Chapter 4

Kinetic Ballooning Modes and Finite $\beta$ Effects on Drift Type Modes

4.1 Introduction

In Chapter 3, we have seen that how low frequency, electrostatic ion acoustic mode described by $\omega = k || c_s$ is coupled to drift mode, $\omega = \omega_{ce}$. The drift mode can also couple to the electromagnetic Alfven mode $\omega = k || V_A$. The Alfven mode is one of the fundamental, low frequency electromagnetic modes. In a low $\beta$ plasma, the Alfven velocity far exceeds the ion acoustic velocity, $V_A \gg c_s$. We have already seen in Chapter 2 that the ballooning instability caused by the combination of pressure gradient and (bad) magnetic curvature is essentially the destabilized Alfven mode. There, our treatment was hydrodynamic, with no account taken of the effects of finite ion Larmor radius and kinetic resonances. In this Chapter, the gyrokinetic equation derived in Chapter 3 will be generalized to implement electromagnetic fields in order to assess kinetic modifications to the MHD ballooning mode and finite $\beta$ effects on predominantly electrostatic modes, such as the drift and ion temperature gradient modes.

4.2 Kinetic Equations (Two-Potential Approximation)

Electrostatic waves can be fully described in terms of the scalar potential $\phi$ alone as seen in Chapter 3. Electromagnetic modes, on the other hand, involve magnetic perturbations which can be deduced from the vector potential $A$. The electric and magnetic fields are then fully described by the two potentials,

$$E = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t}, \quad (4.1)$$

$$B = \nabla \times A. \quad (4.2)$$
In the Vlasov equation,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m} \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \nabla f = 0,$$

(4.3)

the perturbed magnetic field $\mathbf{B}$ can be eliminated via Faraday’s law

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},$$

(4.4)

and considering the three independent components of the electric field $\mathbf{E}$ is sufficient. In analysis using the two potentials $\phi$ and $\mathbf{A}$, there is a total of four components. However, in either Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) or Lorentz gauge

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0,$$

one of the four components can be eliminated, and we still have three independent components to work with.

In analyzing low frequency ($\omega \ll \Omega_i$) electromagnetic waves in a plasma, it is often convenient to adopt the following three fields as the independent field components,

$$\phi, \mathbf{A}_\parallel \text{ and } \mathbf{B}_\parallel,$$

(4.5)

where $\mathbf{A}_\parallel$ and $\mathbf{B}_\parallel$ are the perturbed vector potential and magnetic field both parallel to the unperturbed magnetic field $\mathbf{B}$. $\mathbf{A}_\parallel$ produces a magnetic field predominantly perpendicular to $\mathbf{B}$,

$$\mathbf{B}_\perp = \nabla_\perp \times \mathbf{A}_\parallel,$$

(4.6)

where $\nabla_\perp$ is the perpendicular gradient operator. $\mathbf{A}_\parallel$ can therefore be identified as the vector potential associated with the shear Alfvén mode. The parallel magnetic field $\mathbf{B}_\parallel$ obviously describes the compressional Alfvén wave (magnetosonic wave) which should be taken into account when plasma $\beta$ is not negligible. In a low $\beta$ plasma, the magnetosonic perturbation is ignorable and low frequency electromagnetic modes can be described in terms of the two potentials, $\phi$ and $\mathbf{A}_\parallel$ with sufficient accuracy.

The Alfvén mode is characterized by magnetic field line bending which causes particle drift

$$v_\parallel \frac{\mathbf{B}_\perp}{B}.$$  

(4.7)

This is still an $\mathbf{E} \times \mathbf{B}$ drift with a motional electric field given by

$$\mathbf{E}' = \frac{1}{c} \mathbf{v}_\parallel \times \mathbf{B}_\perp,$$

(4.8)

which yields

$$c \frac{\mathbf{E}' \times \mathbf{B}}{B^2} = v_\parallel \frac{\mathbf{B}_\perp}{B}.$$  

(4.9)
Combining with the electrostatic $\mathbf{E} \times \mathbf{B}$ drift, we thus find the total perturbed drift,

$$\mathbf{v}_D = c \frac{\mathbf{B} \times \nabla \phi}{B^2} + v_\parallel \frac{\mathbf{B}_\perp}{B}.$$  \hfill (4.10)

After linearization, the Vlasov equation becomes

$$\frac{df}{dt} + \mathbf{v}_D \cdot \nabla f_0 + \frac{e}{m} \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0,$$  \hfill (4.11)

where as before

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}_\parallel + \mathbf{V}_D) \cdot \nabla,$$

with $\mathbf{V}_D$ the magnetic drift velocity. If the unperturbed distribution $f_0$ is Maxwellian, this reduces to

$$\frac{df}{dt} + \mathbf{v}_D \cdot \nabla f_M - \frac{e}{T} \mathbf{E} \cdot (\mathbf{v}_\parallel + \mathbf{V}_D) f_M = 0,$$  \hfill (4.12)

where

$$\mathbf{E} \approx -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}_\parallel}{\partial t}.$$  \hfill (4.13)

Following the same procedure as developed in Chapter 3, we obtain

$$f = -\frac{\phi}{T} f_M + \frac{\omega - \bar{\omega}_*}{\omega - \bar{\omega}_D - k_\parallel v_\parallel} \left( \phi - \frac{v_\parallel}{c} A_\parallel \right) \frac{J_0^2(\Lambda)}{T^2} f_M,$$  \hfill (4.14)

with $\Lambda = k_\perp v_\perp / \Omega$. If needed, the magnetosonic perturbation can be readily implemented in a similar manner,

$$f = -\frac{\phi}{T} f_M + \frac{\omega - \bar{\omega}_*}{\omega - \bar{\omega}_D - k_\parallel v_\parallel} \left\{ \left( \phi - \frac{v_\parallel}{c} A_\parallel \right) J_0^2 - i \frac{v_\perp}{c} J_0 J_1 A_\perp \right\} \frac{e}{T} f_M,$$  \hfill (4.15)

where the Bessel functions $J_{0,1}$ have the same argument $\Lambda$.

### 4.3 Kinetic Ballooning Mode Equation

The basic equations to govern low frequency modes are the charge neutrality condition

$$n_i = n_e, \text{ or } \int f_i d\mathbf{v} = \int f_e d\mathbf{v},$$  \hfill (4.16)

and parallel Ampere's law,

$$\nabla^2 A_\parallel = -\frac{4\pi}{c} \int f_\parallel d\mathbf{v} = -\frac{4\pi n_0}{c} \int v_\parallel (f_i - f_e) d\mathbf{v},$$  \hfill (4.17)
where the perturbed ion distribution function $f_i$ can be found from Eq. (4.14),

$$f_i = -\frac{e\phi}{T_i} f_{Mi} + \frac{\omega + \tilde{\omega}_{si}}{\omega - k_{\|}v_{\|} + \tilde{\omega}_{Di}} f_{0}^i \left( \frac{k_{\perp}v_{\perp}}{\Omega_i} \right) \left( \phi - \frac{v_{\|}}{c} A_{\|} \right) \frac{e}{T_i} f_{Mi},$$

(4.18)

with

$$\tilde{\omega}_{si}(v^2) = \frac{cT_i}{eB^2} \left[ 1 + \eta_i \left( \frac{Mv_{\perp}^2}{2T_i} - \frac{3}{2} \right) \right] \left[ \nabla \ln n_0 \times B_0 \right] \cdot k_{\perp},$$

(4.19)

$$\tilde{\omega}_{Di}(v) = \frac{cM}{eB^2} \left( \frac{1}{2} v_{\perp}^2 + v_{\|}^2 \right) (\nabla B \times B_0) \cdot k_{\perp}.$$

(4.20)

For the ballooning mode, we may assume that the mode frequency is much larger than the ion transit frequency $|\omega| \gg k_{\|}v_{Ti}$. Then the ion density perturbation becomes electrostatic and the ion current parallel to the magnetic field is ignorable,

$$n_i \simeq (-1 + I_i) \frac{e\phi}{T_i} n_0,$$

(4.21)

where the function $I_i$ defined by

$$I_i = \int \frac{\omega + \tilde{\omega}_{si}}{\omega + \tilde{\omega}_{Di}} f_{0}^i \left( \frac{k_{\perp}v_{\perp}}{\Omega_i} \right) f_{Mi} dv,$$

(4.22)

involves ion kinetic resonance at $\omega + \tilde{\omega}_{Di}(v) = 0$.

For the electrons, the finite Larmor radius effect may be ignored. The perturbed electron distribution function can be written down analogously,

$$f_e = \frac{e\phi}{T_e} f_{Me} - \frac{\omega - \tilde{\omega}_{se}}{\omega - k_{\|}v_{\|} - \tilde{\omega}_{De}} \left( \phi - \frac{v_{\|}}{c} A_{\|} \right) \frac{e}{T_e} f_{Me},$$

(4.23)

where

$$\tilde{\omega}_{se}(v^2) = \frac{cT_e}{eB^2} \left[ 1 + \eta_e \left( \frac{Mv_{\perp}^2}{2T_e} - \frac{3}{2} \right) \right] \left[ \nabla \ln n_0 \times B_0 \right] \cdot k_{\perp},$$

(4.24)

$$\tilde{\omega}_{De}(v) = \frac{cm}{eB^2} \left( \frac{1}{2} v_{\perp}^2 + v_{\|}^2 \right) (\nabla B \times B_0) \cdot k_{\perp}.$$

(4.25)

In Eq. (4.23), effects of trapped electrons are ignored. They have relatively weak stabilizing influence on the ballooning mode through a reduction in the electron parallel current.

In the low frequency limit $|\omega| \ll k_{\|}v_{Te}$, the electron density perturbation can be approximated by

$$n_e = \frac{e\phi}{T_e} n_0 - \left( \frac{\omega - \tilde{\omega}_{se}(v^2)}{\omega + \tilde{\omega}_{De}(v) - k_{\|}v_{\|}} \left( \phi - \frac{v_{\|}}{c} A_{\|} \right) \right) \frac{e}{T_e} n_0$$

$$\simeq \left( \phi - \frac{\omega - \tilde{\omega}_{se}}{ck_{\|} A_{\|}} \right) \frac{e}{T_e} n_0.$$

(4.26)

The parallel current is largely carried by the electrons, and can be evaluated from the 1st order moment of
the perturbed electron velocity distribution function,
\[ J_{||e} = -e \int v_{||} f_e \, dv \]
\[ \simeq \frac{n_0 e^2}{k|| T_e} \left[ (\omega_{se} - \omega) \phi + \frac{(\omega - \omega_{se})(\omega - \omega_{De}) + \eta_e \omega_{se} \omega_{De}}{c k||} A|| \right]. \] (4.27)

Substituting the ion and electron density perturbations into the charge neutrality condition,
\[ n_e = n_i, \]
and the parallel electron current into Ampere’s law,
\[ \nabla^2 A|| = -\frac{4\pi}{c} J||, \]
we obtain
\[ (-1 + I_i) \frac{e \phi}{T_i} n_0 = \left( \phi - \frac{\omega - \omega_{se}}{c k||} A|| \right) \frac{e \phi}{T_e} n_0, \] (4.28)
and
\[ \nabla^2 A|| = -\frac{4\pi}{c} \frac{n_0 e^2}{k|| T_e} \left[ (\omega_{se} - \omega) \phi + \frac{(\omega - \omega_{se})(\omega - \omega_{De}) + \eta_e \omega_{se} \omega_{De}}{c k||} A|| \right]. \] (4.29)
These two equations form a closed set for the two unknowns, \( \phi \) and \( A|| \). The parallel Ampere’s law can be rearranged as
\[ k|| k_{||} A|| = \frac{k_{De}^2}{c^2} \left\{ (\omega_{se} - \omega) \phi + \frac{(\omega - \omega_{se})(\omega - \omega_{De}) + \eta_e \omega_{se} \omega_{De}}{c k||} A|| \right\}. \] (4.30)
where \( k_{De}^2 = 4\pi n_0 e^2 / T_e \) is the square of the electron Debye wavenumber. Eliminating \( A|| \) between Eqs. (4.28) and (4.30) yields the following kinetic ballooning mode equation,
\[ k_{||} k_{||} k_{||} \tilde{\phi} + \frac{k_{De}^2}{c^2} \left[ \frac{(\omega - \omega_{se})^2}{1 + \tau - \tau I_i} - (\omega - \omega_{De})(\omega - \omega_{se}) - \eta_e \omega_{se} \omega_{De} \right] \tilde{\phi} = 0, \] (4.31)
where \( \tilde{\phi} \) is a reduced scalar potential defined by
\[ \tilde{\phi} = (1 + \tau - \tau I_i) \phi. \] (4.32)

After ballooning transformation, Eq. (4.31) is converted into a differential equation,
\[ \frac{d}{d\theta} \left\{ [1 + (s \theta - \alpha \sin \theta)^2] \frac{d \phi}{d\theta} \right\} + \frac{\alpha}{4 \epsilon_n (1 + \eta)} \left\{ (\Omega - 1)[\Omega - f(\theta)] + \eta f(\theta) - \frac{(\Omega - 1)^2}{2 - I_i(\theta)} \right\} \phi = 0, \] (4.33)
where \( T_i = T_e \) and \( \eta_e = \eta_e = \eta \) have been assumed and the mode frequency is normalized by electron
diamagnetic drift frequency, $\Omega = \omega/\omega_{ce}$. In the MHD limit $|\omega| \gg \omega_s$, $\omega_D$ and $(k_{\perp} \rho)^2 \ll 1$, we readily recover the MHD ballooning mode equation analyzed in Chapter 2. It is noted that the safety factor $q$ is absorbed in the ballooning parameter defined by

$$\alpha = q^2 \frac{R}{L_n} [\beta_i (1 + \eta_i) + \beta_e (1 + \eta_e)],$$

and does not appear explicitly in the mode equation. This is because the ion acoustic transit effect has been ignored by assuming $\omega \gg k_{\parallel} c_s$. When this condition becomes marginal, an integral equation approach must be taken as will be discussed in Section 4.5.

The ion integral, $I_i (\theta)$, involves double integrations over $v_{\perp}$ and $v_{\parallel}$. Its explicit form is

$$I_i (\theta) = \frac{4}{\sqrt{\pi}} \int_0^\infty \frac{\Omega + 1 + \eta_i (x^2 + y^2 - \frac{3}{2})}{\Omega + f(\theta) (\frac{1}{2} x^2 + y^2)} J_0 \left[ \sqrt{2\xi(\theta)} x \right] e^{-x^2 - y^2} x dx dy,$$

(4.34)

where $x = v_{\perp}/v_{Ti}$, $y = v_{\parallel}/v_{Ti}$ ($v_{Ti} = \sqrt{2T_i/M}$) are the normalized velocities, and $\xi(\theta)$ is defined by,

$$\xi(\theta) = k_{\parallel} \rho \sqrt{1 + (s\theta - \alpha \sin \theta)^2}.$$ (4.35)

The integral is to be evaluated numerically within a shooting code at every point in the ballooning space $\theta$. Exact numerical evaluation of the integral is time consuming. An efficient means to evaluate the velocity integral is the Gaussian-Hermite quadrature method which exploits the exponential function, $e^{-x^2 - y^2}$, contained in the equilibrium Maxwellian velocity distribution function. The one-dimensional Gaussian-Hermite quadrature formula is,

$$\int_{-\infty}^{\infty} e^{-x^2} f(x) dx = \sum_i A_i f(x_i),$$

(4.36)

where $\{x_i, i = 1, \cdots, n\}$ are the integral nodes and $\{A_i, i = 1, \cdots, n\}$ the weighting coefficients. Both $x_i$ and $A_i$ are independent of the integrand $f(x)$. $x_i$ are the roots of the $n-$th order Hermite polynomial, $H_n(x)$, and the coefficients $A_i$ are given by

$$A_i = \frac{2^{n-1} (n-1)! \sqrt{\pi}}{2n [H_{n-1}(x)]^2}.$$ (4.37)

Both $x_i$ and $A_i$ are tabulated, or can be generated by the following recurrence relations,

$$\begin{align*}
H_{n+1} (x) &= 2x H_n (x) - 2n H_{n-1} (x), \\
H_0 (x) &= 1, \\
H_1 (x) &= 2x.
\end{align*}$$

(4.38)
The integral $I_i(\theta)$ in the ballooning mode equation involves two dimensional integrations of the form

$$\int_0^\infty \int_0^\infty f(x, y)e^{-x^2-y^2} dx dy \approx \sum_i \sum_j A_i A_j f(x_i, y_j).$$  \hspace{1cm} (4.39)$$

For the integral $I_i(\theta)$ it has been found that $n \sim 40$ is sufficient to provide the required accuracy.

An alternative is to find an analytic approximation for the integral. Guided by the two fluid ion density perturbation in the long wavelength limit worked out in Chapter 3, it has been shown that the following expression provides a reasonable approximation for $I_i(\theta)$ which agrees qualitatively with the exact velocity integral,

$$I_i(\theta) = \frac{(\omega + \frac{5}{3} \omega_D i) (\omega + \omega_s i) - \eta_i \omega_D i \omega_s i}{(\omega + \frac{5}{3} \omega_D i)^2 - \frac{10}{9} \omega_D^2} e^{-\lambda I_0(\lambda)}$$

$$- \frac{\eta_i (\omega + \frac{5}{3} \omega_D i) \omega_D i \lambda}{(\omega + \frac{5}{3} \omega_D i)^2 - \frac{10}{9} \omega_D^2} e^{-\lambda [I_0(\lambda) - I_1(\lambda)]},$$

where $\lambda(\theta) = (k_\theta \rho)^2 [1 + (\alpha \sin \theta - \alpha)^2]$, and $I_{0,1}$ are the zero-th and first order modified Bessel function, respectively. The approximation in Eq. (4.40) qualitatively agrees with the exact integral and may be useful for analytic stability analysis of low frequency electrostatic and electromagnetic modes.

### 4.4 Analysis of the Kinetic Ballooning Mode

With the numerical techniques described in the preceding section, we now present the results of stability analysis of the kinetic ballooning mode. In Fig. 4.1 (a) and (b), the growth rate and mode frequency normalized by the Alfvén frequency $\gamma/\omega_A$ and $\omega_r/\omega_A$ ($\omega_A = V_A/qR$) are shown when $s = 0.4$, $b_0 = (k_\theta \rho)^2 = 0.01$, $\eta_i = \eta_e = 2$, $\epsilon_n = L_n/R = 0.175$. The growth rates predicted by the ideal MHD and two-fluid analyses are also shown (dotted line). As far as the maximum growth rate is concerned, the MHD and kinetic theories agree well. The critical $\alpha$ for the onset of the ballooning mode from the kinetic theory is $\alpha_c = 0.35$, which is somewhat smaller than that obtained from the ideal MHD theory, $\alpha_c = 0.39$. The dashed line shows a second mode revealed by the kinetic analysis.

The second stability regime predicted by the ideal MHD essentially disappears in kinetic analyses. The growth rate revealed from the kinetic analysis persists in the MHD second stability region. The kinetic ballooning mode in the MHD second stability regime requires a finite ion temperature gradient, $\eta_i \gtrsim 1$, and is driven through the resonance contained in the non-adiabatic ion density perturbation, $I_i(\theta)$.

At small shear, the critical $\alpha$ for the (kinetic) ballooning mode becomes small and the instability becomes threshold-less and remains unstable at any $\alpha$. Fig. 4.2 shows the case $s = 0.2$ with other parameters unchanged from those in Fig. 4.1.
Figure 4-1: Growth rate $\gamma/\omega_A$ and frequency $\omega_r/\omega_A$ of the kinetic ballooning mode vs. $\alpha$ when $s = 0.4$, $q = 2$, $b_0 = 0.01$, $T_i = T_e$, $L_n/R = 0.175$, $n_i = n_e = 2$. The dotted line shows the growth rate of the ideal MHD ballooning mode and dashes lines show the second kinetic ballooning mode.

4.5 Finite $\beta$ Effects on the Drift Type Modes

The toroidal drift mode and ion temperature gradient mode are predominantly electrostatic. In the analyses presented in Chapter 3, the magnetic perturbation $A_\parallel$ was ignored which, however, is justifiable for discharges with negligible $\beta$. The magnetic perturbation $A_\parallel$ is proportional to $\beta \phi$. This may be seen from the parallel Ampere’s law

$$\nabla^2 A_\parallel = -\frac{4\pi}{c} J_\parallel. \quad (4.41)$$

Taking divergence of both sides and recalling the charge neutrality condition $\nabla \cdot J_\perp + \nabla \cdot J_\parallel = 0$, we obtain

$$\nabla \cdot \nabla^2 A_\parallel = \frac{4\pi}{c} \nabla \cdot J_\perp, \quad (4.42)$$
where the divergence of the cross field current may be approximated by

$$\nabla \cdot \mathbf{J} \approx -i n_0 e^2 \left( \omega + (1 + \eta_i) \omega_{se} \right) (k_L p_k)^2 + \frac{\omega_{se}(\omega_D e + \omega_D i)}{\omega} \right) \phi. \quad (4.43)$$

For $\beta$ well below the MHD ballooning limit, the ion polarization current is dominant and the parallel vector potential is related to the scalar potential through

$$A_{||} \approx \frac{\omega + \omega_{pi}}{ck_{||}} \left( \frac{\omega_{pi}}{\Omega_i} \right)^2 \phi \propto \beta \phi. \quad (4.44)$$

Substituting this into the electron density perturbation, we find

$$n_e = \left( \phi - \frac{\omega - \omega_{se}}{ck_{||}} A_{||} \right) e T_{e} n_0 = \left( 1 - \frac{(\omega - \omega_{se})(\omega + (1 + \eta_i) \omega_{se})}{(ck_{||})^2} \left( \frac{\omega_{pi}}{\Omega_i} \right)^2 \right) \frac{e \phi}{T_{e}} n_0. \quad (4.45)$$

For modes having a frequency close to the electron diamagnetic frequency, electromagnetic corrections are small. For the long wavelength ion acoustic drift mode characterized by $\omega > \omega_{se}$, the electron density
perturbation becomes smaller than that in the electrostatic limit,

\[ n_e < \frac{e\phi}{T_e} n_0, \quad (4.46) \]

that is, electron response is less adiabatic which should be further destabilizing. In contrast, the toroidal \( \eta_i \) mode is characterized by \( \omega < 0 \) and \( |\omega| < (1 + \eta_i)\omega_{\text{ci}} \). Then, the electron response is more adiabatic in this case,

\[ n_e > \frac{e\phi}{T_e} n_0, \quad (4.47) \]

and electromagnetic (finite \( \beta \)) corrections to the \( \eta_i \) mode are expected to be stabilizing.

Finite \( \beta \) effects on the toroidal \( \eta_i \) mode can still be analyzed by Eq. (4.33) as long as the ion transit frequency is negligible, \( |\omega| > k_\parallel v_{Ti} \). In this limit, the ion dynamics remains electrostatic and electromagnetic effects enter mainly through electron dynamics. Eq. (4.33) describes both kinetic ballooning mode and \( \eta_i \) mode corrected for finite \( \beta \) effects. For the ballooning mode having a frequency \( |\omega| \simeq k_\parallel V_A \), the condition is well satisfied relatively independent of the finite Larmor radius parameter \( k_\perp \rho \). However, for the toroidal \( \eta_i \) mode, the eigenvalue \( \omega \) scales with \( \omega \propto k_\perp \rho \) and the condition \( |\omega| > k_\parallel v_{Ti} \) is satisfied only for comparatively short wavelengths.

In contrast to the \( \eta_i \) mode, the ion acoustic drift mode is further destabilized by \( \beta \). Eq. (4.33) is, unfortunately, inapplicable to analyzing the ion acoustic mode because the mode frequency in this case is close to the ion acoustic transit frequency \( k_\parallel c_s \simeq k_\parallel v_{Ti} (T_e \simeq T_i) \) and the assumption \( \omega \gg k_\parallel v_{Ti} \) breaks down. A rigorous analysis on finite \( \beta \) effects on the ion acoustic mode requires integral equation formulation which has recently been developed in the Plasma Physics Laboratory.

Figure 4.3 shows stabilizing effect of total \( \alpha \) (ballooning parameter) on the toroidal \( \eta_i \) mode in various conditions. In (a), \( L_n/R = 0.2 \), \( \eta_i = \eta_e = 2 \), in (b), \( L_n/R = 0.5 \) (nearly flat density profile), \( \eta_i = \eta_e = 4 \), and in (c), same condition as in (a) except trapped electrons are included (\( r/R = 0.2 \)). Common parameters are: \( (k_\phi \rho)^2 = 0.1 \), \( s = 1 \), \( T_i = T_e \). Stabilization of the toroidal \( \eta_i \) mode occurs when the ballooning parameter \( \alpha \) exceeds a threshold which depends on discharge parameters. Trapped electrons have destabilizing effect on the \( \eta_i \) mode. (This is in contrast to the case of the kinetic ballooning mode which tends to be stabilized by trapped electrons.) The critical \( \alpha_e \) required for stabilization of the \( \eta_i \) mode is approximately given by

\[ \alpha_e \gtrsim \frac{1 + \eta_e}{3 (1 - \sqrt{\varepsilon}) (1 + 2\varepsilon \eta_e) (\tau + 1) + \tau^2 \eta_e}, \quad (4.48) \]

where \( \varepsilon = r/R \), \( \tau = T_e/T_i \), \( \varepsilon_n = L_n/R \).
When the mode frequency approaches either the ion transit frequency \( k_i v_{Ti} \) or the electron transit frequency \( k_i v_{Te} \), the differential formulation breaks down and formulation based on integral equations in the ballooning space \( \theta \) must be used. \( \omega \approx k_i v_{Ti} \) (\( \approx k_i c_s \) if \( T_i \approx T_e \)) occurs in long wavelength regime \( (k_i \rho_i)^2 \ll 1 \), and \( \omega \approx k_i v_{Te} \) in short wavelength regime \( (k_i \rho_i)^2 > 1 \). In this Section, derivation of integral equations suitable for stability analysis of tokamaks will be outlined. Unfortunately, resultant integral equations can only be solved numerically.

The perturbed distribution \( f \) derived in Section 4.1,

\[
f = - \frac{e \phi}{T} f_M + \frac{\omega - \tilde{\omega}_s}{\omega - \tilde{\omega}_D - k_i v_i} \left( \phi - \frac{v_i}{c} A_i \right) J_i^0(\Lambda) \frac{e}{T} f_M, \tag{4.49}
\]

is, strictly speaking, valid only if the operator nature of the parallel wavenumber \( k_i \) is ignorable. If not, the
nonadiabatic part \( h \) in \( f \),

\[
f = -\frac{e\phi}{T} f_M + h \langle e^{i\mathbf{k} \cdot \mathbf{r}} \rangle = -\frac{e\phi}{T} f_M + h J_0 (k_{\perp} \rho), \tag{4.50}
\]

should be calculated by solving the gyro-kinetic equation for \( h \),

\[
\frac{\partial h}{\partial t} + \mathbf{V}_{D0} \cdot \nabla h + \langle \mathbf{V}_{D1} \rangle \cdot \nabla f_M + \langle \frac{\partial \varepsilon_1}{\partial t} \rangle \langle \frac{\partial f_M}{\partial \varepsilon} \rangle = 0, \tag{4.51}
\]

where \( \mathbf{V}_{D0} \) is the zero-th order guiding center drift velocity,

\[
\mathbf{V}_{D0} = v_\parallel + \frac{mc}{eB^2} \left( \frac{1}{2} v_\perp^2 + v_\parallel^2 \right) \mathbf{B} \times \nabla B, \tag{4.52}
\]

\( \mathbf{V}_{D1} \) is the perturbed guiding center drift due to wave motion,

\[
\mathbf{V}_{D1} = \frac{c}{B} \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2} = i \frac{c}{B^2} \left( \phi - \frac{v_\parallel}{c} A_\parallel \right) \mathbf{B} \times \mathbf{k}, \tag{4.53}
\]

and \( \varepsilon_1 \) is the energy perturbation,

\[
\varepsilon_1 = e \left( \phi - \frac{v_\parallel}{c} A_\parallel \right). \tag{4.54}
\]

Note that the perturbed electric field \( \mathbf{E}_\perp \) is

\[
\mathbf{E}_\perp = -\nabla_{\perp} \phi + \frac{1}{c} \left( \mathbf{v} \times \mathbf{B} \right)_{\perp} = -\nabla_{\perp} \phi + \frac{i}{c} \mathbf{v}_\parallel \times \left( k_{\perp} \times A_\parallel \right) = -i k_{\perp} \left( \phi - \frac{v_\parallel}{c} A_\parallel \right). \tag{4.55}
\]

(\( \cdots \)) indicates gyro-phase averaging. For example,

\[
\langle \phi_0 e^{i\mathbf{k} \cdot \mathbf{r}} \rangle = \phi_0 e^{i\mathbf{k} \cdot \mathbf{r}_C} \langle e^{i\mathbf{k} \cdot \mathbf{r}} \rangle = \phi_0 e^{i\mathbf{k} \cdot \mathbf{r}_C} J (k \rho),
\]

where \( \mathbf{r}_C \) is the coordinates of the guiding center.

With these preparations, Eq. (4.51) can now be cast into the form

\[
\frac{v_\parallel}{qR} \frac{\partial h}{\partial \theta} - i [\omega - \omega_D (v, \theta)] h + i (\omega - \omega_\star) \left( \phi - \frac{v_\parallel}{c} A_\parallel \right) J (\Lambda) \frac{c}{T} f_M = 0. \tag{4.55}
\]
where

\[
\omega_D (v, \theta) = \frac{mc}{eB^3} \left( \frac{1}{2} v_{\perp}^2 + v_{\parallel}^2 \right) k \cdot (B \times \nabla B)
\]

\[
= \frac{mc}{eBR} \left( \frac{1}{2} v_{\perp}^2 + v_{\parallel}^2 \right) k_\theta \left[ \cos \theta + (s\theta - \alpha \sin \theta) \sin \theta \right], \quad (4.56)
\]

\[
\omega_* = \omega_0 \left[ 1 + \eta \left( \frac{mv^2}{2T} - \frac{3}{2} \right) \right], \quad \omega_0 = \frac{eT \kappa}{eBL_n}.
\]}

and

\[
\Lambda (\theta) = \frac{k_\theta v_{\perp}}{\Omega} \sqrt{1 + (s\theta - \alpha \sin \theta)^2}.
\]}

Integration of Eq. (4.55) over \( \theta \) can be performed separately for circulating (untrapped) particles and trapped particles. For circulating particles, we obtain

\[
v_{\parallel} > 0 : \quad h_1^C (\theta) = -i \int_{-\infty}^{\theta} \exp \left( -i \Xi_{\theta}^D \right) \frac{qR}{v_{\parallel}} (\omega - \omega_*) \left( \phi - \frac{|v_{\parallel}|}{c} A_\parallel \right) J_0 (\Lambda') d\theta' \frac{e}{T} f_M,
\]

\[
v_{\parallel} < 0 : \quad h_2^C (\theta) = -i \int_{\theta}^{\infty} \exp \left( i \Xi_{\theta}^D \right) \frac{qR}{v_{\parallel}} (\omega - \omega_*) \left( \phi + \frac{|v_{\parallel}|}{c} A_\parallel \right) J_0 (\Lambda') d\theta' \frac{e}{T} f_M,
\]}

where

\[
\Xi_{\theta}^D = \int_{\theta}^{\theta' \prime} \frac{qR}{v_{\parallel}} \left[ \omega - \omega_D (\theta'' \prime) \right] d\theta''.
\]}

For trapped particles with turning point angles \( \theta_1 \) and \( \theta_2 \),

\[
h_T^\sigma (\theta) = \exp \left( i \sigma \Xi_{\theta_0}^D \right) h^\sigma (\theta_0) - i \sigma \int_{\theta_0}^{\theta} \exp \left( i \sigma \Xi_{\theta_0}^D \right) \Psi^\sigma (\theta'),
\]

where \( \sigma = 1 \) for \( v_{\parallel} > 0 \) and \( \sigma = -1 \) for \( v_{\parallel} < 0 \),

\[
\Psi^\sigma (\theta') = \Psi_{\phi} + \sigma \Psi_A,
\]

\[
\Psi_{\phi} = -\frac{qR}{v_{\parallel}} (\omega - \omega_*) J_0 (\Lambda') \frac{e \phi}{T} f_M,
\]

\[
\Psi_A = \frac{qR}{v_{\parallel}} (\omega - \omega_*) J_0 (\Lambda') \frac{|v_{\parallel}|}{c} \frac{e A_\parallel}{T} f_M,
\]}

and \( \theta_0 \) is arbitrary angular location between the turning points, \( \theta_1 < \theta < \theta_2 \). The boundary conditions are:

\[
h_T^+ (\theta_1) = h_T^- (\theta_1) \equiv h_T (\theta_1),
\]

\[
h_T^+ (\theta_2) = h_T^- (\theta_2) \equiv h_T (\theta_2).
\]
Since
\[ h_T^+ (\theta_1) = \exp \left( i \Xi^0_{\theta_1} \right) h_T^+ (\theta_2) - i \int_{\theta_1}^{\theta_2} \exp \left( i \Xi^0_{\theta'} \right) \Psi^+ (\theta') d\theta', \]
and
\[ h_T^- (\theta_2) = \exp \left( -i \Xi^0_{\theta_2} \right) h_T^- (\theta_1) + i \int_{\theta_1}^{\theta_2} \exp \left( -i \Xi^0_{\theta'} \right) \Psi^- (\theta') d\theta', \]
we find
\[ h_T (\theta_1) = \frac{\exp \left( i \Xi^0_{\theta_2} \right)}{1 - \exp \left( 2i \Xi^0_{\theta_2} \right)} \int_{\theta_1}^{\theta_2} \left[ \exp \left( -i \Xi^0_{\theta'} \right) \Psi^- (\theta') + \exp \left( i \Xi^0_{\theta'} \right) \Psi^+ (\theta') \right] d\theta'. \]

Since \( \theta_0 \) is arbitrary, we set it equal to \( \theta_1 \). Then
\[ h_T^\sigma (\theta) = \frac{\exp \left( i \sigma \Xi^0_{\theta_1} \right)}{2 \sin \Xi^0_{\theta_1}} \int_{\theta_1}^{\theta_2} \left[ \exp \left( -i \Xi^0_{\theta'} \right) \Psi^- (\theta') + \exp \left( i \Xi^0_{\theta'} \right) \Psi^+ (\theta') \right] d\theta' \]
and
\[ -i \sigma \int_{\theta_1}^{\theta_2} \exp \left( i \sigma \Xi^0_{\theta'} \right) \Psi^\sigma (\theta') d\theta'. \]

Most pressure gradient driven modes of interest fall in the frequency regime \( |\omega| \ll \omega_{be} \) (the bounce frequency of trapped electrons \( \omega_{be} \simeq v_{Te}/qR \)), and we may simplify \( h_T^\sigma (\theta) \) for trapped electrons as
\[ h_T^\sigma (\theta) \simeq \frac{1}{\Xi^0_{\theta_1}} \int_{\theta_1}^{\theta_2} \left\{ \Psi_0 (\theta') + i \left[ \Xi^0_{\theta'} - H (\theta - \theta') \Xi^0_{\theta_1} \right] \Psi_A (\theta') \right\} d\theta' + O \left( \frac{\omega}{\omega_{be}} \right), \]
where \( H (\theta - \theta') \) is the Heaviside step function. Trapped ion modes pertain to the frequency regime well below the ion bounce frequency \( |\omega| \ll \omega_i \) and are ignored.

Substitution of the perturbed distribution functions into the Poisson’s equation and parallel Ampere’s law yields the following integral equations,
\[ k^2 \phi = 4\pi n_0 \sum_j e_j \left( -\frac{e_j \phi}{T_j} + \int \left[ g^+_{Cj} + g^+_{Tj} \right] J_0 (\Lambda_j) d\nu \right), \]
\[ k^2 A_\parallel (\theta) = \frac{4\pi n_0}{c} \sum_j e_j \int \left| v_{||} \right| \left[ g^+_{Cj} + g^+_{Tj} \right] J_0 (\Lambda_j) d\nu, \]
where
\[ g^+_{C(T)j} = \frac{1}{2} \left[ h^+_{C(T)j} (\theta') + \sigma h^-_{C(T)j} (\theta') \right]. \]

An efficient numerical algorithm to solve the set of integral equations has been developed by M. Elia (Ph.D. thesis, 2000).

Bench-marking studies with the previously found eigenvalues (\( \omega = \omega_r + i\gamma \)), particularly those by Rewoldt and co-workers, have shown that eigenvalues in low beta regime agree. However, the growth rate of the ballooning mode found by Rewoldt seems overestimated. Fig. 4.4 shows comparison among the eigenvalues
from five different methods, ideal MHD, Rewoldt’s comprehensive integral equation code, kinetic differential equation method, semilocal kinetic code, and Elia-Hirose’s integral equation code (labelled “Nonlocal”) for discharge parameters pertinent to the Doublet III tokamak, \( r/R = 0.16 \), \( T_i = T_e \), \( q = 1.14 \), \( \eta_i = 0.87 \), \( \eta_e = 0.93 \), \( L_n/R = 0.07 \), when \( k_0\rho_i = 0.28 \) (\( n = 20 \)). The growth rate found by Rewoldt is comparable with that of ideal MHD, while all other methods predict growth rates considerably smaller.

Figure 4-4: Comparison among the eigenvalues found from 5 methods for discharge parameters pertinent to the Doublet III tokamak, \( r/R = 0.16 \), \( T_i = T_e \), \( q = 1.14 \), \( \eta_i = 0.87 \), \( \eta_e = 0.93 \), \( L_n/R = 0.07 \), when \( k_0\rho_i = 0.28 \) (\( n = 20 \)).