrt{2\lambda}x) e^{-x^2} dx = \frac{\sqrt{\lambda}}{2\sqrt{2}} \left[I_n'(\lambda) - I_n(\lambda)\right] e^{-\lambda}, \tag{6.49a}$$

which has been used in calculation of $\varepsilon_{xy} = -\varepsilon_{yx}$ and $\varepsilon_{yz} = -\varepsilon_{zy}$. Differentiating Eq.(6.49a) further with respect to b and substituting $a = b = \sqrt{2\lambda}$, we also obtain

$$\int_0^\infty x^3 \left[J_n'(\sqrt{2\lambda}x)\right]^2 e^{-x^2} dx = \left\{ \frac{n^2}{4\lambda} I_n(\lambda) + \frac{\lambda}{2}\left(I_n - I_n'\right) \right\} e^{-\lambda}, \tag{6.50}$$

where use is made of the differential equation satisfied by $I_n(\lambda)$,

$$I_n'' + \frac{I_n'}{\lambda} - \left(1 + \frac{n^2}{\lambda^2}\right) I_n = 0. \tag{6.51}$$

This is needed in calculating ε_{yy}. Also, it is noted that the plasma dispersion function $Z(\zeta)$ satisfies the following differential equation,

$$Z'(\zeta) + 2[1 + \zeta Z(\zeta)] = 0, \tag{6.52}$$

since

$$\begin{aligned} Z'(\zeta) &= \frac{1}{\sqrt{\pi}} \frac{d}{d\zeta} \int_{-\infty}^\infty \frac{e^{-x^2}}{x - \zeta} dx \\ &= -\frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \left(\frac{d}{dx} \frac{1}{x - \zeta} \right) e^{-x^2} dx \\ &= -\frac{2}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{x e^{-x^2}}{x - \zeta} dx = -2\left[1 + \zeta Z(\zeta)\right]. \end{aligned} \tag{6.53}$$

6.4 Plasma Dispersion Function $Z(\zeta)$

When the particle velocity distribution function $f_0(v_\perp^2, v_z)$ is characterized by isotropic Maxwellian or bi-Maxwellian

$$f(v_\perp^2, v_z) = \frac{m^{3/2}}{2\pi T_\perp (2\pi T_\parallel)^{1/2}} \exp\left[-\frac{m v_\perp^2}{2T_\perp} - \frac{m v_\parallel^2}{2T_\parallel} \right],$$

the dielectric tensor ϵ_{ij} contains the plasma dispersion function

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{e^{-x^2}}{x - \zeta}\, dx, \tag{6.54}$$

and its derivative, $Z'(\zeta)$. Since this important function so often appears in analyzing plasma waves, it may be appropriate to devote one Section. Fried and Conte have tabulated numerical values of $Z(\zeta)$. Mathematically speaking, the plasma dispersion function is the Hilbert transform of the function $1/(x - \zeta)$.

Approximate series and asymptotic expansions of $Z(\zeta)$ can be found in two limiting cases, $|\zeta| \ll 1$ and $|\zeta| \gg 1$. We first note that $Z(\zeta)$ satisfies the following differential equation

$$\frac{dZ}{d\zeta} = -2\left[1 + \zeta Z(\zeta)\right], \tag{6.55}$$

as shown in the preceding section. This can be integrated as

$$Z(\zeta) = Z(0)e^{-\zeta^2} - 2e^{-\zeta^2}\int_0^{\zeta} e^{x^2}dx, \tag{6.56}$$

with $Z(0)e^{-\zeta^2}$ being the general solution and the last term the particular solution. To find the "initial value" $Z(0)$, we evaluate

$$Z(0) = \frac{1}{\sqrt{\pi}}\lim_{\varsigma\to 0}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{x-\zeta}\,dx. \tag{6.57}$$

Letting $\zeta = \alpha + i\beta$, we find

$$\begin{aligned} Z(0) &= \frac{1}{\sqrt{\pi}}\lim_{\alpha,\ \beta\to 0}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{x-\alpha-i\beta}\,dx \\ &= \frac{1}{\sqrt{\pi}}\lim_{\beta\to 0}\int_{-\infty}^{\infty}\frac{x+i\beta}{x^2+\beta^2}e^{-x^2}dx \\ &= i\sqrt{\pi} \end{aligned} \tag{6.58}$$

Therefore,

$$Z(\zeta) = i\sqrt{\pi}e^{-\zeta^2} - 2e^{-\zeta^2}\int_0^{\zeta} e^{x^2}dx, \tag{6.59}$$

which can be used for any values of the complex quantity ζ. As we have seen in the preceding Section, ζ is given by

$$\zeta = \frac{\omega - n\Omega}{k_{\parallel}v_T},$$

and in general a complex number since ω and/or $k_{\parallel}$ can be complex depending on growing ($\omega_i > 0$) or damped ($\omega_i < 0$) wave. Even if Im ζ is small, the plasma dispersion function is intrinsically complex as indicated in Eq. (6.59). Physically, this means that plasma waves in general suffer damping through wave-particle interaction. Under certain circumstances, the intrinsic dissipation can be the source of plasma instability as we have already seen in Chapter 3.

When $|\zeta| \ll 1$, Eq. (6.59) yields the following series expansion for $Z(\zeta)$,

$$
\begin{aligned}
Z(\zeta) &\simeq i\sqrt{\pi}e^{-\zeta^2} - 2\left(1 - \zeta^2 + \frac{1}{2}\zeta^4 - \cdots\right)\int_0^{\zeta}\left(1 + x^2 + \frac{1}{2}x^4 + \cdots\right)dx \\
&= -2\zeta + \frac{4}{3}\zeta^3 - \cdots + i\sqrt{\pi}e^{-\zeta^2}\ , \quad (|\zeta| \ll 1)\,. \qquad (6.60)
\end{aligned}
$$

In the opposite limit $|\zeta| \gg 1$, we directly expand the defining equation for $Z(\zeta)$,

$$
\begin{aligned}
Z(\zeta) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{x-\zeta}\,dx \\
&\simeq -\frac{1}{\sqrt{\pi}\zeta}\int_{-\infty}^{\infty}\left(1 - \frac{x}{\zeta}\right)^{-1}e^{-x^2}dx \\
&= -\frac{1}{\zeta} - \frac{1}{2\zeta^3} - \cdots + i\sqrt{\pi}e^{-\zeta^2}\ , \quad (|\zeta| \gg 1)\,. \qquad (6.61)
\end{aligned}
$$

The plasma dispersion function $Z(x+iy)$ satisfies the following symmetry properties,

$$\text{Re}\ Z(-x+iy) = -\text{Re}\ Z(x+iy), \qquad (6.62)$$

$$\text{Im}\ Z(-x+iy) = \text{Im}\ Z(x+iy), \qquad (6.63)$$

or

$$Z(-x+iy) = -Z^*(x+iy), \qquad (6.64)$$

which can be seen from the definition of $Z(\zeta)$,

$$Z(x+iy) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{t-(x+iy)}\,dt = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{t-x+iy}{(t-x)^2+y^2}e^{-t^2}dt. \qquad (6.65)$$

Also, for $y > 0$,

$$Z(x-iy) = Z^*(x+iy) + 2i\sqrt{\pi}e^{-(x-iy)^2}. \qquad (6.66)$$

Therefore, knowing $Z(x+iy)$ for $x,\ y > 0$ is sufficient to evaluate $Z(x+iy)$ for arbitrary x and y.

6.5 Unmagnetized Plasma

In the absence of external magnetic field, the perturbed distribution function satisfies

$$i(\mathbf{k}\cdot\mathbf{v}-\omega)f_1(\mathbf{v}) = -\frac{e}{m}\left[\left(1 - \frac{\mathbf{k}\cdot\mathbf{v}}{\omega}\right)\mathbf{E}\cdot\frac{\partial f_0}{\partial\mathbf{v}} + \frac{\mathbf{v}\cdot\mathbf{E}}{\omega}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{v}}\right]. \qquad (6.67)$$

If the velocity distribution is isotropic, Eq. (6.67) simplifies as

$$i(\mathbf{k}\cdot\mathbf{v}-\omega)f_1(\mathbf{v}) = -\frac{e}{m}\mathbf{E}\cdot\frac{\partial f_0}{\partial \mathbf{v}}. \tag{6.68}$$

The wave vector may be assumed to be in arbitrary direction, say, $\mathbf{k} = k\mathbf{e}_z$. The electric field may be assumed to be

$$\mathbf{E} = \mathbf{E}_x + \mathbf{E}_z. \tag{6.69}$$

Evidently, $\mathbf{E}_x$ is associated with the transverse wave while $\mathbf{E}_z$ is associated with the longitudinal wave. We further assume that f_0 is Maxwellian,

$$f_M(\mathbf{v}) = \left(\frac{m}{2\pi T}\right)^{3/2}\exp\left(-\frac{mv^2}{2T}\right). \tag{6.70}$$

The current density is

$$\begin{aligned}
\mathbf{J} &= ne\int \mathbf{v}f_1 d^3v \\
&= -i\frac{ne^2}{T}\int\frac{\mathbf{v}}{kv_z-\omega}\mathbf{E}\cdot\mathbf{v}f_M(\mathbf{v}) \\
&= -i\frac{ne^2}{mkv_T}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{t-\zeta}dt\mathbf{E}_x - 2i\frac{ne^2}{mkv_T}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{t^2e^{-t^2}}{t-\zeta}dt\mathbf{E}_z \\
&= -i\frac{ne^2}{mkv_T}Z(\zeta)\mathbf{E}_x - 2i\frac{ne^2}{mkv_T}\zeta\left[1+\zeta Z(\zeta)\right]\mathbf{E}_z.
\end{aligned} \tag{6.71}$$

The permittivity pertinent to the transverse wave E_x is

$$\begin{aligned}
\varepsilon_T &= 1 + i\frac{4\pi}{\omega}\left(-i\frac{ne^2}{mkv_T}Z(\zeta)\right) \\
&= 1 + \frac{\omega_p^2}{\omega k v_T}Z(\zeta).
\end{aligned} \tag{6.72}$$

The dispersion relation is given by

$$\left(\frac{\omega}{k}\right)^2 = \frac{c^2}{\epsilon_T} = \frac{c^2}{1+\dfrac{\omega_{pe}^2}{\omega k v_{Te}}Z(\zeta_e)}, \tag{6.73}$$

where the ion contribution has been ignored. When $\omega \gg kv_{Te}$,

$$Z(\zeta_e) \simeq -\frac{1}{\zeta_e},$$

and

$$\left(\frac{\omega}{k}\right)^2 = \frac{c^2}{1-(\omega_{pe}/\omega)^2},$$

$$\omega^2 = \omega_{pe}^2 + (ck)^2. \tag{6.74}$$

In the opposite limit, $\omega \ll kv_{Te}$,using

$$Z(\zeta_e) \simeq -2\zeta_e + i\sqrt{\pi}e^{-\zeta_e^2} \simeq i\sqrt{\pi},$$

we find

$$k^3 = i\sqrt{\pi}\frac{\omega_{pe}^2\omega}{c^2 v_{Te}},\ k = \frac{\sqrt{3}+i}{2}\pi^{1/6}\left(\frac{\omega_{pe}^2\omega}{c^2 v_{Te}}\right)^{1/3}. \tag{6.75}$$

The wavenumber becomes complex which indicates spatial damping or evanescence. The inverse of the damping factor

$$\delta = \frac{1}{k_i} = \frac{2}{\pi^{1/6}}\left(\frac{c^2 v_{Te}}{\omega_{pe}^2\omega}\right)^{1/3},$$

is often called anomalous skin depth since it can be larger than the classical skin depth

$$\delta_c = \frac{c}{\omega_{pe}}, \tag{6.76}$$

provided

$$\omega < \frac{v_{Te}}{c}\omega_{pe}.$$

The longitudinal wave E_z is characterized by the dispersion relation

$$\varepsilon_L = 1 + 2\frac{4\pi ne^2}{mkv_T}\zeta\left[1+\zeta Z(\zeta)\right] = 0, \tag{6.77}$$

or

$$1 + \left(\frac{k_D}{k}\right)^2\left[1+\zeta Z(\zeta)\right] = 0, \tag{6.78}$$

where

$$k_D = \sqrt{\frac{4\pi ne^2}{T}},$$

is the inverse Debye length. For an electron plasma,

$$1 + \left(\frac{k_{De}}{k}\right)^2\left[1+\zeta_e Z(\zeta_e)\right] = 0, \tag{6.79}$$

and for electron-ion plasma,

$$1 + \left(\frac{k_{De}}{k}\right)^2\left[1+\zeta_e Z(\zeta_e)\right] + \left(\frac{k_{Di}}{k}\right)^2\left[1+\zeta_i Z(\zeta_i)\right] = 0. \tag{6.80}$$

These dispersion relations will be discussed in detail in Chapter 7.

6.6 Waves in a Cold Plasma

In Section 6.2, we have seen that all components of the dielectric tensor ϵ_{ij} contain harmonics of the cyclotron frequency Ω in the argument of the plasma dispersion function

$$\zeta_n = (\omega - n\Omega)/k_\parallel v_T.$$

The appearance of harmonics is due to deviation of particle orbit from complete circular motion when acted by wave electric field perpendicular to the external magnetic field. When the Larmor radius is small (cold plasma), such deviation becomes ignorable, and harmonics are expected to disappear. Only the fundamental cyclotron frequency Ω will enter the dispersion relation.

Another manifestation of particle thermal motion is collisionless wave damping. Both Landau and cyclotron damping require that waves find particles which travel along the magnetic field with the speed corresponding to the phase velocity (or Doppler shifted phase velocity) of the waves. In cold plasmas, the number of these resonant particles is very small, and all damping mechanisms are expected to disappear. The dielectric tensor in this case should become Hermitian, $\epsilon_{ij} = \epsilon_{ji}^*$ since no absorption of wave energy by plasma is involved.

Let us first assume that both electrons and ions are characterized by delta function distribution,

$$f(\mathbf{v}) = \delta(\mathbf{v}) = \frac{\delta(v_\perp)}{2\pi v_\perp}\delta(v_\parallel), \tag{6.81}$$

where $\delta(\mathbf{v})$ is the three-dimensional delta function. This assumption is equivalent to the condition that the phase velocity of concerned wave be much larger than the thermal velocities of both electrons and ions. (For high frequency waves, such assumption is often appropriate. However, for low-frequency waves such as Alfven wave, the condition can easily be violated. In Chapter 3, we have in fact assumed that $\omega/k_\parallel \ll v_{Te}$ for Alfven waves when analyzing the MHD ballooning instability.) Then, the dielectric tensor can be evaluated from Eq. (6.32).

For example, ϵ_{xx} becomes

$$\begin{aligned}
\epsilon_{xx} &= 1 + \sum_s \frac{\omega_p^2}{\omega} \sum_{n=-\infty}^{\infty} 2\pi \int_0^\infty v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z \frac{v_\perp \left(\frac{n}{\Lambda}\right)^2 J_n^2(\Lambda)}{\omega - k_\parallel v_\parallel - n\Omega} \frac{\partial}{\partial v_\perp} \left[\frac{\delta(v_\perp)}{2\pi v_\perp} \delta(v_\parallel)\right] \\
&= 1 - \frac{1}{2} \sum_s \frac{\omega_p^2}{\omega} \left(\frac{1}{\omega - \Omega} + \frac{1}{\omega + \Omega}\right) \\
&= 1 - \sum_s \frac{\omega_p^2}{\omega^2 - \Omega^2} \\
&= 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2},
\end{aligned} \tag{6.82}$$

where only $n = \pm 1$ terms remain finite because of the delta function distribution. Also note that $J_1(x) \simeq x/2$ for $x \ll 1$. Similarly, we find

$$\epsilon_{xy} = -\epsilon_{yx} = -i \sum_s \frac{\Omega \omega_p^2}{\omega(\omega^2 - \Omega^2)}, \tag{6.83}$$

$$\epsilon_{xz} = \epsilon_{zx} = \epsilon_{yz} = \epsilon_{zy} = 0, \tag{6.84}$$

$$\epsilon_{yy} = \epsilon_{xx} = 1 - \sum_s \frac{\omega_p^2}{\omega^2 - \Omega^2}, \tag{6.85}$$

$$\epsilon_{zz} = 1 - \sum_s \frac{\omega_p^2}{\omega^2}. \tag{6.86}$$

Substituting these ϵ_{ij} in Eq. (6.31), we obtain the following dispersion relation,

$$\begin{vmatrix}
k_\parallel^2 - \frac{\omega^2}{c^2}\left(1 - \sum_s \frac{\omega_p^2}{\omega^2 - \Omega^2}\right) & i\frac{\omega^2}{c^2} \sum_s \frac{\Omega\omega_p^2}{\omega(\omega^2 - \Omega^2)} & -k_\parallel k_\perp \\
-i\frac{\omega^2}{c^2} \sum_s \frac{\Omega\omega_p^2}{\omega(\omega^2 - \Omega^2)} & k^2 - \frac{\omega^2}{c^2}\left(1 - \sum_s \frac{\omega_p^2}{\omega^2 - \Omega^2}\right) & 0 \\
-k_\perp k_\parallel & 0 & k_\perp^2 - \frac{\omega^2}{c^2}\left(1 - \sum_s \frac{\omega_p^2}{\omega^2}\right)
\end{vmatrix} = 0. \tag{6.87}$$

As expected, the dielectric tensor is Hermitian, $\epsilon_{ij} = \epsilon_{ji}^*$, indicating no absorption of wave energy in a cold plasma.

The components

$$\epsilon_{xx} = \epsilon_{yy} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}, \tag{6.88}$$

very much resemble the dielectric constant ϵ (scalar) of an isotropic dielectric medium if the cyclotron frequency Ω_s is replaced with the frequency of bound harmonic electron motion, ω_0. As we will see shortly, the dispersion relation of electromagnetic waves propagating along the magnetic field is given by

$$\left(\frac{ck_\parallel}{\omega}\right)^2 = \epsilon_{xx}, \tag{6.89}$$

with electric fields perpendicular to the magnetic field. This also resembles the dispersion relation of electromagnetic waves in an isotropic dielectric medium,

$$\left(\frac{ck}{\omega}\right)^2 = \epsilon = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2 - \omega_{0j}^2}, \tag{6.90}$$

where ω_{0j} is the frequency of harmonic motion at the $j-$th bound state. Therefore, cyclotron motion of charged free particles in a plasma can be considered as bound harmonic motion, and this analogy holds particularly well in a cold plasma.

The dispersion relation, Eq. (6.87), is a sixth order algebraic equation for ω and in principle can be solved with the propagation angle

$$\theta = \tan^{-1}\left(k_\perp/k_\parallel\right),$$

with respect to the external magnetic field as a parameter. However, fundamental modes can be revealed by considering two particular angles, parallel propagation $\theta = 0$ ($k_\perp = 0$) and perpendicular propagation $\theta = \pi/2$ ($k_\parallel = 0$). Propagation at arbitrary angles can be regarded as linear combination of the fundamental modes.

6.7 Parallel Propagation ($k_\perp = 0$)

If $k_\perp = 0$ in Eq. (6.87), we obtain two independent solutions,

$$\left(\frac{c^2k_\parallel^2}{\omega^2} - \epsilon_{xx}\right)^2 + \epsilon_{xy}^2 = 0, \tag{6.91}$$

$$\epsilon_{zz} = 0. \tag{6.92}$$

In terms of electric field components E_i, these solutions correspond, respectively, to

$$\left(\frac{c^2k_\parallel^2}{\omega^2} - \epsilon_{xx}\right)E_x - \epsilon_{xy}E_y = 0, \tag{6.93}$$

$$\epsilon_{xy}E_x + \left(\frac{c^2k_\parallel^2}{\omega^2} - \epsilon_{xx}\right)E_y = 0, \tag{6.94}$$

$$\epsilon_{zz}E_z = 0. \tag{6.95}$$

Equation (6.91) yields

$$\frac{c^2k_\parallel^2}{\omega^2} - \epsilon_{xx} = \pm i\epsilon_{xy}. \tag{6.96}$$

In this case, the electric fields E_x and E_y are related through

$$E_x = \mp iE_y,$$

that is, the wave is circularly polarized with either positive ($E_x = -iE_y$) or negative ($E_x = +iE_y$) helicity. Superposition of these two modes yields a plane wave having E_x only. The dispersion relation of the plane wave is given by

$$\frac{c^2k_\parallel^2}{\omega^2} = \epsilon_{xx} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}.$$

6.7.1 Modes with Positive Helicity

Electron Cyclotron and Whistler Modes

We first consider the mode described by

$$\frac{c^2k_\parallel^2}{\omega^2} - \epsilon_{xx} = +i\epsilon_{xy}. \tag{6.97}$$

The two components of the electric field E_x and E_y are related through $E_x = -iE_y$, that is, the field is circularly polarized with positive helicity (circulation in the same sense as the electron gyration motion). Substituting

$$\epsilon_{xx} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}, \tag{6.98}$$

$$\epsilon_{xy} = -i\sum_s \frac{\Omega\omega_{ps}^2}{\omega(\omega^2 - \Omega_s^2)} = i\frac{|\Omega_e|\,\omega_{pe}^2}{\omega(\omega^2 - \Omega_e^2)} - \frac{\Omega_i\omega_{pe}^2}{\omega(\omega^2 - \Omega_i^2)}, \tag{6.99}$$

we obtain

$$\frac{c^2k_\parallel^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega\left(\omega - |\Omega_e|\right)} - \frac{\omega_{pi}^2}{\omega\left(\omega + \Omega_i\right)}, \tag{6.100}$$

where subscripts e and i refer to electrons and (singly ionized) ions. As ω approaches $|\Omega_e|$ from below, the second term in RHS becomes large. The wavenumber $k_\parallel$ correspondingly becomes large, too, and such phenomenon on the $\omega - k_\parallel$ space is called resonance. Resonance indicates possibility of strong absorption, and in the present case, absorption at the electron cyclotron frequency is suggested. The wave damping rate will be calculated in Section 6.9 after removing the cold plasma assumption.

When $k_\parallel \to 0$, Eq. (6.100) gives

$$\omega = 0 \quad \text{and} \quad \frac{1}{2}\left(|\Omega_e| \pm \sqrt{\Omega_e^2 + 4\omega_{pe}^2}\right), \tag{6.101}$$

provided $\omega_{pe}^2 \gg \omega_{pi}^2$, $|\Omega_e| \gg \Omega_i$ as in conventional plasmas. The solution with the negative sign must be discarded since for $k_\parallel \to +0$, it corresponds to a wave having negative helicity thus violating the assumption made earlier. The positive solution,

$$\omega_{c1} = \frac{1}{2}\left(|\Omega_e| + \sqrt{\Omega_e^2 + 4\omega_{pe}^2}\right), \tag{6.102}$$

is called a cut-off frequency since waves become evanescent below this frequency (but above $|\Omega_e|$). The solution $\omega \to 0$ with $k_\parallel = 0$ corresponds to the Alfven wave branch, $\omega \simeq k_\parallel V_A$ where V_A is the Alfven velocity. This mode exists in very low frequency regime, even lower than the ion cyclotron frequency, $\omega \ll \Omega_i$. The Alfven mode will be discussed in more detail in the section to follow.

In the intermediate frequency range $\Omega_i \ll \omega \ll |\Omega_e|$, Eq. (6.100) yields

$$\omega \simeq \frac{|\Omega_e|}{\omega_{pe}^2} c^2 k_\parallel^2, \tag{6.103}$$

which is known as the *whistler* or *helicon wave*. This wave is strongly dispersive. Both phase and group velocities are proportional to $\sqrt{\omega}$. As we will see later, the whistler wave can become unstable when the electron distribution function is characterized by bi-Maxwellian having $T_\perp > T_\parallel$. Such anisotropic distribution function is believed to prevail in the space plasma trapped by the earth magnetic field. The name "whistler" is coined after whistling tones accompanying electromagnetic radiation emitted from the ionospheric plasma. The whistler wave can be excited by lightning and propagate ove a long distance in the ionosphere along the earth magnetic field.

Kinetic Alfven Mode

In the low frequency limit, $\omega \ll \Omega_i$, Eq. (6.100) yields

$$\begin{aligned}\frac{c^2k_\parallel^2}{\omega^2} &\simeq 1+\frac{\omega_{pe}^2}{\omega\,|\Omega_e|}\left(1+\frac{\omega}{|\Omega_e|}\right)-\frac{\omega_{pi}^2}{\omega\Omega_i}\left(1-\frac{\omega}{\Omega_i}\right)\\ &= 1+\frac{\omega_{pe}^2}{\Omega_e^2}+\frac{\omega_{pi}^2}{\Omega_i^2}\simeq\frac{\omega_{pi}^2}{\Omega_i^2},\end{aligned} \tag{6.104}$$

in typical laboratory plasmas with $\omega_{pi}^2 \gg \Omega_i^2$. Solving for ω, we obtain

$$\omega = k_\parallel V_A, \tag{6.105}$$

where

$$V_A = c\frac{\Omega_i}{\omega_{pi}} = \frac{B_0}{\sqrt{4\pi M n_0}}, \tag{6.106}$$

is the Alfven velocity with M the ion mass and Mn_0 the mass density of the plasma. The Alfven wave described by Eq. (6.105) is nondispersive. The Alfven mode with negative helicity is also allowed as will be shown in the following section. If two Alfven modes with positive and negative helicities are superposed, a plane Alfven wave is realized provided they have the same amplitude.

In order to find finite ion Larmor radius effects on the Alfven mode, the assumption $k_\perp = 0$ must be removed. Let us go back to the starting equation, Eq. (6.97). When $\omega \ll \Omega_i$, $\epsilon_{xy} \simeq 0$ if summation over electron and ion is taken. Also, ϵ_{xx} may be approximated by ω_{pi}^2/Ω_i^2. Therefore, the Alfven wave is essentially described by

$$\frac{c^2k_\parallel^2}{\omega^2} \simeq \epsilon_{xx}, \tag{6.107}$$

and $\epsilon_{xx} \simeq \omega_{pi}^2/\Omega_i^2$ plays the role of plasma permittivity perpendicular to the external magnetic field as discussed in Chapter 1. The dispersion relation Eq. (6.107) holds even when the condition $k_\perp = 0$ is relaxed. In Section 6.2, we have derived an expression for ϵ_{xx} in the case of isotropic Maxwellian distribution. If we neglect electron contribution, and assume $|\zeta_{in}| \gg 1$ for ions in Eq. (6.36), we obtain

$$\epsilon_{xx} \simeq \frac{\omega_{pi}^2}{\Omega_i^2}\frac{1}{\lambda_i}\left(1-e^{-\lambda_i}I_0(\lambda_i)\right), \tag{6.108}$$

where use has been made of the following identity

$$2\sum_{n=1}^{\infty} e^{-\lambda} I_n(\lambda) = \sum_{n=-\infty}^{\infty} e^{-\lambda} I_n - e^{-\lambda} I_0(\lambda) = 1 - e^{-\lambda} I_0(\lambda).$$

Since $I_0(\lambda) \simeq 1 + \frac{1}{4}\lambda^2$ for $\lambda \ll 1$, we find the correction to ϵ_{xx} due to the finite ion Larmor radius,

$$\epsilon_{xx} = \frac{\omega_{pi}^2}{\Omega_i^2}\left(1 - \frac{3}{4}\lambda_i\right). \tag{6.109}$$

Physically, the correction term λ_i is due to change in the effective electric field experienced by the ion as discussed in Chapter 3,

$$\mathbf{E}_{\perp\ \mathrm{eff}} \simeq [1 - \mathcal{O}(\lambda_i)]\mathbf{E}_\perp. \tag{6.110}$$

The ion polarization drift corrected for the ion finite Larmor radius is

$$\mathbf{v}_{pi} = \frac{e}{M\Omega_i^2}\frac{\partial}{\partial t}\left([1 - \mathcal{O}(\lambda_i)]\mathbf{E}_\perp\right). \tag{6.111}$$

Therefore, the cross-field ion permittivity is subject to a correction of order λ_i,

$$\epsilon_\perp \simeq \frac{\omega_{pi}^2}{\Omega_i^2}[1 - \mathcal{O}(\lambda_i)]. \tag{6.112}$$

The dispersion relation of Alfven wave based on the permittivity in Eq. (6.109) is

$$\omega^2 \simeq k_\parallel^2 V_A^2\left(1 + \frac{3}{4}\lambda_i\right),\ \ \lambda_i \ll 1 \tag{6.113}$$

which can hold even when $k_\perp \gg k_\parallel$ as long as the condition $\lambda_i \ll 1$ is satisfied. In a plasma with an electron temperature comparable with the ion temperature, this should be modified as

$$\omega^2 \simeq k_\parallel^2 V_A^2\left(1 + \frac{3}{4}\lambda_i + \lambda_i\frac{T_e}{T_i}\right) = k_\parallel^2 V_A^2\left[1 + k_\perp^2\rho_s^2\left(1 + \frac{3}{4}\frac{T_i}{T_e}\right)\right], \tag{6.114}$$

where $\rho_s^2 = (T_e/M)/\Omega_i^2$ (ion Larmor radius with the electron temperature) which often appeared in Chapters 3 and 4. Alfven waves can therefore have long parallel, but short perpendicular, wavelengths. The Alfven wave described by Eq. (6.114) is often called *kinetic Alfven wave.*

The correction $k_\perp^2\rho_s^2$ in the kinetic Alfven mode is due to the deviation from the ideal MHD. In ideal MHD, the parallel electric field is vanishingly small,

$$E_\parallel = -ik_\parallel\phi + i\frac{\omega}{c}A_\parallel = 0, \tag{6.115}$$

and the parallel Ampere's law combined with the charge neutrality condition $\nabla\cdot(\mathbf{J}_\perp+\mathbf{J}_\parallel)=0$,

$$\nabla_\parallel \nabla^2 A_\parallel = \frac{4\pi}{c}\nabla\cdot\mathbf{J}_\perp, \tag{6.116}$$

readily yields the dispersion relation $\omega^2=(k_\parallel V_A)^2$ if for the perpendicular current the lowest order ion polarization current

$$\mathbf{J}_\perp = \frac{n_0 e^2}{M\Omega_i^2}\frac{\partial}{\partial t}\mathbf{E}_\perp, \tag{6.117}$$

is substituted. The charge neutrality condition itself is not useful in ideal MHD because both electron and ion density perturbations are vanishing. In order to find corrections of order $(k_\perp\rho_s)^2$, the ideal MHD assumption should be removed and one has to use two-fluid approximation or kinetic analysis as done for the ballooning mode in Chapter 4. In a uniform plasma, the ion density perturbation in the frequency regime $k_\parallel v_{Ti}\ll\omega\ll\Omega_i$ is (*cf.* Eq. (4. 21))

$$n_i = -\frac{e\phi}{T_i}n_0 + e^{-\lambda_i}I_0(\lambda_i)\frac{e\phi}{T_i}n_0 = -\lambda_i\left(1-\frac{3}{4}\lambda_i\right)\frac{e\phi}{T_i}n_0, \tag{6.118}$$

and that of electrons in the low frequency limit $\omega\ll k_\parallel v_{Te}$ is

$$n_e = \left(\phi-\frac{\omega}{ck_\parallel}A_\parallel\right)\frac{e}{T_e}n_0. \tag{6.119}$$

From the charge neutrality condition $n_i=n_e$ and Ampere's law

$$\nabla^2 A_\parallel = -\frac{4\pi}{c}J_\parallel,$$

where the parallel current is largely carried by the electrons,

$$J_\parallel \simeq J_{\parallel e} = \frac{n_0 e^2}{k_\parallel T_e}(-\omega)\left(\phi-\frac{\omega}{ck_\parallel}A_\parallel\right), \tag{6.120}$$

one readily finds the dispersion relation in Eq. (6.114). The parallel electric field associated with the Alfven mode is of order $(k_\perp\rho)^2 k_\parallel\phi$ $(\ll k_\parallel\phi)$.

The kinetic Alfven mode can of course be recovered from the general dispersion relation in Eq. (6.30) provided the following assumption is made, $E_y=0$ because the cross-field electric field is essentially curl-free (electrostatic) in a low β plasma. (Recall that we have assumed $\mathbf{k}_\perp=k_\perp\mathbf{e}_x$.) Then, the dispersion relation in Eq. (6.114) readily follows from

$$\left(k_\parallel^2-\frac{\omega^2}{c^2}\epsilon_{xx}\right)\left(k_\perp^2-\frac{\omega^2}{c^2}\epsilon_{zz}\right)-(k_\perp k_\parallel)^2=0, \tag{6.121}$$

where

$$\epsilon_{xx} \simeq \frac{\omega_{pi}^2}{\Omega_i^2}\left(1 - \frac{3}{4}\lambda_i\right), \quad \epsilon_{zz} \simeq \frac{k_{De}^2}{k_\parallel^2} - \frac{\omega_{pi}^2}{\omega^2} \simeq \frac{k_{De}^2}{k_\parallel^2}. \tag{6.122}$$

In ϵ_{zz}, the ion term is ignorable for the Alfven mode in a low β plasma since $\omega/k_\parallel \simeq V_A \gg c_s$ (the ion acoustic velocity).

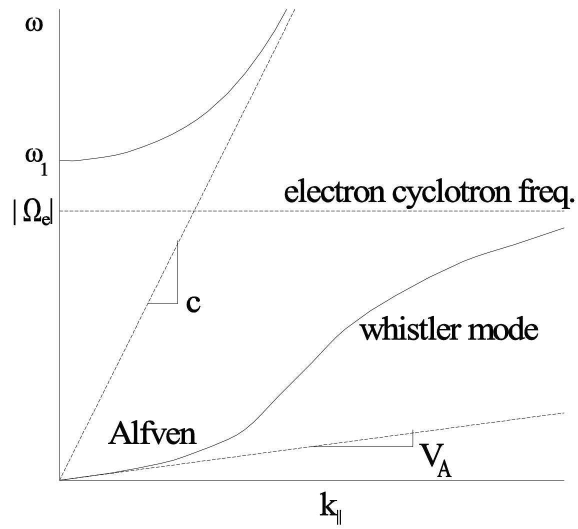

Figure 6.2: Parallel modes with positive helicity. Electron cyclotron resonance $\left(k_\parallel \to \infty\right)$ occurs at $\omega \lesssim |\Omega_e|$. The non-dispersive low frequency mode $\omega \ll \Omega_i$ ($\ll |\Omega_e|$) is the Alfven mode described by $\omega = k_\parallel V_A$.

6.7.2 Modes with Negative Helicity

Ion Cyclotron Mode

The mode described by

$$\frac{c^2 k_\parallel^2}{\omega^2} - \epsilon_{xx} = -i\epsilon_{xy}, \tag{6.123}$$

has negative helicity ($E_x = +iE_y$). Again in the cold plasma approximation, Eq. (6.123) becomes

$$\frac{c^2 k_\parallel^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega\left(\omega + |\Omega_e|\right)} - \frac{\omega_{pi}^2}{\omega\left(\omega - \Omega_i\right)}, \tag{6.124}$$

This exhibits a resonance ($k_\parallel \to \infty$) at the ion cyclotron frequency Ω_i. For $\omega \ll \Omega_i$, Eq. (6.124) also reduces to Alfven mode,

$$\frac{c^2 k_\parallel^2}{\omega^2} \simeq 1 + \frac{\omega_{pe}^2}{\Omega_e^2} + \frac{\omega_{pi}^2}{\Omega_i^2} \simeq \frac{\omega_{pi}^2}{\Omega_i^2},$$

$$\omega^2 = V_A^2 k_\parallel^2.$$

Therefore, in the low frequency limit, Alfven wave can have either helicity, negative or positive, and a plane, linearly polarized Alfven wave can thus be constructed.

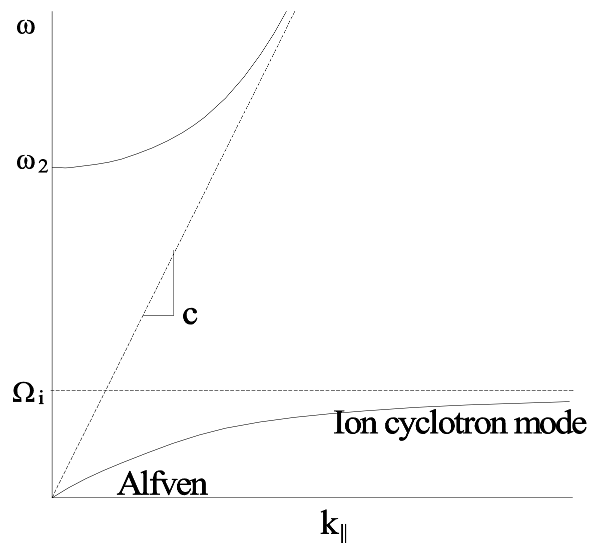

Figure 6.3: Parallel modes with negative helicity. The Alfven mode $\omega = V_A k_\parallel$ with negative helicity exists in the low frequency region $\omega \ll \Omega_i$. The ion cyclotron resonance occurs at $\omega \lesssim \Omega_i$. The cutoff frequency ω_2 is given by $\omega_2 = \left(\sqrt{4\omega_{pe}^2 + \Omega_e^2} - |\Omega_e|\right)/2$.

The resonance at the ion cyclotron frequency is approached from below as in the case of electron cyclotron resonance. Usually in plasma heating by cyclotron resonance, waves are launched from the region where the magnetic field is strong so that $\omega < \Omega_i$. As they propagate along the magnetic field toward weaker field region, waves eventually hit the resonance point (or resonance plane) $\omega = \Omega_i$ where strong absorption takes place. Absorption mechanism (cyclotron damping) is due to thermal effects and will be discussed later. It should be noted that the cutoff frequency $\omega \to 0$ in the limit $k_\parallel \to 0$ corresponds to the Alfven mode.

Electron Mode

In the regime $\omega \gg \omega_{pi} \left(\gg \Omega_i\right)$, Eq. (6.124) yields

$$\frac{c^2 k_\parallel^2}{\omega^2} \simeq 1 - \frac{\omega_{pe}^2}{\omega\left(\omega + |\Omega_e|\right)}.$$

The cutoff frequency of this mode is

$$\omega_2 = \frac{1}{2}\left(\sqrt{\Omega_e^2 + 4\omega_{pe}^2} - |\Omega_e|\right).$$

There exist no modes with negative helicity in the frequency domain

$$\Omega_i < \omega < \frac{1}{2}\left(\sqrt{\Omega_e^2 + 4\omega_{pe}^2} - |\Omega_e|\right).$$

6.7.3 Plasma Oscillation

Finally, the mode described by

$$\epsilon_{zz} = 0, \tag{6.125}$$

is purely electrostatic, since $\mathbf{E} \parallel \mathbf{k}$ both along the external magnetic field. In a cold plasma,

$$\epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2} \simeq 1 - \frac{\omega_{pe}^2}{\omega^2} = 0, \tag{6.126}$$

which indicates electron plasma oscillation having arbitrary wavenumber $k_\parallel$. Electrostatic waves in a magnetized plasma will be discussed in detail in Chapter 7.

6.8 Perpendicular Propagation ($k_\parallel = 0$)

When $k_\parallel = 0$, Eq. (6.31) yields

$$\begin{bmatrix} -\epsilon_{xx} & -\epsilon_{xy} \\ \epsilon_{xy} & \dfrac{k_\perp^2 c^2}{\omega^2} - \epsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = 0, \tag{6.127}$$

and

$$\left(\frac{k_\perp^2 c^2}{\omega^2} - \epsilon_{zz}\right) E_z = 0. \tag{6.128}$$

Corresponding dispersion relations are

$$\left(\frac{ck_\perp}{\omega}\right)^2 = \epsilon_{yy} + \frac{\epsilon_{xy}^2}{\epsilon_{xx}}, \tag{6.129}$$

and

$$\frac{k_\perp^2 c^2}{\omega^2} = \epsilon_{zz}. \tag{6.130}$$

The mode described by Eq. (6.129) is called extraordinary mode while the mode described by Eq. (6.130) is called ordinary mode in analogy to optical waves in anisotropic crystals which exhibit double refraction.

If thermal effects are negligible, the expression obtained earlier for cold plasma may be substituted into the dielectric components, ϵ_{ij}. Eq. (6.129) reduces to

$$\left(\frac{ck_\perp}{\omega}\right)^2 = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} - \frac{\left(\dfrac{\Omega_e \omega_{pe}^2}{\omega^2 - \Omega_e^2} + \dfrac{\Omega_i \omega_{pi}^2}{\omega^2 - \Omega_i^2}\right)^2}{\omega^2 \left(1 - \dfrac{\omega_{pe}^2}{\omega^2 - \Omega_{pe}^2} - \dfrac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}\right)}. \tag{6.131}$$

This does *not* exhibit resonance at $\omega = |\Omega_e|$ and Ω_i. (Showing this is tedious but straightforward. Please try.) Resonance occurs when $\epsilon_{xx} = 0$, or

$$1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} = 0, \tag{6.132}$$

which yields

$$\omega^2 = \omega_{pe}^2 + \Omega_e^2 = \omega_{UH}^2, \tag{6.133}$$

and

$$\omega^2 = \frac{\omega_{pi}^2}{1 + \dfrac{\omega_{pe}^2}{\Omega_e^2}} = \omega_{LH}^2, \tag{6.134}$$

where ω_{UH} is the upper hybrid frequency, and ω_{LH} is the lower hybrid frequency. At these resonance frequencies, the waves essentially become electrostatic. To see this, we take the first equation in Eq. (6.127),

$$\epsilon_{xx} E_x + \epsilon_{xy} E_y = 0. \tag{6.135}$$

When $\epsilon_{xx} = 0$, we must have $E_y = 0$, since ϵ_{xy} remains finite at the resonance frequencies. Recalling that we have assumed $\mathbf{k}_\perp = k_\perp \mathbf{e}_x$, we observe that

$$\mathbf{k} \parallel \mathbf{E}, \text{ and } \boldsymbol{\nabla} \times \mathbf{E} = 0,$$

which indicates that upper and lower hybrid waves are electrostatic.

The cutoff frequencies are the same as those in the parallel modes,

$$\omega_{1,2} = \frac{1}{2}\left(\sqrt{4\omega_{pe}^2 + \Omega_e^2} \pm |\Omega_e|\right).$$

This is because when $k_\perp = 0$ (cutoff condition), Eq. (6.127) requires that

$$\epsilon_{xx}^2 + \epsilon_{xy}^2 = 0,$$

or

$$\epsilon_{xx} = \pm i \epsilon_{xy}.$$

These relationships are identical to Eq. (6.96) corresponding to the cutoff condition of the parallel modes.

At very low frequencies such that $\omega \ll \Omega_i$, Eq. (6.131) becomes

$$\left(\frac{ck_\perp}{\omega}\right)^2 = 1 + \frac{\omega_{pi}^2}{\Omega_i^2} - 2\frac{\omega_{pi}^2}{\omega^2}(k_\perp \rho_i)^2 - 2\frac{\omega_{pe}^2}{\omega^2}(k_\perp \rho_e)^2, \tag{6.136}$$

$$\omega^2 = \frac{V_A^2 + 2\dfrac{T_i + T_e}{M}}{1 + (V_A/c)^2} k_\perp^2. \tag{6.137}$$

This mode is called the magnetosonic mode or compressional Alfven mode corrected for the sound speed V_s,

$$V_{s\perp}^2 = 2\frac{T_i + T_e}{M}, \;\; k_\parallel = 0. \tag{6.138}$$

It is noted that the adiabatic coefficients are $\gamma_i = \gamma_e = 2$ for waves propagating strictly normal to the magnetic field $k_\parallel = 0$. In the case $\omega \ll k_\parallel v_{Te}$, which may occurs for samll but finite $k_\parallel$ (peopagation slightly tilted from $\theta = 90°$), $\gamma_e = 1$, and the sound speed for $k_\perp > k_\parallel$ becomes

$$V_{s\perp}^2 = \frac{2T_i + T_e}{M}, \;\; \omega \ll k_\parallel v_{Te}.$$

Except for the sound speed, the dispersion relation is formally identical to that of the shear Alfven wave propagating along the magnetic field. However, field polarization is entirely different. In order to see the characteristic difference between shear Alfven wave propagating along the magnetic field and compressional Alfven wave propagating perpendicular to the magnetic field, we go back to the field equation,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{6.139}$$

or

$$\mathbf{k} \times \mathbf{E} = \frac{\omega}{c}\mathbf{B}. \tag{6.140}$$

For shear Alfven wave, $\mathbf{k}_{\parallel} = k_{\parallel}\mathbf{e}_z$, both the electric and magnetic field are perpendicular to the unperturbed magnetic field $\mathbf{B}_0$. Physically, such field configuration corresponds to bending of the magnetic field lines. On the other hand, the magnetosonic wave is associated with $\mathbf{k} = \mathbf{k}_{\perp}$, and $\mathbf{E}_{\perp}$, with $\mathbf{k}_{\perp}$ and $\mathbf{E}_{\perp}$ being normal to each other. Then, from Eq. (6.140), we observe that the perturbed magnetic field associated with compressional Alfven wave is parallel to the unperturbed field $\mathbf{B}_0$,

$$\mathbf{B} \parallel \mathbf{B}_0,$$

which creates compression and rarefaction of magnetic field lines.

In Chapter 2, we have seen that the most dangerous MHD modes are incompressible characterized by $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$. Compressional Alfven (magnetosonic) mode is obviously accompanied by field line and thus plasma compression, and $\boldsymbol{\nabla} \cdot \mathbf{v}$ remains finite. The corresponding plasma density perturbation may be found from the continuity equation,

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{v}) = 0, \tag{6.141}$$

where for $\omega \ll \Omega_i$, the dominant cross-field velocity is the $E \times B$ drift,

$$\mathbf{v}_E = c\frac{\mathbf{E} \times \mathbf{B}_0}{B_0^2}, \tag{6.142}$$

where $\mathbf{E}$ is the induction (rather than electrostatic) electric field related to the magnetic perturbation through

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{6.143}$$

If the density n_0 and the magnetic field B_0 are uniform, the perturbed density n_1 can be found as

$$\frac{\partial n_1}{\partial t} = -n_0 \boldsymbol{\nabla} \cdot \mathbf{v} = \frac{n_0}{B_0^2} \mathbf{B}_0 \cdot \frac{\partial \mathbf{B}}{\partial t}, \tag{6.144}$$

or

$$\frac{n_1}{n_0} = \frac{B_\parallel}{B_0}. \tag{6.145}$$

This indicates that plasma density and magnetic perturbations are in phase and equal in relative magnitude which is another manifestation of frozen-in nature of plasma to the magnetic field lines. The appearance of sound speed in the dispersion relation of the magnetosonic mode is thus understandable.

The mode given in Eq. (6.130) is the ordinary electromagnetic wave propagating across the magnetic field. If thermal effects are neglected, $\epsilon_{zz} \simeq 1 - \dfrac{\omega_{pe}^2}{\omega^2}$, and Eq. (6.130) becomes

$$\omega^2 = \omega_{pe}^2 + c^2 k_\perp^2. \tag{6.146}$$

This mode is unaffected by the external magnetic field because of the fact that the electric field assocaited with the mode is along the magnetic field $\mathbf{B}_0$ and the corresponding perturbed current is also along $\mathbf{B}_0$. The Lorentz $\mathbf{v} \times \mathbf{B}_0$ force vanishes in this case, and particle motion remains unaffected by the magnetic field. That the phase velocity is independent of the magnetic field makes this particular mode an extremely convenient probe in measuring the electron density. In typical fusion devices, $f_{pe} = \omega_{pe}/2\pi$ is of order of tens of GHz, and millimeter wave or shorter wavelength infrared laser is required in interferometric plasma density measurements.

Figure 6.4 summarizes the perpendicular modes.

6.9 Propagation at Arbitrary Angle

When both $k_\parallel$ and $k_\perp$ are nonzero, the dispersion relation, Eq. (6.87), must be analyzed as it is. One of the important applications of wave analysis is in radio frequency (rf) heating of plasma, and the discussion presented here will be developed having this particular application in mind. In rf heating, the wave frequency ω is given. (It is the frequency determined by

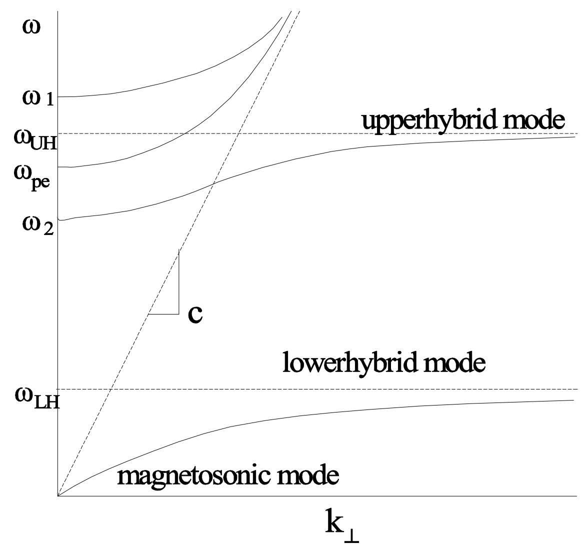

Figure 6.4: Perpendicular modes. Resonance ($k_\perp \to \infty$) occurs at the upperhybrid frequency $\omega_{UH} = \sqrt{\omega_{pe}^2 + \Omega_e^2}$ and the lowerhybrid frequency $\omega_{LH} = \omega_{pi}/\sqrt{1 + (\omega_{pe}/\Omega_e)^2}$ which are slow electrostatic modes and allow efficient wave absorption. The cutoff frequencies ω_1 and ω_2 are identical to those found in parallel propagation.

wave sources.) Therefore, rather than solving Eq. (6.87) for ω, we attempt to find $k_\parallel$ and $k_\perp$ for a given frequency ω. Wave propagation is allowed if real solution for k exists. Otherwise, waves become evanescent being cut off by the plasma itself. However, in several practical applications, waves must go through an evanescent region before fully penetrating deep into plasma core. If the spatial damping due to evanescent region is tolerably small, such excitation mechanism is still a useful method in plasma heating. A typical example is the lower hybrid wave which has successfully been developed for plasma heating and current drive.

Let us introduce the parallel and perpendicular indices of refraction,

$$n_\parallel = \frac{ck_\parallel}{\omega} \quad , \quad n_\perp = \frac{ck_\perp}{\omega}. \tag{6.147}$$

Also, to save subscripts, we let

$$\begin{cases} \epsilon_\perp = \epsilon_{xx} = \epsilon_{yy}, \text{ (in cold plasma)} \\ \\ \epsilon_\parallel = \epsilon_{zz}, \\ \\ \epsilon_X = \epsilon_{xy} = -\epsilon_{yx}. \end{cases} \tag{6.148}$$

Expanding the determinant in Eq.(6.87), we obtain

$$\epsilon_\perp n_\perp^4 + \left(\epsilon_\perp n_\parallel^2 + \epsilon_\parallel n_\parallel^2 - \epsilon_\perp \epsilon_\parallel - \epsilon_\perp^2 - \epsilon_X^2\right) n_\perp^2 + \epsilon_\parallel \left(n_\parallel^2 - \epsilon_\perp\right)^2 + \epsilon_\parallel \epsilon_X^2 = 0. \tag{6.149}$$

The direction of wave injection in most heating applications is perpendicular to the magnetic field or in the radial direction. For this reason, we assume that $n_\parallel$ as well as ω is prescribed ($k_\parallel$ is usually determined by wave excitation mechanisms, such as grill structure in lower hybrid wave exciters. Even if $k_\parallel$ is not well defined, one can always Fourier analyze along the direction of magnetic field.) Eq. (6.149) is an algebraic equation for $n_\perp^2$, and the condition that $n_\perp^2$ be real is given by

$$\left(\epsilon_\perp n_\parallel^2 + \epsilon_\parallel n_\parallel^2 - \epsilon_\perp \epsilon_\parallel - \epsilon_\perp^2 - \epsilon_X^2\right)^2 - 4\epsilon_\perp \left[\epsilon_\parallel \left(n_\parallel^2 - \epsilon_\perp\right)^2 + \epsilon_\parallel \epsilon_X^2\right] > 0. \tag{6.150}$$

$n_\perp^2$ may become negative. In this case, wave becomes evanescent. However, as explained earlier, in a nonuniform plasma, evanescence does not necessarily mean complete cutoff, and if evanescence region is confined at the plasma-vacuum boundary, waves can still tunnel through and penetrate into a plasma. If $n_\perp^2$ becomes complex, on the other hand, wave accessibility is usually much reduced. Also, two otherwise independent modes become degenerate when $n_\perp^2$ is complex. For these reasons, Eq. (6.150) may be regarded as the accessibility condition. In general, the condition is necessary, but not always sufficient, and accessibility of a particular mode should be examined carefully for given experimental conditions, such as the plasma density profile. In the following, accessibility problems of some typical modes of practical interest will be discussed.

6.9.1 Electron Cyclotron Mode

In Section 6.2, we have seen that electron cyclotron wave propagating along the magnetic field is described by the dispersion relation,

$$n_{\parallel}^2 = \frac{c^2 k_{\parallel}^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega\left(\omega - |\Omega_e|\right)}. \tag{6.151}$$

The wave has positive helicity with circular polarization. When the propagation angle θ is slightly titled from the magnetic field, the dispersion relation is modified as

$$n^2 = n_{\parallel}^2 + n_{\perp}^2 = 1 - \frac{\omega_{pe}^2}{\omega\left(\omega - |\Omega_e|\cos\theta\right)}. \tag{6.152}$$

To see this modification, we directly solve the dispersion relation which is rewritten in terms of the total index of refraction n and propagation angle θ as

$$n^4\left(\epsilon_{\perp}\sin^2\theta + \epsilon_{\parallel}\cos^2\theta\right) - n^2\left[\left(\epsilon_{\perp}^2 + \epsilon_X^2\right)\sin^2\theta + \epsilon_{\perp}\epsilon_{\parallel}\left(1+\cos^2\theta\right)\right] + \epsilon_{\parallel}\left(\epsilon_{\perp}^2 + \epsilon_X^2\right) = 0. \tag{6.153}$$

This can readily be solved for n^2,

$$n^2 = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}, \tag{6.154}$$

where

$$\begin{cases} A = \epsilon_{\perp}\sin^2\theta + \epsilon_{\parallel}\cos^2\theta \\ B = (\epsilon_{\perp}^2 + \epsilon_x^2)\sin^2\theta + \epsilon_{\perp}\epsilon_{\parallel}(1+\cos^2\theta) \\ C = \epsilon_{\parallel}\left(\epsilon_{\perp}^2 + \epsilon_x^2\right) \end{cases} \tag{6.155}$$

Note that in Figs. 6.2 and 6.3, there exist at most two solutions for $k_{\parallel}$ or $k_{\perp}$ for a given frequency ω. The two solutions given in Eq. (6.154) indicate that at arbitrary propagation angle θ, we still have two propagation modes.

It is convenient to rewrite Eq. (6.154) as

$$n^2 = 1 - \frac{2(A - B + C)}{2A - B \pm \sqrt{B^2 - 4AC}}. \tag{6.156}$$

For the electron cyclotron wave, we may ignore ion dynamics since $\omega \gg \omega_{pi}$ ($\gg \Omega_i$). Then, the dielectric components assume

$$\epsilon_\perp = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2}, \tag{6.157}$$

$$\epsilon_X = i\frac{|\Omega_e|\,\omega_{pe}^2}{\omega\left(\omega^2 - \Omega_e^2\right)}, \tag{6.158}$$

and

$$\epsilon_\parallel = 1 - \frac{\omega_{pe}^2}{\omega^2}. \tag{6.159}$$

Substituting these into A, B and C, we find

$$n^2 = 1 - \frac{2\omega_{pe}^2\left(1 - \omega_{pe}^2/\Omega_e^2\right)}{2\omega^2\left(1 - \omega_{pe}^2/\Omega_e^2\right) - \Omega_e^2 \sin^2\theta \pm |\Omega_e|\sqrt{D}}, \tag{6.160}$$

where

$$D = \Omega_e^2 \sin^4\theta + 4\omega^2\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^2 \cos^2\theta. \tag{6.161}$$

If $\theta = 0$ (parallel propagation), we recover the familiar result for the parallel electron cyclotron mode,

$$n_\parallel^2 = \left(\frac{ck_\parallel}{\omega}\right)^2 = 1 - \frac{\omega_{pe}^2}{\omega\left(\omega \pm |\Omega_e|\right)}. \tag{6.162}$$

Also, if $\theta = \pi/2$,

$$n_\perp^2 = \left(\frac{ck_\perp}{\omega}\right)^2 = 1 - \frac{\omega_p^2\left(\omega^2 - \omega_p^2\right)}{\omega^2\left(\omega^2 - \omega_p^2 - \Omega_e^2\right)}, \tag{6.163}$$

and

$$n_\perp^2 = 1 - \frac{\omega_{pe}^2}{\omega^2}, \tag{6.164}$$

which are also consistent with those found in Section 6.3.

The dispersion relation, Eq. (6.152) derives from Eq. (6.160) if $\sin^2\theta$ is sufficiently small. Since D is positive definite, the accessibility condition for electron cyclotron wave may be imposed by

$$n^2 > 0, \tag{6.165}$$

everywhere along the wave trajectory, that is, from the plasma edge where $\omega_p = 0$ to the plasma core where ω_p is maximum, and cyclotron resonance is aimed at. In toroidal devices such as tokamaks and stellarators, the toroidal magnetic field varies being proportional to

$1/R$, where R is the radius from the toroidal axis. Therefore, if the wave frequency ω is so chosen that cyclotron resonance is to take place at the plasma minor center, accessibility can be achieved only from the inner side. Often this requirement causes technical difficulties since the inner side of toroidal devices is crowded with mechanical structures (toroidal coils, poloidal windings, *etc.*).

Electron cyclotron resonance heating has become feasible rather recently after development of high power ($\gtrsim$ 100 kW), high frequency ($\simeq$ 30 GHz) microwave sources (*e.g.* gyrotron). Although heating reactor scale devices will require a large total power at much higher frequencies (because of higher magnetic fields), ECR heating is one of the promising auxiliary heating methods to achieve ignition temperatures. One restriction of ECR heating is its density limit at the region when cyclotron absorption takes place. The frequency ω must be above the lower cutoff frequency

$$\omega > \omega_{c1} = \frac{\sqrt{4\omega_{pe}^2 + \Omega_e^2} - |\Omega_e|}{2}$$

everywhere along the wave trajectory. (We note that the cutoff frequencies to make $n^2 = 0$ in Eq. (6.160) is independent of the propagation angle, θ.) Therefore, the maximum allowable density in terms of the plasma frequency ω_{pe} is

$$\omega_{pe}^2 \text{ (at resonance) } \leq 2\Omega_e^2.$$

6.9.2 Lowerhybrid Wave

As seen in Section 6.6, the lower hybrid resonance frequency

$$\omega_{LH} = \frac{\omega_{pi}}{\sqrt{1 + (\omega_{pe}/\Omega_e)^2}}, \tag{6.166}$$

is essentially proportional to the ion plasma frequency and resonance region is a point rather than a plane as in the case of cyclotron resonance. Fortunately, in the sense of geometrical optics, any waves tend to refract toward lower phase velocity region. At the resonance region, the wavenumber k increases and thus waves tend to converge.

In the case of lower hybrid wave, it is impossible to completely avoid evanescent region. However, as we will see shortly, the evanescent region is limited to the plasma edge where the plasma density is low. Waves can easily tunnel through the evanescent region and penetrate into the plasma core.

Accessibility of lower hybrid waves has been first analyzed by Golant, and we follow the analysis developed by him. If we assume $|\Omega_e| \gg \omega \gg \Omega_i$ appropriate for the lower hybrid wave, the dielectric components become

$$\begin{cases} \epsilon_\perp = 1 + \dfrac{\omega_{pe}^2}{\Omega_e^2} - \dfrac{\omega_{pi}^2}{\omega^2} \ \ (\geq 0) \\ \\ \epsilon_X = i\dfrac{\omega_{pe}^2}{\omega\left|\Omega_e\right|} - i\dfrac{\Omega_i \omega_{pi}^2}{\omega^3} \simeq i\dfrac{\omega_{pe}^2}{\omega\left|\Omega_e\right|} \\ \\ \epsilon_\parallel = 1 - \dfrac{\omega_{pe}^2}{\omega^2} \simeq \dfrac{\omega_{pe}^2}{\omega^2} \ \ (< 0) \end{cases} \tag{6.167}$$

As the accessibility condition, we adopt Eq. (6.150) which only ensures that $n_\perp^2$ be real, but not necessarily $n_\perp^2$ be positive. The inequality, Eq. (6.150), can easily be solved for the range of $n_\parallel^2$,

$$n_\parallel^2 > \frac{\epsilon_\perp\left(\epsilon_\perp - \epsilon_\parallel\right)^2 + \left(\epsilon_\perp - \epsilon_\parallel\right)\epsilon_{xy}^2 + 2\sqrt{\epsilon_\perp \epsilon_\parallel \epsilon_X^2\left[\left(\epsilon_\perp - \epsilon_\parallel\right)^2 + \epsilon_X^2\right]}}{\left(\epsilon_\perp - \epsilon_\parallel\right)^2}. \tag{6.168}$$

Since

$$\left|\epsilon_\parallel\right| \gg \left|\epsilon_\perp\right| \ \ , \ \ \left|\epsilon_X\right|$$

we may approximate Eq. (6.168) as

$$n_\parallel^2 > \epsilon_\perp + \frac{\epsilon_X^2}{\epsilon_\parallel} + 2\sqrt{\frac{\epsilon_\perp}{\epsilon_\parallel}}\epsilon_X = \left(\sqrt{\epsilon_\perp} + \frac{\left|\epsilon_X\right|}{\sqrt{\left|\epsilon_\parallel\right|}}\right)^2. \tag{6.169}$$

The RHS of the above equation becomes maximum at a distance x_m from the plasma edge where the plasma density $n_0(x)$ satisfies

$$\omega_{pe}^2(x_m) = \frac{\omega_r^4}{\Omega_e^2 + \omega_r^2}. \tag{6.170}$$

Here, ω_r (= const.) is given by

$$\omega_r^2 = \frac{\omega^2 \Omega_e^2}{\dfrac{m}{M}\Omega_e^2 - \omega^2}, \tag{6.171}$$

which is the electron plasma frequency at the resonance point, $x = x_r$,

$$\omega_r = \omega_{pe}(x_r) \;\;,\;\; x_m < x_r. \tag{6.172}$$

The maximum value of RHS takes a simple value and is given by

$$n_\parallel^2 > 1 + \frac{\omega_{pe}^2(x_r)}{\Omega_e^2}. \tag{6.173}$$

This is the condition to avoid complex solutions of $n_\perp^2$, and can be regarded as the accessibility condition for lower hybrid waves.

At the plasma edge where the plasma density is zero, we have the vacuum solution

$$n_\perp^2 + n_\parallel^2 = 1.$$

Therefore, when the accessibility condition is satisfied ($n_\parallel^2 > 1$), the wave at the edge region should be evanescent with respect to transverse propagation,

$$n_\perp^2 = 1 - n_\parallel^2 < 0.$$

The distance from the plasma edge beyond which the wave becomes transparent $n_\perp^2 > 0$ can readily be found from the dispersion relation, Eq. (6.149). In the edge region, $\epsilon_\perp \simeq 1$ can still be assumed, but $\epsilon_\parallel$, ϵ_x can quickly become large. When, Eq. (6.149) is written in the form

$$n_\perp^4 + Bn_\perp^2 + C = 0, \tag{6.174}$$

the condition for $n_\perp^2$ to have a positive real solution is simply

$$C < 0. \tag{6.175}$$

Note that the lower hybrid wave corresponds to the "slow wave" solution,

$$n_\perp^2 = \frac{1}{2}\left[-B + \sqrt{B^2 - 4C}\right]. \tag{6.176}$$

Therefore, the transition position becomes

$$\omega^2 = \omega_{pe}^2(x_t). \tag{6.177}$$

Since ω is of the order of ion plasma frequency at the plasma core where the density is maximum, we find that the transition distance from the plasma edge is extremely small. The inevitable existence of evanescent region in launching lower hybrid waves into a plasma is not expected to be problematic as successfully demonstrated in several recent experiments.

6.9.3 Ion Cyclotron Wave

RF heating with the frequency in the range of ion cyclotron frequency is one of the oldest methods in fusion research. This is due to the obvious reason that high power rf sources at typically tens of MHz are readily available. ICR heating is still in active use and can well compete with neutral beam injection in attaining high ion temperatures. Since the frequency is considerably lower than electron cyclotron and lower hybrid schemes, wave excitation based on waveguides cannot be used. Ion cyclotron waves are usually excited by antennas located near the plasma edge. Coupling efficiency is expected to be higher, the closer the antenna is to the plasma. This may cause impurity problems, for it is practically impossible to completely insulate the antenna from the plasma. It is commonly observed the impurity level significantly increases when attempts are made to increase rf power. Of course, impurities (high Z ions) directly contribute to plasma energy loss through radiation, and deteriorate energy confinement times.

Analysis on accessibility of ion cyclotron resonance is less involved than ECR and LHR. $n_\perp^2$ remains real in the approximation to be used. As in the case of LHR, mild evanescence exists, but it can be tolerated.

Ion cyclotron wave is characterized by the frequency

$$\omega \lesssim \Omega_i \ll |\Omega_e|$$

Therefore, $\epsilon_\perp$, ϵ_X, and $\epsilon_\parallel$ in Eq. (6.149) may be assumed to be

$$\begin{cases} \epsilon_\perp \simeq 1 + \dfrac{\omega_{pe}^2}{\Omega_e^2} - \dfrac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} \simeq -\dfrac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}, \\ \epsilon_X \simeq -i\dfrac{\Omega_i \omega_{pi}^2}{\omega\left(\omega^2 - \Omega_i^2\right)} - i\dfrac{\omega_{pi}^2}{\omega \Omega_i}, \\ \epsilon_\parallel \simeq -\dfrac{\omega_{pe}^2}{\omega^2}, \end{cases} \tag{6.178}$$

and the solutions for $n_\perp^2$ are given by

$$\begin{aligned} n_\perp^2 \;=\;& \frac{1}{2\epsilon_\perp}\frac{\omega_{pe}^2}{\omega^2}\left(n_\parallel^2 - \epsilon_\perp\right) \\ & \pm \frac{1}{2\epsilon_\perp}\left[\frac{\omega_{pe}^4}{\omega^4}\left(n_\parallel^2 - \epsilon_\perp\right)^2 - \frac{4\omega_{pi}^2}{\omega^2 - \Omega_i^2}\frac{\omega_{pe}^2}{\omega^2}\left(n_\parallel^4 + \frac{2\omega_{pi}^2}{\omega^2 - \Omega_i^2}n_\parallel^2 - \frac{\omega_{pi}^4}{\Omega_i^2\left(\omega^2 - \Omega_i^2\right)}\right)\right]^{1/2} \end{aligned} \tag{6.179}$$

In the square root, the first term dominates over the rest. Then, the two solutions are

$$n_\perp^2 \simeq \frac{1}{\epsilon_\perp}\frac{\omega_{pe}^2}{\omega^2}\left(n_\parallel^2 - \epsilon_\perp\right), \tag{6.180}$$

and

$$n_\perp^2 \simeq \frac{n_\parallel^4\left(\Omega_i^2 - \omega^2\right) - 2n_\parallel^2\omega_{pi}^2 + \omega_{pi}^4/\Omega_i^2}{n_\parallel^2\left(\omega^2 - \Omega_i^2\right) + \omega_{pi}^2}. \tag{6.181}$$

The first solution is evanescent unless $n_\parallel^2$ is very large. The second solution corresponds to ion cyclotron wave propagating at an angle with respect to the magnetic field. When $n_\perp = 0$ (parallel propagation), we recover the solution previously found,

$$n_\parallel^2 \simeq \frac{\omega_{pi}^2}{\Omega_i(\Omega_i \pm \omega)}, \tag{6.182}$$

which is in the form of

$$n_\parallel^2 = \epsilon_\perp \pm i\epsilon_X. \tag{6.183}$$

From Eq. (6.181), it can be seen that cutoff ($n_\perp^2 = 0$) occurs when

$$\omega_{pi}^2(x_c) = \omega(\Omega_i \pm \omega)n_\parallel^2, \tag{6.184}$$

and the resonance when

$$\omega_{pi}^2(x_r) = \left(\Omega_i^2 - \omega^2\right)n_\parallel^2. \tag{6.185}$$

At the plasma edge where $\omega_{pi} = 0$, the wave is evanescent, $n_\perp^2 = -n_\parallel^2 < 0$. (We do not recover the vacuum solution $n^2 = 1$ since in the expression for $\epsilon_\perp$ and $\epsilon_\parallel$, the vacuum term 1 has been omitted.) Wave propagation is allowed ($n_\perp^2 > 0$) in the region

$$\omega\left(\Omega_i - \omega\right)n_\parallel^2 < \omega_{pi}^2(x) < \left(\Omega_i^2 - \omega^2\right)n_\parallel^2. \tag{6.186}$$

When $\omega \lesssim \Omega_i$, the ratio between the density at the resonance and that at the cutoff is approximately equal to 2. This should be compared with $\sqrt{M/m}$ (M/m being the ion-electron mass ratio) found for the lower hybrid wave. The evanescent region in ion cyclotron wave is substantially larger than that in lower hybrid wave. However, this does not necessarily mean that the wave damping due to evanescence is correspondingly strong. In the evanescent region, $n_\perp^2$ may be approximated by

$$n_\perp^2 \simeq -n_\parallel^2 + \frac{3\omega_{pi}^2}{\Omega_i^2 - \omega^2}\ (< 0). \tag{6.187}$$

Thus, the spatial damping factor $|k_\perp|$ in exp $(-|k_\perp|x)$ is substantially reduced from the vacuum value $(|k_y| = k_\parallel)$.

Ion cyclotron resonance heating based on the mechanism described above is best suited to cylindrical plasmas, such as those in tandem mirror experiments. In addition to transverse resonance, longitudinal resonance $(n_\parallel^2 \to \infty)$ can be achieved if the magnetic field itself is nonuniform along the plasma column (concept of magnetic beach).

In toroidal plasmas, the toroidal magnetic field is strongly nonuniform and accessibility of ion cyclotron waves encounters similar difficulties as electron cyclotron waves. For this reason, ICR heating based on slow wave (the branch of conventional ion cyclotron mode) is seldom used particularly in tokamak research. Instead, fast wave (magnetosonic wave) with $\omega \simeq 2\Omega_i$ is commonly used. Since fast wave heating requires knowledge of wave propagation in a plasma with finite ion temperature, we defer this important topic until a later chapter.

6.10 Kinetic Effects

The cold plasma approximation employed in the preceding Sections is applicable when the phase velocity (or the Doppler shifted phase velocity) is sufficiently remote from the thermal velocities of electrons and ions. The components of the dielectric tensor ϵ_{ij} all contain the plasma dispersion function $Z(\zeta_n)$ or its derivative where

$$\zeta_n = \frac{\omega - n\Omega}{k_\parallel v_T}.$$

Cold plasma approximation pertains to the limit $|\zeta_n| \gg 1$ which allows us to use the asymptotic expansion of the plasma dispersion function,

$$Z(\zeta_n) \simeq -\frac{1}{\zeta_n} - \frac{1}{2\zeta_n^3} \cdots + i\sqrt{\pi}e^{-\zeta_n^2}.$$

The imaginary residue term, though small, indicates wave damping through Landau $(n = 0)$ and cyclotron $(n \neq 0)$ resonance.

6.10.1 Cyclotron Damping

In this Section, we analyze the dispersion relation of the electron cyclotron mode

$$\left(\frac{ck_\parallel}{\omega}\right)^2 = \epsilon_{xx} + i\epsilon_{xy}, \tag{6.188}$$

by retaining kinetic resonance. Ion contributions to ϵ_{xx} and ϵ_{xy} can be ignored and we approximate them by

$$\epsilon_{xx} \simeq 1 + \frac{1}{2}\left(\frac{\omega_{pe}}{\omega}\right)^2 \frac{\omega}{k_\parallel v_{Te}}\left[Z(\zeta_{e,+1}) + Z(\zeta_{e,-1})\right],$$

$$\epsilon_{xy} \simeq i\frac{1}{2}\left(\frac{\omega_{pe}}{\omega}\right)^2 \frac{\omega}{k_\parallel v_{Te}}\left[Z(\zeta_{e,+1}) - Z(\zeta_{e,-1})\right].$$

Substitution into Eq. (6.188) yields

$$\left(\frac{ck_\parallel}{\omega}\right)^2 = 1 + \frac{\omega_{pe}^2}{\omega k_\parallel v_{Te}} Z(\zeta_{e,-1}), \tag{6.189}$$

where

$$\zeta_{e,-1} = \frac{\omega - |\Omega_e|}{k_\parallel v_{Te}}.$$

If $|\zeta_{e,-1}| \gg 1$, Eq. (6.189) reduces to

$$\left(\frac{ck_\parallel}{\omega}\right)^2 \simeq 1 - \frac{\omega_{pe}^2}{\omega(\omega - |\Omega_e|)} + i\sqrt{\pi}\frac{\omega_{pe}^2}{\omega k_\parallel v_{Te}} \exp\left[-\left(\frac{\omega - |\Omega_e|}{k_\parallel v_{Te}}\right)^2\right]. \tag{6.190}$$

Solutions for ω or $k_\parallel$ must be complex which indicates wave damping. (Recall that we have assumed a propagation function $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$.) In steady state plasma heating experiments, ω is real and Im $k_\parallel$ (> 0) becomes the spatial damping factor. In the lowest order, the solution for $k_\parallel$ is given by

$$k_\parallel = k_{\parallel 0} + ik_i,$$

where

$$k_{\parallel 0} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{pe}^2}{\omega(\omega - |\Omega_e|)}}, \tag{6.191}$$

$$k_i = \frac{\sqrt{\pi}}{2}\frac{\omega\omega_{pe}^2}{c^2 k_{\parallel 0}^2 v_{Te}} \exp\left[-\left(\frac{\omega - |\Omega_e|}{k_\parallel v_{Te}}\right)^2\right]. \tag{6.192}$$

At the electron cyclotron resonance, $\omega = |\Omega_e|$, the plasma dispersion function takes a simple value $Z(0) = i\sqrt{\pi}$. Then Eq. (6.189) becomes

$$\left(\frac{ck_\parallel}{\omega}\right)^2 \simeq 1 + i\sqrt{\pi}\frac{\omega_{pe}^2}{|\Omega_e|k_\parallel v_{Te}}. \tag{6.193}$$

In magnetic fusion plasmas, $|\Omega_e| \simeq \omega_{pe}$. Also, $k_\parallel \ll k_{De} = \sqrt{2}\omega_{pe}/v_{Te}$. Then the unity can be ignored and we have

$$k_\parallel = \pi^{1/6} e^{i\pi/6} \frac{(|\Omega_e|\,\omega_{pe}^2)^{1/3}}{(c^2 v_{Te})^{1/3}}. \tag{6.194}$$

Im $k_\parallel$ is comparable with the real part indicating strong spatial damping.

In the case of second harmonic electron cyclotron resonance heating, $\omega \simeq 2|\Omega_e|$, the term $Z(\zeta_{e,-2})$ contained in ϵ_{xx} and ϵ_{xy} plays the dominant role. The damping mechanism is the same as in the fundamental mode except that a target plasma should have appreciable electron temperature for the second harmonic heating to be effective. This is because for a cold plasma, the finite electron Larmor radius effect is vanishingly small, $I_2(\lambda_e)/\lambda_e \propto \lambda_e$.

The ion cyclotron resonance can be analyzed in a similar manner.

6.10.2 Whistler Instability due to Temperature Anisotropy

The Whistler mode has been first observed as naturally occurring radiation from the ionospheric plasma. The mechanism of self-excitation (instability) of the mode is generally attributed to the anisotropic electron temperature $T_{e\perp} > T_{e_\parallel}$ which is plausible in the ionospheric plasma confined by the earth mirror magnetic field. In a mirror field, particles can freely escape through the mirror throats and the velocity distribution is in general non Maxwellian. Here we assume that the electron velocity distribution is bi Maxwellian characterized by two temperatures, $T_{e\perp}$ and $T_{e_\parallel}$,

$$f_e(v_\perp^2, v_\parallel) = \frac{m}{2\pi T_{e\perp}} \exp\left(-\frac{mv_\perp^2}{2T_{e\perp}}\right)\sqrt{\frac{m}{2\pi T_{e\parallel}}} \exp\left(-\frac{mv_\parallel^2}{2T_{e\parallel}}\right). \tag{6.195}$$

The dielectric components ϵ_{xx} and ϵ_{xy} in Eqs. (6.31) and (6.32) were for Maxwellian distribution and cannot be used for the present purpose. Instead, ϵ_{xx} and ϵ_{xy} have to be calculated from Eq. (6.26),

$$\epsilon_{xx} = 1 + \frac{\omega_{pe}^2}{\omega^2} \sum_n \int d^3v \frac{v_\perp \left(\frac{n}{\Lambda_e}\right)^2 J_n^2(\Lambda_e)}{\omega - k_\parallel v_\parallel - n\Omega_e} \left((\omega - k_\parallel v_\parallel) \frac{\partial f_e}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_e}{\partial v_\parallel} \right), \tag{6.196}$$

where ion contributions have been ignored. For $k_\perp = 0$, we have

$$\epsilon_{xx} = 1 - \left(\frac{\omega_{pe}}{\omega}\right)^2 + \frac{1}{2}\left(\frac{\omega_{pe}}{\omega}\right)^2 \left[\frac{|\Omega_e|}{k_\parallel v_{T_\parallel e}} Z(\zeta_{e,-1}) - \frac{T_{e\perp}}{2T_{e\parallel}} Z'(\zeta_{e,-1}) \right]$$

$$-\frac{1}{2}\left(\frac{\omega_{pe}}{\omega}\right)^2 \left[\frac{|\Omega_e|}{k_\parallel v_{T_\parallel e}} Z(\zeta_{e,1}) + \frac{T_{e\perp}}{2T_{e\parallel}} Z'(\zeta_{e,1}) \right], \tag{6.197}$$

where $v_{T_\parallel e} = \sqrt{2T_{e\parallel}/m}$ and $Z'(\zeta) = -2[1 + \zeta Z(\zeta)]$ is the derivative of the plasma dispersion function. Similarly,

$$\epsilon_{xy} = i\frac{1}{2}\left(\frac{\omega_{pe}}{\omega}\right)^2 \left[-\frac{|\Omega_e|}{k_\parallel v_{T_\parallel e}} Z(\zeta_{e,1}) - \frac{T_{e\perp}}{2T_{e\parallel}} Z'(\zeta_{e,1}) - \frac{|\Omega_e|}{k_\parallel v_{T_\parallel e}} Z(\zeta_{e,-1}) + \frac{T_{e\perp}}{2T_{e\parallel}} Z'(\zeta_{e,-1}) \right]. \tag{6.198}$$

Substituting these into the dispersion relation

$$\left(\frac{ck_\parallel}{\omega}\right)^2 = \epsilon_{xx} + i\epsilon_{xy},$$

yields

$$\left(\frac{ck_\parallel}{\omega}\right)^2 = 1 + \left(\frac{\omega_{pe}}{\omega}\right)^2 \left[\frac{\omega}{k_\parallel v_{T_\parallel e}} Z(\zeta_{e,-1}) + \frac{1}{2}\left(1 - \frac{T_{e\perp}}{T_{e\parallel}}\right) Z'(\zeta_{e,-1}) \right]. \tag{6.199}$$

Assuming $|\zeta_{e,-1}| \gg 1$, we thus obtain

$$\left(\frac{ck_\parallel}{\omega}\right)^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega - |\Omega_e|)} + i\sqrt{\pi}\left(\frac{\omega_{pe}}{\omega}\right)^2 \left[|\Omega_e| \left(1 - \frac{T_{e\perp}}{T_{e\parallel}}\right) + \omega \frac{T_{e\perp}}{T_{e\parallel}} \right] \frac{e^{-\zeta_{e,-1}^2}}{k_\parallel v_{T_\parallel e}}. \tag{6.200}$$

If $T_{e\perp} > T_{e\parallel}$, the last resonance term can change its sign from positive to negative and a whistler instability (Im $\omega > 0$) occurs. Since the frequency ω is much smaller than the electron cyclotron frequency, a slight temperature anisotropy is sufficient to excite the whistler mode.

An anisotropic distribution function is thermodynamically unstable in the sense that it has freedom to relax to more stable isotropic Maxwellian distribution. A plasma can approach thermodynamic equilibrium through particle collisions. However, if the growth rate of plasma instability is much larger than the collision frequency, the relaxation process is accelerated by the instability. In the case of the whistler instability, it enhances the temperature equilibriation by transferring the perpendicular energy ($T_{e\perp}$) to parallel energy ($T_{e\parallel}$) at a rate much faster than expected from the collisional process.

6.10.3 Weibel Instability

Another well known instability caused by temperature anisotropy is the Weibel instability. This is an electromagnetic instability in *unmagnetized* plasma and may occur when a plasma is heated anisotropically as in strong rf heating. To analyze the instability, we assume the geometry shown in Fig. 6.4. An electromagnetic wave propagates in the z direction with a wavevector k and electric field in the x direction, $\mathbf{E} = E_x\mathbf{e}_x$. The unperturbed electron distribution function integrated over the ignorable v_y is assumed to be bi Maxwellian,

$$f_0(v_x, v_z) = \sqrt{\frac{m}{2\pi T_\perp}} \exp\left(-\frac{mv_x^2}{2T_\perp}\right) \sqrt{\frac{m}{2\pi T_\parallel}} \exp\left(-\frac{mv_z^2}{2T_\parallel}\right), \tag{6.201}$$

where $\perp$ and $\parallel$ are with respect to the direction of wave propagation. After eliminating the perturbed magnetic field from the linearized Vlasov equation for the electron, we have

$$-i(\omega - kv_z)f_1 - \frac{e}{m}\left[\mathbf{E}\left(1 - \frac{\mathbf{k}\cdot\mathbf{v}}{\omega}\right) + \frac{\mathbf{k}}{\omega}(\mathbf{E}\cdot\mathbf{v})\right]\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{6.202}$$

where

$$\frac{\partial f_0}{\partial \mathbf{v}} = -\left(\frac{m}{T_\perp}\mathbf{v}_x + \frac{m}{T_\parallel}\mathbf{v}_z\right) f_0. \tag{6.203}$$

Solving Eq. (6.202) for f_1,

$$f_1 = -i\frac{e}{T_\perp}\frac{E_x v_x}{\omega - kv_z}\left[1 + \left(\frac{T_\perp}{T_\parallel} - 1\right)\frac{kv_z}{\omega}\right] f_0. \tag{6.204}$$

Only the current density in the x direction is non vanishing and given by

$$\begin{aligned} J_x &= -en_0 \textstyle\int v_x f_1 dv_x dv_z \\ &= -i\frac{n_0 e^2}{mkv_{Te}}\left[\frac{T_\perp}{T_\parallel}Z(\zeta_e) + \frac{1}{\zeta_e}\left(\frac{T_\perp}{T_\parallel} - 1\right)\right] E_x, \end{aligned} \tag{6.205}$$

where $v_{Te} = \sqrt{2T_\parallel/m}$ and $\zeta_e = \omega/kv_{Te}$. This defines a scalar conductivity

$$\sigma = \frac{J_x}{E_x} = -i\frac{n_0 e^2}{mkv_{Te}}\left[\frac{T_\perp}{T_\parallel}Z(\zeta_e) + \frac{1}{\zeta_e}\left(\frac{T_\perp}{T_\parallel} - 1\right)\right], \tag{6.206}$$

and the dielectric constant,

$$\epsilon = 1 + i\frac{4\pi}{\omega}\sigma = 1 + \frac{\omega_{pe}^2}{\omega kv_{Te}}\left[\frac{T_\perp}{T_\parallel}Z(\zeta_e) + \frac{1}{\zeta_e}\left(\frac{T_\perp}{T_\parallel} - 1\right)\right]. \tag{6.207}$$

Then the dispersion relation for electromagnetic mode is given by

$$\left(\frac{ck}{\omega}\right)^2 = \epsilon = 1 + \frac{\omega_{pe}^2}{\omega kv_{Te}}\left[\frac{T_\perp}{T_\parallel}Z(\zeta_e) + \frac{1}{\zeta_e}\left(\frac{T_\perp}{T_\parallel} - 1\right)\right]. \tag{6.208}$$

Let us first consider isotropic case, $T_\perp = T_\parallel$,

$$\left(\frac{ck}{\omega}\right)^2 = 1 + \frac{\omega_{pe}^2}{\omega kv_{Te}}Z(\zeta_e). \tag{6.209}$$

This describes electromagnetic modes in an unmagnetized plasma with kinetic corrections. Evidently, it formally agrees with the dispersion relation of the ordinary mode in a magnetized plasma in Eq. (6.146). For $\zeta_e \gg 1$, we indeed recover the familiar form,

$$\left(\frac{ck}{\omega}\right)^2 = 1 - \left(\frac{\omega_{pe}}{\omega}\right)^2 \text{ or } \omega^2 = \omega_{pe}^2 + (ck)^2.$$

In the opposite limit $\zeta_e \ll 1$, we have

$$\left(\frac{ck}{\omega}\right)^2 = 1 + i\sqrt{\pi}\frac{\omega_{pe}^2}{\omega kv_{Te}} \simeq i\sqrt{\pi}\frac{\omega_{pe}^2}{\omega kv_{Te}},$$

which yields

$$k = \pi^{1/6}e^{i\pi/6}\left(\frac{\omega\omega_{pe}^2}{c^2 v_{Te}}\right)^{1/3}. \tag{6.210}$$

The quantity

$$\delta_a = \left(\frac{c^2 v_{Te}}{\omega\omega_{pe}^2}\right)^{1/3}, \tag{6.211}$$

is known as the anomalous (or kinetic) skin depth. It can exceed the conventional collisionless skin depth

$$\delta = \frac{c}{\omega_{pe}}, \ (\omega \ll \omega_{pe}) \tag{6.212}$$

provided $\omega < v_{Te}\omega_{pe}/c$. (Of course, at extremely low frequency $\omega < \nu_c$ (the electron collision frequency), the collisional skin depth becomes dominant,

$$\delta_c = \frac{c}{\omega_{pe}}\sqrt{\frac{\nu_c}{\omega}}, \tag{6.213}$$

which emerges from the dispersion relation

$$\left(\frac{ck}{\omega}\right)^2 = 1 + i\frac{4\pi}{\omega}\sigma_c = 1 + i\frac{\omega_{pe}^2}{\omega\nu_c}, \ (\omega \ll \nu_c)$$

where $\sigma_c = n_0e^2/m\nu_c$ is the collisional conductivity.)

We now return to the Weibel instability. When $\zeta_e \gg 1$, Eq. (6.208) reduces to

$$\left(\frac{ck}{\omega}\right)^2 \simeq 1 - \left(\frac{\omega_{pe}}{\omega}\right)^2 - \frac{\omega_{pe}^2k^2v_{Te}^2}{\omega^4}\frac{T_\perp}{T_\parallel}. \tag{6.214}$$

There always exists a negative solution for ω^2 which indicates a purely growing instability. The maximum growth rate is of the order of

$$\gamma_{\max} \simeq \frac{\sqrt{T_\perp/m}}{c}\omega_{pe}. \tag{6.215}$$

The distribution function with anisotropic temperatures may be replaced by two cold electron clouds drifting in opposite directions along the x axis,

$$f_0(\mathbf{v}) = \frac{1}{2}\left[\delta(v_x - V) + \delta(v_x + V)\right]\delta(v_y)\delta(v_z). \tag{6.216}$$

(The factor 1/2 is for the normalization, $\int f_0 d^3v = 1$.) In this case, the growth rate is given by

$$\gamma_{\max} \simeq \frac{V}{c}\omega_{pe}. \tag{6.217}$$

However, as we will see in Chapter 8, such two-stream distribution function is unstable against rapidly growing electrostatic instability with a growth rate far exceeding that of the Weibel instability.

The physical mechanism of the Weibel instability is in the magnetic Lorentz force. In the geometry assumed, the perturbed magnetic field is in the y direction and the Lorentz force $\mathbf{v} \times \mathbf{B}$ directed in the z direction causes the filamentation of the plasma. If the unperturbed distribution is isotropic, the magnetic force term

$$(\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_0}{\partial \mathbf{v}}, \tag{6.218}$$

in the Vlasov equation identically vanishes. For the anisotropic distribution assumed, this term remains finite,

$$m\left(\frac{1}{T_\perp}-\frac{1}{T_\parallel}\right)v_x v_z B_y f_0, \tag{6.219}$$

and contributes to the perturbation in the distribution function.