Chapter 5

Short Wavelength Modes

5.1 Introduction

In short wavelength regime \((k_\perp \rho_i)^2 > 1\) (cross field wavelength smaller than the ion Larmor radius), the toroidal ITG mode tends to be stabilized because ion dynamics tends to be adiabatic,

\[ n_i \rightarrow -\frac{e\phi}{T_i} n_0, \quad (k_\perp \rho_i)^2 \gg 1. \]

However, the slab ITG mode persists in toroidal geometry provided the toroidicity is not too strong, \(L_n/R \lesssim 0.15\). The ion Larmor radius \(\rho_i\) and the electron skin depth \(c/\omega_{pe}\) are related through

\[ \frac{\rho_i \omega_{pe}}{c} = \sqrt{\frac{m_i}{m_e} \frac{4\pi n T_i}{B^2}} = \sqrt{\frac{m_i}{2m_e} \beta_i}, \]

where \(\beta_i = 8\pi n T_i / B^2\) is the ion \(\beta\) factor. In deuterium plasma with \(\beta_i = 1\%\), the ratio is approximately 4.3. As the wavelength decreases or \(k_\perp \rho_i\) increases, the lower unstable edge of electron mode should appear at \(k_\perp \simeq \omega_{pe}/c\). In this regime, neither ions nor electrons are adiabatic. In addition, stability analysis must be done in terms of electromagnetic mode equation as evident from the appearance of the skin depth.

Further decrease in the wavelength brings in the regime \(k_\perp \lambda_{De} \simeq 1\) where \(\lambda_{De}\) is the electron Debye length. It is noted that in tokamaks, \(\Omega_e \gtrsim \omega_{pe}\) holds in general, and mode with \(k_\perp \lambda_{De} \simeq 1\) satisfies \(k_\perp \rho_e < 1\). The well known mode in this regime is the electron temperature gradient (ETG) mode. Charge neutrality breaks down in the ETG mode in practical tokamak discharges characterized by \(\lambda_{De} > \rho_e\). In this case, the maximum growth rate of the ETG mode becomes strongly dependent on the \(\beta_e\) factor, not because of finite \(\beta\) destabilization but because of charge non-neutrality.

The maximum growth rate of the ETG mode is of the order of the electron transit frequency \(k_{||} v_{Te}\). This is another major difference from the ITG mode in which ion transit frequency is subdominant, \(|\omega| > k_{||} v_{Ti}\).
The electron thermal diffusivity due to the ETG mode is much larger than

$$\chi_e \simeq \frac{v_{Te} \rho_e^2}{L_T} = \sqrt{\frac{m_e c_s \rho_e^2}{m_i L_T}},$$

which would be expected if the ITG and ETG modes were completely dual, namely, if charge neutrality holds in both modes, ions are adiabatic in the ETG mode, as electrons are in the ITG mode, and $|\omega| > k\|v_{Ti}$, $\omega > k\|v_{Te}$ hold respectively in the ITG and ETG mode. However, as noted, charge neutrality breaks down in the ETG mode which is characterized by $\omega < k\|v_{Te}$. The duality does not hold in tokamaks, and the following electron thermal diffusivity emerges from the ETG mode,

$$\chi_e \simeq \frac{q_{Te}}{L_T} \left( \frac{e}{\omega_{pe}} \right)^2 \sqrt{\beta_e}.$$

### 5.2 Local Kinetic Dispersion Relation

We continue to employ the gyro-kinetic equations subject to the conditions that $\omega \ll \Omega_i (\ll \Omega_e)$ and plasma nonuniformity scale lengths be much larger than the ion gyro radius. The maximum frequency and growth rate of interest does not exceed the electron transit frequency $\omega_{Te} = \sqrt{T_{i e}/m_e/qR}$. The condition $\omega \simeq \omega_{Te} \ll \Omega_i$ becomes

$$\rho_e \ll \frac{m_e}{m_i} qR,$$

where $\rho_e$ is the electron Larmor radius which is of the order of $10^{-4}$ m or less. In the RHS, $\frac{m_e}{m_i} qR \simeq 10^{-3}$ m. Therefore, the condition $\omega \ll \Omega_i$ is satisfied with a large margin. The second condition $\rho_i \ll L_{Te}$ is also well satisfied even in the internal transport barrier (ITB) characterized by steep density and temperature gradient. In the ETG mode, ions are essentially adiabatic,

$$n_i \simeq -\frac{e \phi}{T_i} n_0,$$

particularly in the regime where the growth rate peaks. As will be shown, this occurs at $k_\perp \simeq 0.7k_{De}$ where $k_{De}$ is the electron Debye wavenumber. However, in the lower end of the $k$ spectrum, the wavelength approaches the electron skin depth, $k_\perp \simeq \omega_{pe}/c$, where ions are not adiabatic. Note that $c/\omega_{pe}$ is comparable with the ion Larmor radius $\rho_i$. In order to cover the entire spectrum of the ITG and ETG modes satisfactorily, fully kinetic, electromagnetic ion and electron responses without the assumption of adiabatic ions or electrons must be employed.

As in Chapter 4, we assume that the unperturbed distribution functions are Maxwellian. The perturbed ion and electron distribution functions are:

$$f_i = -\frac{e \phi}{T_i} f_{Mi} + \frac{\omega + \tilde{\omega}_{xi}}{\omega - k\|v\| + \tilde{\omega}_{Di}} \alpha_i \left( \frac{k_\perp v_\perp}{\Omega_i} \right) (\phi - \frac{v\|}{c} A\|) \frac{e}{T_i} f_{Mi},$$

(5.1)
\[ f_e = \frac{e^2}{T_e} f_{M_e} - \frac{\omega - \omega_{se}}{\omega - k_{||} v_{||} - \omega_{De}} J_0^2 \left( \kappa_{\perp} v_{\perp} \right) \left( \phi - \frac{v_{||}}{c} A_{||} \right) \frac{e^2}{T_e} f_{M_e}, \]  

(5.2)

where \( \phi \) is the scalar potential and \( A_{||} \) is the vector potential parallel to the magnetic field,

\[ \tilde{\omega}_{si,e}(v^2) = \frac{c T_{i,e}}{e B^2} \left[ 1 + \eta_{i,e} \left( \frac{m_{i,e} v_i^2}{2 T_{i,e}} - \frac{3}{2} \right) \right] [\nabla (\ln n_0) \times B] \cdot k_{\perp}, \]

(5.3)

\[ \tilde{\omega}_{Di,e}(v) = \frac{c m_{i,e}}{e B^2} \left( \frac{1}{2} v_{\perp}^2 + v_{||}^2 \right) \left( \nabla B \times B \right) \cdot k_{\perp}, \]

(5.4)

\( J_0 \) is the Bessel function, and \( k_{\perp} \) is gradient operator along the magnetic field. The magnetosonic perturbation is ignored in light of low \( \beta \) tokamak discharges.

The Poisson’s equation,

\[ \nabla^2 \phi = -4 \pi e \int (f_i - f_e) \, dv, \]

(5.5)

and the parallel Ampere’s law,

\[ \nabla_{\perp}^2 A_{||} = -\frac{4 \pi e}{c} \int v_{||} (f_i - f_e) \, dv, \]

(5.6)

yields the following electromagnetic dispersion relation,

\[ \left\{ k_{\perp}^2 + 2 \left( \frac{\omega_{pe}}{c} \right)^2 F_{e2} + 2 \left( \frac{\omega_{pi}}{c} \right)^2 F_{e2} \right\} \left[ F_{e0} - 1 - \left( \frac{k}{k_{De}} \right)^2 - \tau (1 - F_{ii}) \right] \]

\[ = 2 \left( \frac{\omega_{pe}}{c} \right)^2 \left( F_{e1} + \sqrt{\tau m_e} F_{e1} \right)^2, \]

(5.7)

where

\[ F_{ej} = \left\langle \frac{v_{||}}{v_{Te}} \right\rangle^j \frac{\omega - \omega_{se}}{\omega - \omega_{De} - k_{||} v_{||}} J_0^2 \left( \Lambda_e \right), \]

(5.8)

\[ F_{ij} = \left\langle \frac{v_{||}}{v_{Ti}} \right\rangle^j \frac{\omega + \omega_{si}}{\omega + \omega_{Di} - k_{||} v_{||}} J_0^2 \left( \Lambda_i \right), \]

(5.9)

\( \langle \cdot \cdot \cdot \rangle \) indicating averaging over the velocity with Maxwellian weighting. The norms of the differential operators based on a simple trial eigenfunction \( \phi (\theta) = 1 + \cos \theta, \, |\theta| \leq \pi \) are:

\[ \langle k_{\perp}^2 \rangle = k_{\perp}^2 \left( 1 + \frac{\pi^2 - 7.5}{3} s^2 - \frac{10}{9} s \alpha + \frac{5}{12} \alpha^2 \right), \]

(5.10)

\[ \langle \omega_D \rangle = 2 \varepsilon_n \omega_s \left( \frac{2}{3} + \frac{5}{9} s - \frac{5}{12} \alpha \right), \, \langle \omega_{eb} \rangle = \frac{1}{3(qR)^2} f(s, \alpha), \]

(5.11)

where

\[ f(s, \alpha) = \frac{1 + (\pi^2 / 3 - 0.5) s^2 - 8 s \alpha / 3 + 3 \alpha^2 / 4}{1 + (\pi^2 / 3 - 2.5) s^2 - 10 s \alpha / 9 + 5 \alpha^2 / 12}. \]

(5.12)

The validity of the local dispersion relation has been checked by comparing the mode frequency \( \omega = \omega_r + i \gamma \).
of the ITG with that found from the method based on integral equations.

### 5.3 Integral Equation Approach

Although the local kinetic dispersion relation is expected to describe low frequency modes in tokamaks qualitatively, its validity and accuracy should be checked against a more rigorous approach based on integral equations in the ballooning space. In particular, the norm of the parallel gradient operator \( k_\parallel \) has to be justified through comparison between eigenvalues found from the local kinetic dispersion relation and from nonlocal analysis. For long wavelength modes including ITG and kinetic ballooning modes, the validity of the local kinetic dispersion relation has been well established. More recently, nonlocal analysis has been carried out of shorter wavelength (skin size) drift mode. As the wavelength becomes even shorter, charge neutrality breaks down and the set of integral equations are to be modified to implement the Poisson’s equation for the scalar potential.

We again consider a high temperature, low \( \beta \) tokamak discharge with eccentric circular magnetic surfaces. The frequency regime of interest is \( \omega_{bi} < \omega \lesssim \omega_{be} \), where \( \omega_{bi(e)} \) is the trapped ion (electron) bounce frequency. The magnetosonic perturbation \( (A_{-\perp}) \) is ignored in light of the low \( \beta \) assumption and we employ the two-potential \((\phi \text{ and } A_\parallel)\) approximation to describe electromagnetic modes. The basic field equations are the Poisson’s equation,

\[
\nabla^2 \phi = -4\pi e \left[ n_i(\phi, A_\parallel) - n_e(\phi, A_\parallel) \right],
\]

and the parallel Ampere’s law,

\[
\nabla^2_\perp A_\parallel = -\frac{4\pi}{c} J_\parallel(\phi, A_\parallel),
\]

where the density perturbations are given in terms of the perturbed velocity distribution functions \( f_i \) and \( f_e \) as

\[
n_i = \int f_i d\mathbf{v}, \quad n_e = \int f_e d\mathbf{v},
\]

and the parallel current by

\[
J_\parallel = e \int v_\parallel (f_i - f_e) d\mathbf{v}.
\]

The distribution functions \( f_i \) and \( f_e \) are given by

\[
f_i = -\frac{e\phi}{T_i} f_{Mi} + g_i(v, \theta) J_\parallel(\Lambda_i),
\]

and

\[
f_e = \frac{e\phi}{T_e} f_{Me} + g_e(v, \theta) J_\parallel(\Lambda_e),
\]
where \( g_{i,e} \) are the nonadiabatic parts that satisfy

\[
\left( \frac{i v^i_i (\theta)}{q R} \frac{\partial}{\partial \theta} + \omega + \tilde{\omega}_{Di} \right) g_i = (\omega + \tilde{\omega}_{si}) J_0 (\Lambda_i) \left( \phi - \frac{v^i_i}{c} A^i_i \right) \frac{e}{T_i} f_{Mi},
\]

(5.19)

\[
\left( \frac{i v^e_e (\theta)}{q R} \frac{\partial}{\partial \theta} + \omega - \tilde{\omega}_{De} \right) g_e = -(\omega - \tilde{\omega}_{se}) J_0 (\Lambda_e) \left( \phi - \frac{v^e_e}{c} A^e_i \right) \frac{e}{T_e} f_{Me}.
\]

(5.20)

Here, \( \theta \) is the extended poloidal angle, \( \phi \) is the scalar potential, \( A^i_i \) is the parallel vector potential, \( J_0 \) is the Bessel function with argument \( \Lambda_i, e, k = k_{i,e} v_{ci,e} / \omega_{ci,e} \), and \( q R \) is the connection length.

For circulating particles, \( g_j \) \( (j = i, e) \) can be integrated as

\[
v^j_j > 0, \quad g_j^+ = -i \frac{e_j f_{Mj}}{T_j} \int_{\infty}^{\theta} d\theta' \frac{q R}{|v^j_j|} e^{i \beta_j (\omega - \tilde{\omega}_{sj})} J_0 (\Lambda_j') \left( \phi(\theta') - \frac{|v^j_j|}{c} A^j_j(\theta') \right),
\]

(5.21)

\[
v^j_j < 0, \quad g_j^- = -i \frac{e_j f_{Mj}}{T_j} \int_{\theta}^{\infty} d\theta' \frac{q R}{|v^j_j|} e^{-i \beta_j (\omega - \tilde{\omega}_{sj})} J_0 (\Lambda_j') \left( \phi(\theta') + \frac{|v^j_j|}{c} A^j_j(\theta') \right),
\]

(5.22)

where

\[
\beta_j(\theta, \theta') = \int_{\theta'}^{\theta} \frac{q R}{|v^j_j|} \frac{1}{\omega - \tilde{\omega}_{Di}(\theta')} d\theta''.
\]

For trapped particles with turning points \( \theta_1 \) and \( \theta_2 \) \( (\theta_2 > \theta_1) \), the solution is

\[
g^\sigma = \frac{e^{i \sigma \beta_j(\theta, \theta') \sin(\theta - \theta')}}{2 \sin[\beta(\theta, \theta')]} \int_{\theta_1}^{\theta_2} \left( e^{-i \beta_j(\theta_2', \theta') \text{sgn}(\theta')} + e^{i \beta_j(\theta_2', \theta') \text{sgn}(\theta')} \right) d\theta' - i \sigma \int_{\theta_1}^{\theta} e^{i \sigma \beta(\theta, \theta') \gamma} \gamma d\theta',
\]

(5.23)

where \( \sigma = \text{sgn}(v^j_j) \),

\[
\gamma = \gamma_\phi + \sigma \gamma_A,
\]

(5.24)

\[
\gamma_\phi = \frac{e_j q R}{T_j |v^j_j|} (\omega - \omega_{sj}) J_0 (\Lambda_j) \phi(\theta') f_{Mj},
\]

(5.25)

\[
\gamma_A = -\frac{e_j q R}{T_j |v^j_j|} (\omega - \omega_{sj}) J_0 (\Lambda_j) \frac{|v^j_j|}{c} A^j_j(\theta') f_{Mj}.
\]

(5.26)

Since for electrons, \( \beta_j(\theta_2, \theta_1) \) is of order of \( \omega / \omega_{be} \ll 1 \) where \( \omega_{be} \) is the electron bounce frequency, trapped electron response may be approximated by

\[
g^e_e \simeq \frac{1}{2 \beta(\theta_2, \theta_1)} \int_{\theta_1}^{\theta_2} (\gamma_\phi + i \beta_j(\theta_2, \theta') \gamma_A) d\theta' - i \int_{\theta_1}^{\theta} \gamma_A d\theta'.
\]

(5.27)

In this analysis, we ignore trapped ions since the frequency regime of interest is at least of order of the ion transit frequency. Substitution of perturbed distribution functions into charge neutrality and parallel
Ampere’s law yields
\[ \nabla^2 \phi = -4\pi \sum_j e_j \left( -\frac{e_j}{T_j} \phi + \int [g_j^+ (\theta) + g_j^- (\theta)] J_0(\Lambda_j) d\nu \right), \tag{5.28} \]

\[ \nabla^2_{\perp} A_k (\theta) = -\frac{4\pi}{c} \sum_j e_j \int v_{\parallel} [g_j^+ (\theta) - g_j^- (\theta)] J_0(\Lambda_j) d\nu, \tag{5.29} \]

where \(\int d\nu = 2\pi \int_0^\infty v_{\perp} d\nu \int_0^\infty d\nu_{\parallel}\). This system of inhomogeneous integral equations can be solved by employing the method of Fredholm in which the integral equations are viewed as a system of linear algebraic equations.

### 5.4 Short Wavelength ITG Modes

It is generally conjectured that the ITG mode should be deactivated in the regime \((k_{\perp} \rho_i)^2 \gg 1\) since ion dynamics tends to be adiabatic. This is not the case if \(\omega_{Di} < |\omega| < \omega_{\ast i}\), where \(\omega_{Di}\) is the ion magnetic drift frequency and \(\omega_{\ast i}\) is the ion diamagnetic drift frequency. Consider the ion perturbation in slab geometry \((\omega_{Di} = 0)\) with negligible ion transit frequency \(|\omega| \gg k_{\perp} v_{Ti}\),

\[ f_i = -\frac{e_0}{T_i} f_{Mi} + \frac{\omega + \omega_{\ast i}}{\omega} \int J_0^2 (\Lambda_i) \frac{e_0}{T_i} f_{Mi}, \quad \omega_{\ast i} = \omega_{\ast i} \left[ 1 + \frac{\eta_i}{2} \left( \frac{Mv^2}{2T_i} - \frac{3}{2} \right) \right], \tag{5.30} \]

or its integral

\[ n_i \simeq \frac{e_0}{T_i} n_0 + \left[ \frac{\omega + \omega_{\ast i}}{\omega} e^{-b_i \eta_i} \right] \frac{e_0}{T_i} n_0, \quad \omega_{\ast i} = \omega_{\ast i} \left[ 1 + \frac{\eta_i}{2} \frac{\partial}{\partial T_i} \right], \quad b_i = (k_{\perp} \rho_i)^2. \tag{5.31} \]

If \(|\omega| \ll \omega_{\ast i}\), the ion density perturbation in the limit \(b_i \gg 1\) approaches

\[ n_i \simeq -1 + \frac{v_{Ti}/L_n}{\sqrt{2\pi} \omega} \left( 1 - \frac{1}{2} \eta_i \right) \frac{e_0}{T_i} n_0, \quad v_{Ti} = \sqrt{T_i/M}, \tag{5.32} \]

which is far from adiabatic. If \(|\omega| \ll k_{\parallel} v_{Te}\) (electron transit frequency), the following stable mode emerges,

\[ \omega \simeq \frac{v_{Te}/L_n}{2\sqrt{2\pi}} \left( 1 - \frac{1}{2} \eta_i \right) \frac{\tau}{1 + \tau}, \quad \tau = T_e/T_i. \tag{5.33} \]

In short wavelength regime, the mode frequency becomes non-negligible compared with the electron transit frequency and electron Landau damping can destabilize the mode.

In toroidal geometry, stability analysis of short wavelength ITG mode requires use of the integral equations in order to implement the ion and electron transit effects (ion and electron Landau damping) in a satisfactory manner. This has been done recently in A. Hirose, Phys. Rev. Lett. 92, 025001 (2004). Main findings are as follows. (a) In the slab limit (small toroidicity \(\varepsilon_n = L_n/R\), a strong temperature gradient driven
ion mode persists in the regime $b_i \gg 1$. The instability requires both $\eta_i$ and $\eta_e$ above critical values. (b) Toroidicity has significant stabilizing influence on the mode. Stabilization occurs for $L_n/R \gtrsim 0.15$. (c) The instability is driven by magnetic shear and the growth rate is approximately proportional to $\sqrt{|s|}$ where $s$ is the shear parameter, either positive or negative. (d) As in the case of long wavelength $\eta_i$ mode in the regime $b_i < 1$, the mode is stabilized by a modest $\alpha$, the ballooning parameter. And, (e) trapped electrons have no significant influence on the mode in short wavelength regime.

Figure 5-1: Mode frequency (a) and growth rate (b) normalized by $c_s/L_n$ as functions of $b_s = (k_\theta \rho_s)^2$ when $\tau = T_e/T_i = 1$, $\varepsilon_n = L_n/R = 0.1$, $\eta_i = \eta_e = 2.5$, $m_e/m_i = 1/1837$ (hydrogen), $s = 1.5$, $q = 1.5$, $\beta_i = \beta_e = 0.1\%$.

Figure 1 shows the mode frequency normalized by $c_s/L_n$ as a function of $b_s = (k_\theta \rho_s)^2 = k_\theta^2 T_e/m_i \Omega_i^2$ when $\varepsilon_n = L_n/R = 0.1$, $\eta_e = \eta_i = 2.5$, $s = q = 1.5$, $T_i = T_e$, $\beta_i = \beta_e = 0.1\%$ and $m_i/m_e = 1837$ (hydrogen discharge). Negative frequency $\omega_r < 0$ indicates propagation in the ion diamagnetic direction. The first peak
in the growth rate at small $b_s$ ($\simeq 0.5$) is the conventional long wavelength toroidal $\eta_i$ mode. It is deactivated as $b_s$ increases due to finite ion Larmor radius effect. However, the growth rate exhibits a second peak at shorter wavelength $b_s \simeq 6$. For the parameters assumed, $b_s = 6$ corresponds to $k_0c/\omega_{pe} \simeq 2.6$. (Note that when $\beta_i = \beta_e$,

$$\left( \frac{k_0c}{\omega_{pe}} \right)^2 = 2b_s \frac{m_e}{m_i\beta_e},$$

where $m_e/m_i$ is the electron/ion mass ratio and $\beta_e$ is the electron beta factor $\beta_e = 8\pi n_0 T_e / B^2$.) The mode frequency $\omega_r$ above $b_s = 2$ becomes constant. In terms of the ion acoustic transit frequency $\omega_s = c_s/qR$, the normalized frequency is $\omega_r/\omega_s \simeq -9$. Therefore, parallel ion dynamics should not play a role. For electrons, $|\omega| \simeq \omega_{De} \simeq 0.2\omega_{Te}$ where $\omega_{Te} = \sqrt{T_e/m_e/qR}$ is the electron transit frequency. Electron parallel resonance (Landau damping) is thus expected to be important. Since $\omega_r < 0$, there is no resonance with the electron magnetic drift $\omega_{De}$. Therefore, the instability is largely due to electron parallel resonance (Landau damping). Too large toroidicity ($\omega_{De} = 2\varepsilon_n\omega_{se}$) should deactivates the instability because ion dynamics tends to be adiabatic.

![Graph](image)

**Figure 5-2:** (a) Dependence of $\gamma/\omega_s$ ($\omega_s = c_s/qR$) on $\varepsilon_n = L_n/R$. (b) Dependence of $\gamma/\omega_{se}$ on $\eta_i = \eta_e$. Other parameters are: $b_s = 6$, $\eta_e = \eta_i = 2.5$, $\tau = 1$, $s = 1.5$, $q = 1.5$, $\beta_i = \beta_e = 0.1 \%$.

Dependence of the growth rate on the toroidicity $\varepsilon_n = L_n/R$ and the temperature gradient ($\eta_i = \eta_e$ assumed) is shown in Fig. 2 (a) and (b), respectively. As expected, the instability is deactivates at large toroidicity.
The instability persists in the limit $\varepsilon_n \to 0$ which clearly indicates that the mode is of slab nature. Stabilization of the predominantly slab mode by toroidicity may be seen from the toroidal counterpart of Eq. (1),

$$f_i = -\frac{e\phi}{T_i}f_{M_i} + \frac{\omega + \tilde{\omega}_{s_i}}{\omega + \tilde{\omega}_{D_i}} f_0^2(A_i) \frac{e\phi}{T_i} f_{M_i}. \quad (5.34)$$

Since $\tilde{\omega}_{D_i}/\omega_{s_i} \simeq 2\varepsilon_n = 2L_n/R$, for large $\varepsilon_n$ (toroidicity), the ion density perturbation

$$n_i \simeq -\frac{e\phi}{T_i}n_0 + \left[ \frac{\omega + \omega_{s_i} e^{-b_i} I_0(b_i)}{\omega + \tilde{\omega}_{D_i}} \right] \frac{e\phi}{T_i} n_0$$

$$\simeq -\frac{e\phi}{T_i}n_0 + \frac{\omega + \omega_{s_i}}{\omega + \tilde{\omega}_{D_i}} \frac{1}{\sqrt{2\pi b_i T_i}} \frac{e\phi}{T_i} n_0,$$

tends to be adiabatic in short wavelength regime if $|\omega| < \omega_{s_i}$, $\omega_{D_i}$, and the $\eta_i$ slab mode is stabilized by toroidicity.

The critical temperature gradient is approximately $\eta_{cr} \simeq 1.5$ when $\varepsilon_n = 0.1$ and $\eta_i = \eta_e$. Behaviour of the growth rate when $\eta_i$ and $\eta_e$ are varied independently has also been investigated. When $\eta_i = 2.5$, the critical electron temperature gradient is $\eta_{e,cr} \simeq 0.7$ which is smaller than that of ions, $\eta_{i,cr} \simeq 1.3$ when $\eta_e = 2.5$. It is evident that both $\eta_i$ and $\eta_e$ above respective thresholds are required simultaneously for the instability and the instability is of hybrid nature in the sense that neither electrons nor ions are adiabatic.

Dependence of the growth rate on the magnetic shear parameter $s$ is shown in Fig. 3. Shear is destabilizing in both positive and negative regimes and the dependence of the growth rate on $s$ may be approximated by $\gamma \propto \text{const.} |s|$. It is evident that magnetic shear, either positive or negative, is required for this particular instability. The magnetic shear $s$ enters the stability analysis through the magnetic drift frequency $\omega_D$ and $k_\parallel$. Positive (negative) shear enhances (reduces) toroidicity. Since in the instability of concern, toroidicity is relatively unimportant (it reduces the growth rate), the dependence of the growth rate on $s \left( \gamma \propto \sqrt{|s|} \right)$ mainly originates from the shear dependence of the parallel gradient, $k_\parallel$, which depends on the magnitude of $s$. The instability therefore has features of the “universal mode” and is primarily driven by electron parallel Landau damping.

Dependence of the mode frequency and growth rate on the safety factor $q$ has also been investigated. The instability is deactivated in the region $q \lesssim 0.7$ and $q \gtrsim 3.2$. Stabilization at small $q$ may be interpreted as due to too large an electron transit frequency and stabilization at large $q$ is due to finite $\beta$ (or $\alpha$, the ballooning parameter) stabilization, since the ballooning parameter,

$$\alpha = \frac{q^2 R}{L_m} \left( (1 + \eta_e) \beta_e + (1 + \eta_i) \beta_i \right), \quad (5.35)$$

rapidly increases with the safety factor $q$. $\alpha$ stabilization is similar to that in the conventional ITG mode which is caused by coupling of electron dynamics to the magnetic perturbation $A_\parallel$. 


5.5 Stability of the ETG Mode

In this Section, we investigate stability of modes with even shorter wavelength in the regime \( k \perp \rho_e \lesssim 1 \) where \( \rho_e \) is the electron Larmor radius. Ions tend to be adiabatic in such regime and we are primarily concerned with the ETG mode. In tokamaks, \( \rho_e < \lambda_{De} \) (or \( \Omega_e > \omega_{pe} \)) generally holds. Therefore it is not appropriate to assume charge neutrality. If charge neutrality does not hold, that is, if the term \((k/k_{De})^2\) is not negligible, there arises apparent dependence of eigenvalue \( \omega \) on plasma \( \beta \) (or plasma density) even in electrostatic limit.

The electrostatic dispersion relation of the ETG mode with adiabatic ions is

\[
1 + \tau + \left( \frac{k}{k_{De}} \right)^2 = \left( \frac{\omega - \tilde{\omega}_{se}}{\omega - \omega_{De} - k_{||} v_{||}} \right) J_0^2 (\Lambda_e), \quad \tau = \frac{T_e}{T_i}.
\]  

(5.36)

The charge nonneutrality factor \((k/k_{De})^2 \simeq (k_{||}/k_{De})^2\) and the electron finite Larmor radius parameter \((k_{\perp} \rho_e)^2\) are related through

\[
\left( \frac{k_{\perp}}{k_{De}} \right)^2 = \frac{(k_{\perp} \rho_e)^2}{\beta_e} \frac{2 T_e}{\beta_e mc^2},
\]  

(5.37)

where \( T_e/mc^2 \) is the normalized electron temperature. Even in the electrostatic mode equation (and resultant dispersion relation), \( \beta_e \) has to be specified because the ballooning parameter \( \alpha = \alpha_i + \alpha_e \) is one of the
parameters to characterize plasma equilibrium. The electron FLR parameter $k_{\perp}\rho_e$ is of course the key parameter in gyro-kinetic formulation. Therefore, when charge neutrality does not hold, the normalized temperature $T_e/m_e c^2$ has to be specified together with various other dimensionless parameters. For a given electron temperature, charge non-neutrality is evidently more enhanced at lower plasma density. Since the term $(k/k_{De})^2$ is stabilizing, it is expected that the growth rate of the ETG mode becomes dependent on $\beta_e$ (the plasma density). The growth rate of the ETG mode with charge neutrality $(k_{\perp}/k_{De})^2 < 1$ and negligible electron transit frequency $\omega \gg k_{\parallel}v_{Te}$ is approximately given by

$$\gamma \simeq \sqrt{\eta_e \omega_{Te} \omega_{De}/\tau}. \quad (5.38)$$

Charge non-neutrality reduces the growth rate as

$$\gamma \simeq \sqrt{\frac{\eta_e \omega_{Te} \omega_{De}}{\tau + (k_{\perp}/k_{De})^2}}. \quad (5.39)$$

In short wavelength regime $(k_{\perp}/k_{De})^2 > \tau$, the growth rate approaches

$$\gamma \simeq c \sqrt{\frac{\beta_e}{L_T R}}. \quad (5.40)$$

being proportional to $\sqrt{\beta_e}$.

To demonstrate the importance of charge nonneutrality in the ETG mode in terms of the roots of the local kinetic electromagnetic dispersion relation, we show in Fig. 4 the dependence of mode frequency and growth rate, both normalized by the electron transit frequency $\omega_{Te} = v_{Te}/qR$, on the normalized perpendicular wavenumber, $d_e = (k_{\parallel}/k_{De})^2$, for three values of $\beta$, $\beta_e = \beta_i = 0.1\%$, 0.2\% and 0.5\% when $T_e = T_i = 10$ keV ($T_e/m_e c^2 \approx 0.02$). Other parameters assumed are: $L_n/R = 0.2$, $s = 1$, $q = 2$, $\eta_e = \eta_i = 2$, $m_i/m_e = 1836$ (hydrogen). The growth rate peaks at $b_e \simeq 0.02$ when $\beta_e = 0.1\%$ and at $b_e = 0.07$ when $\beta_e = 0.5\%$. The maximum growth rate increases with $\beta_e$ (actually with the plasma density) as expected. The plots in Fig. 5 clearly indicate that the maximum growth rate occurs approximately at a constant value of $(k_{\parallel}/k_{De})^2 \simeq 0.5$ when the plasma density and electron temperature are varied. In this case, the conventional normalization by the electron Larmor radius, $b_e = (k_{\parallel}\rho_e)^2$, is not convenient because $b_e$ at the maximum growth rate shifts as $\beta_e$ is varied. It is then evident that in the ETG mode, the Debye length appears as an important scale length.

It is noted that mode frequency $\omega$ is of the order of the electron transit frequency, $\omega_{Te} = v_{Te}/qR$. This is the major difference from the long wavelength ITG mode in which the ion transit frequency is subdominant, $|\omega| > v_{Ti}/qR$. Electron parallel Landau damping thus plays a major role in the ETG mode. Since $k_{\parallel} \simeq 1/qR$, the mode frequency is expected to be sensitively dependent on the safety factor $q$. This is shown in Fig. 5 for $T_e = 10$ keV. A relatively small value of $\beta_e = 0.2\%$ is chosen to keep the ballooning parameter $\alpha$, which
Figure 5-4: Mode frequency $\omega_r/\omega_{Te}$ (solid line) and growth rate $\gamma/\omega_{Te}$ (dotted line) as functions of $d_e = (k_\theta/k_{De})^2$ when $T_e = T_i = 10$ keV, $s = 1$, $\tau = 1$, $\varepsilon_n = 0.2$, $\eta_i = \eta_e = 2$, $q = 2$, $m_i/m_e = 1836$. $\beta_e (= \beta_i)$ is scanned from 0.1 to 0.5%.

is proportional to $q^2$, below the limit of drift reversal. The maximum growth rate $\gamma/\omega_{Te}$ increases with $q$ in a manner approximately proportional to $q^2$, $\gamma_{\text{max}}/\omega_{Te} \propto q^2$. Since $\omega_{Te} = v_{Te}/qR$, the unfolded growth rate is proportional to $q$.

Dependence on the magnetic shear parameter $s$ has also been investigated. Shear is destabilizing as in the case of the short wavelength ITG mode discussed in the preceding Section. When $q = 2$ ($\alpha = 0.6$ for the assumed parameters), the critical shear parameter is $s_{cr} \simeq -0.5$ and when $q = 3$ ($\alpha = 1.35$), $s_{cr} \simeq 0$. The critical shear parameter is in qualitative agreement with the condition $\omega_{De} > 0$. Negative shear is stabilizing in contrast to the case of the ITG mode. From these observations, it is evident that the ETG mode is driven by both toroidicity and electron parallel resonance.
Figure 5-5: $q$ dependence of $(\omega_r + i\gamma) / \omega_{Te}$ when $T_e = T_i = 5$ keV, $\beta_e = \beta_i = 0.2\%$, $s = 1$, $\tau = 1$, $\varepsilon_n = 0.2$, $\eta_i = \eta_e = 2$, $m_i/m_e = 1836$.

5.6 Mixing Length Estimate of $\chi_e$

In this Section, mixing length estimate of the electron thermal diffusivity,

$$\chi_e = \frac{\gamma^3}{\omega_r^2 + \gamma^2 k_{\perp}^2} \frac{1}{k_{\perp}^2},$$  \hspace{1cm} (5.41)

is presented for the ETG mode. (The short wavelength ITG mode is stabilized by toroidicity and may not be dangerous in practical tokamak discharges.) Fig. 6 shows $\chi_e$ in units of $(v_{Te}/qR)/k_{De}^2$ when $\beta_e = 0.2\%$, $T_e = T_i = 5$ keV and 10 keV. The safety factor $q$ is scanned between $q = 2$ and 4. The maximum value of $\chi_e$ is proportional to $q^2$ but inversely proportional to the temperature. Since $\omega_{Te} \propto 1/q$, this suggests that the electron thermal diffusivity has the following scaling,

$$\chi_e \propto \frac{q v_{Te}}{n_0}.$$
Results of scanning $\beta_e$, electron temperature $T_e$, and the safety factor $q$ can be summarized by the following electron thermal diffusivity,

$$\chi_e \approx \frac{q\nu_{Te}}{L_T} \left( \frac{c}{\omega_{pe}} \right)^2 \sqrt{\beta_e}.$$  \hspace{1cm} (5.42)

Further studies will be needed to go beyond the simple mixing length estimate for the diffusivity. In particular, whether the ETG mode can drive a zonal flow and its consequence on thermal transport remains an open question.

Figure 5-6: $\chi_e$ in units of $\omega_{Te}/k_{De}^2$ vs. $d_e = (k_\perp/\rho_i)^2$ when $\beta_e = \beta_i = 0.2\%$ and $q = 2, 3, 4$. In (a), $T_e = T_i = 5$ keV and in (b), $T_e = T_i = 10$ keV.

### 5.7 Conclusions

In this Chapter, recent investigations of short wavelength collisionless temperature gradient modes in tokamak geometry are reviewed. To summarize the features of the short wavelength ITG mode with $(k_\perp\rho_i)^2 \gg 1$, it has been shown that the instability is driven by parallel electron Landau damping as is the universal mode. The mode is of slab nature and toroidicity has stabilizing effects. The growth rate rapidly decreases with the toroidicity parameter $\varepsilon_n = L_n/R$ and the instability is operative only when the density gradient is steep,
\( \varepsilon_n \lesssim 0.15 \). As in the case of the conventional long wavelength toroidal ITG mode, the instability is subject to finite \( \beta \) (or \( \alpha \), the ballooning parameter) stabilization. The mode is insensitive to trapped electrons.

To summarize the findings made of the ETG mode, it has been shown that for practical tokamak discharges with \( \omega_{pe} < \Omega_e \), charge neutrality breaks down and a natural normalization of the wavenumber for the ETG mode is \((k/k_{De})^2\), rather than \((k\rho_e)^2\). The lower cutoff of the ETG mode occurs at \( k_\perp \gtrsim \omega_{pe}/c \). The maximum electron thermal diffusivity occurs at \((k/k_{De})^2 \simeq 0.1\), namely, at wavelength longer than that corresponding to the maximum growth rate, \((k/k_{De})^2 \simeq 0.5\).