Chapter 5

IMAGES AND GREEN'S FUNCTIONS

5.1 Introduction

The potential due to a point charge q placed above a grounded conducting plane can be found as a sum of two potentials, one due to the charge itself, and the other due to an image charge, $-q$, placed at the mirror point with respect to the plane. This simple example indicates a possibility of greatly simplifying boundary value problems by introducing appropriate images for a given boundary shape.

In this Chapter, image charges for simple boundary shapes will be discussed. Finding an appropriate image enables us to derive a formula for the potential due to a prescribed boundary potential on the surface without necessarily going through the methods developed in Chapter 3. As will be shown, solving a boundary value problem is essentially identical to finding an appropriate scalar function (Green's function) for a given boundary shape.

5.2 Image for Flat Surfaces

Consider the configuration shown in Figure 5.1. The grounded, wide, conducting plane can be replaced by an image $-q$ placed at the mirror point of the real charge $+q$ as long as the potential and electric field in the space above the plane are concerned. The potential above the plane is thus given by

$$\Phi = \Phi_+ + \Phi_- = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) \tag{5.1}$$

where r_+ and r_- are the distances from the observing point to the respective charges. In Eq. (5.1), the first term in RHS is the contribution from the charge q. The potential Φ_+ satisfies the singular Poisson's equation

$$\nabla^2\Phi_+ = -\frac{q}{\epsilon_0}\delta\left(\mathbf{r} - \mathbf{r}_0\right) \tag{5.2}$$

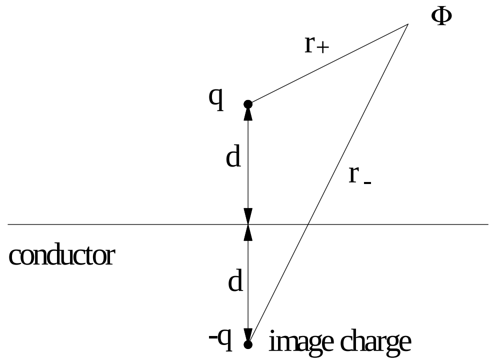

Figure 5-1: Image charge for a flat conducting plate.

where $\mathbf{r}_0 = d\mathbf{e}_z$ is the location of the charge q. The image charge $-q$ is below the plane. Therefore, in the space above the plane, the potential Φ_- due to the image charge should satisfy the Laplace equaion without any singularities,

$$\nabla^2 \Phi_- = 0 \quad (z > 0) \tag{5.3}$$

In other words, the potential Φ_- can be regarded as a general solution to the singular Poisson's equation in Eq. (5.2) which is added in order that the total potential satisfy the boundary condition,

$$\Phi = 0 \qquad \text{on the plane} \tag{5.4}$$

Obviously, the solution Φ_+ is the particular solution to the Poisson's equation. Note that we have a freedom to add general solutions satisfying the Laplace equation to a particular solution satisfying the Poisson's equation.

Images for a grounded wedge can be similarly found. For example, if the wedge angle is 90°, three images appear as shown in Figure 5.2. The potential is given by the sum of four contributions from each charge. The positive image charge is the image of the negative image charges.

The potential due to a charge q placed above a dielectric body can also be worked out by images. In this case, we may assume the potential in the air

$$\Phi_> = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} - \frac{q'}{r_2}\right) \tag{5.5}$$

and the potential in the dielectric

$$\Phi_< = \frac{1}{4\pi\epsilon_0}\frac{q''}{r_3} \tag{5.6}$$

where the distances r_1, r_2 and r_3 are indicated in Fig. 5.3, q' is the image charge in the dielectric, and q'' is another image located at the same position as the real charge q. Note that the potential in the dielectric remains regular (no singularity). To determine q' and q'', we first impose the

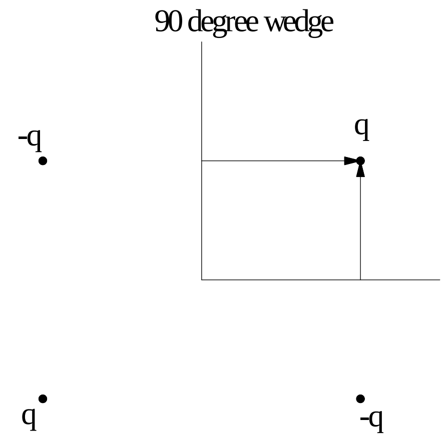

Figure 5-2: Images for 90 degree conductor wedge.

continuity of the potential at the boundary, $r_1 = r_2 = r_3$. This yields

$$\frac{1}{\epsilon_0}(q - q') = \frac{1}{\epsilon} q'' \tag{5.7}$$

The second boundary condition is the continuity in the displacement vector, D_z (normal component). Since

$$r_1 = \sqrt{x^2 + (z-d)^2} \tag{5.8}$$

we have

$$\frac{\partial}{\partial z}\left(\frac{1}{r_1}\right) = -\frac{z-d}{[x^2 + (z-d)]^{3/2}} \tag{5.9}$$

Similarly,

$$\begin{aligned} \frac{\partial}{\partial z}\left(\frac{1}{r_2}\right) &= -\frac{z+d}{[x^2+(z+d)^2]^{3/2}} \\ \frac{\partial}{\partial z}\left(\frac{1}{r_3}\right) & \quad -\frac{z-d}{[x^2+(d-z)^2]^{3/2}} \qquad (z<0) \end{aligned} \tag{5.10}$$

Therefore, the displacement vector just above the boundary surface becomes

$$D_{z1} = \frac{d}{4\pi(x^2+d^2)^{3/2}}(-q' - q) \tag{5.11}$$

and that just below the surface is

$$D_{z2} = \frac{-dq''}{4\pi(x^2+d^2)^{3/2}} \tag{5.12}$$

From $D_{z1} = D_{z2}$, we obtain

$$q'' = q' + q \tag{5.13}$$

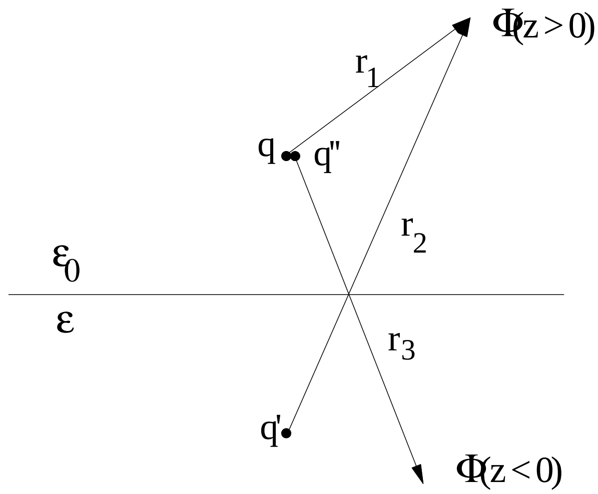

Figure 5-3: Images q' and q'' for a flat dielectric boundary.

Solving Eqs. (5.7) and (5.13) for q' and q'', we find

$$q' = \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \tag{5.14}$$

$$q'' = \frac{2\epsilon}{\epsilon + \epsilon_0} q \tag{5.15}$$

5.3 Image for Cylindrical Surfaces (Two Dimensional)

The potential due to a long line charge λ (C/m) placed parallel to a grounded conducting cylinder can be found by the method of image. In Chapter 2, we saw that two opposite line charges of equal magnitudes, $+\lambda$ and $-\lambda$ create a family of equipotential cylindrical surfaces,

$$\Phi(r_+, r_-) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_-}{r_+}\right) \tag{5.16}$$

where r_+ and r_- are the distances to the respective line charges, $+\lambda$ and $-\lambda$. Consider a line charge at a distance ρ' from the axis of a cylinder having a radius a. A negative line charge $-\lambda$ at a distance $\rho'' = a^2/\rho_1$ and the positive line charge λ make the cylinder surface an equipotential surface

$$\Phi_s = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{a}{\rho'}\right) \tag{5.17}$$

(Check this by considering the potentials at A and B in Figure 5.4.) In order to make the cylinder surface at zero potential, we have to subtract this potential from the potential given in Eq. (5.16) which is based on the choice $\Phi = 0$ on the midplane, $r_+ = r_-$. Therefore, the solution to the potential becomes

$$\Phi(\mathbf{r}) = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\left(\frac{r_-}{r_+}\right) - \ln\left(\frac{a}{\rho_1}\right)\right] \tag{5.18}$$

But

$$r_+ = \left[\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi\right]^{1/2} \tag{5.19}$$

$$r_- = \left[\rho^2 + \left(\frac{a^2}{\rho'}\right)^2 - 2\rho\frac{a^2}{\rho'}\cos\phi\right]^{1/2} \tag{5.20}$$

Substituting these into Eq. (5.18), we finally obtain

$$\Phi(\rho,\phi) = \frac{\lambda}{2\pi\epsilon_0}\ln\left[\frac{\left(\dfrac{\rho^2\rho'^2}{a^2} + a^2 - 2\rho\rho'\cos\phi\right)^{1/2}}{\left(\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi\right)^{1/2}}\right] \tag{5.21}$$

When $\rho = a$, this indeed vanishes.

The surface charge density induced on the cylinder surface is given by

$$\begin{aligned} \sigma &= -\epsilon_0 \left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=a} \\ &= -\frac{\lambda}{2\pi}\frac{\dfrac{\rho_1^2}{a} - a}{\rho'^2 + a^2 - 2a\rho'\cos\phi} \quad (\mathrm{C/m^2}) \end{aligned} \tag{5.22}$$

Therefore, the total charge (per unit length) on the cylinder surface is

$$\frac{q}{l} = -\frac{\lambda}{2\pi}\left(\rho'^2 - a^2\right)\int_0^{2\pi}\frac{d\phi}{\rho'^2 + a^2 - 2a\rho'\cos\phi} \tag{5.23}$$

The integral reduces to

$$\frac{2\pi}{\rho'^2 - a^2} \tag{5.24}$$

when $\rho_1 > a$. Therefore, the charge per unit length of the cylinder is simply $-\lambda$, as expected.

5.4 Image for a Sphere

We consider a charge q at a distance r' from the center of a grounded conducting sphere of radius a. The potential is symmetric about the line connecting the center and the charge. Therefore, an image charge should be on the same line. We assume a negative charge $-q'$ placed at a distance x from the center. The potential at the point A is

$$\Phi_A = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r' - a} - \frac{q'}{a - x}\right) \tag{5.25}$$

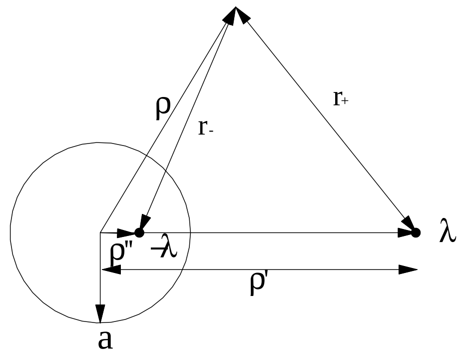

Figure 5-4: A line charge λ at a distance ρ' from the center of a conducting cylinder of radius a and its image line charge $-\lambda$ at $\rho'' = a^2/\rho'$ make the cylinder surface an equipotential surface at $\Phi_s = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(a/\rho'\right)$.

This should vanish if the image is to replace the sphere without affecting the potential outside the sphere. Therefore,

$$\frac{q}{r' - a} = \frac{q'}{a - x} \tag{5.26}$$

Similarly, the potential at B is given by

$$\Phi_B = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{D + a} - \frac{q'}{a + x}\right) = 0 \tag{5.27}$$

$$\frac{q}{r' + a} = \frac{q'}{a + x} \tag{5.28}$$

Solving Eqs. (5.26) and (5.28) for x and q', we find

$$q' = \frac{a}{r'}q \tag{5.29}$$

$$x = \frac{a^2}{r'} \tag{5.30}$$

Therefore, a charge $-aq/r'$ placed at a distance a^2/r' from the center can be used to replace the sphere as far as the potential and electric field <u>outside</u> the sphere are concerned. Outside the sphere, the potential is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{a/r'}{r_-}\right) \tag{5.31}$$

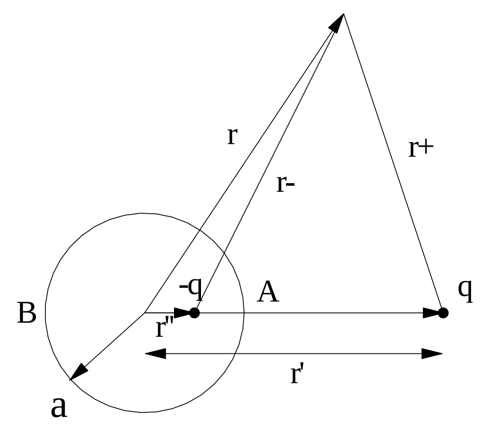

Figure 5-5: Image for a sphere. Charge q at distance r' from the center of a sphere of radius a and its image $-q$ at $r'' = a^2/r'$ make the potential vanish on the sphere surface $r = a$.

where r_1 and r_2 are the distances between the observing point and respective charges, and given, in terms of the coordinate (r, θ) at the observing point, by

$$
\begin{aligned}
r_+ &= \left(r^2 + r'^2 - 2rr'\cos\theta\right)^{1/2} \\
r_- &= \left[r^2 + \left(\frac{a^2}{r'}\right)^2 - 2\frac{a^2}{r'}r\cos\theta\right]^{1/2}
\end{aligned}
\tag{5.32}
$$

The surface charge density on the sphere surface can be evaluated from

$$
\begin{aligned}
\sigma(\theta) &= -\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} \\
&= \frac{1}{4\pi}\frac{\left(a - \frac{r'^2}{a}\right)q}{\left(a^2 + r'^2 - 2ar'\cos\theta\right)^{3/2}}
\end{aligned}
\tag{5.33}
$$

Therefore, the total charge residing on the outer surface of the sphere is given by the integral,

$$
\begin{aligned}
q_s &= \int_0^\pi \sigma(\theta) 2\pi a^2 \sin\theta d\theta \\
&= \left(a - \frac{r'^2}{a}\right) a^2 \int_{-1}^{1} \frac{1}{\left(a^2 + r'^2 - 2ar'\mu\right)^{3/2}} d\mu \quad (\mu = \cos\theta) \\
&= -\frac{a}{r'}q = -q'
\end{aligned}
$$

This result is expected from the Gauss' law because the $-q'$ is the only charge enclosed by the sphere surface.

Example

A charge q is placed at a distance D from the center of an isolated (floating) conducting sphere of a radius a. When the sphere carries no charges, what will the sphere potential be?

We first make the sphere potential zero by placing a charge $-q' = -aq/D$ at $r' = a^2/D$ as in the preceding discussion. Since the sphere carries no net charge, if a charge $+q' = +aq/D$ is placed at the center of the sphere, the sphere potential will be raised (when $q > 0$) to

$$\begin{aligned} \Phi_s &= \frac{1}{4\pi\epsilon_0}\frac{aq/D}{a} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{D} \end{aligned} \tag{5.34}$$

Note that this is independent of the sphere radius a. When the sphere carries a net charge Q, the sphere potential becomes

$$\Phi_s = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{D} + \frac{Q}{a}\right) \tag{5.35}$$

5.5 Green's Theorem and Green's Function

The method of image charges has a more important application in potential boundary value problems. When a potential is prescribed on a closed surface, it uniquely determines the potential in the space surrounding the surface and also in the space surrounded by the surface. The most general formulation for the potential is given in terms of a scalar function called Green's function, which is closely related to the potential due to a charge and its image for a given surface shape.

For arbitrary scalar functions ϕ and ψ, the following identity immediately follows from Gauss' theorem,

$$\int \boldsymbol{\nabla} \cdot (\phi \boldsymbol{\nabla} \psi) dV = \oint \phi \boldsymbol{\nabla} \psi \cdot d\mathbf{S} \tag{5.36}$$

The LHS is equal to

$$\int \left[\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi + \phi\nabla^2\psi\right] dV \tag{5.37}$$

Therefore,

$$\int \left(\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi + \phi\nabla^2\phi\right) dV = \oint \phi \boldsymbol{\nabla}\psi \cdot d\mathbf{S} \tag{5.38}$$

Exchanging ϕ and ψ, we obtain

$$\int \left(\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\phi + \psi\nabla^2\phi\right) dV = \oint \psi \boldsymbol{\nabla}\phi \cdot d\mathbf{S} \tag{5.39}$$

Substracting Eq. (5.39) from Eq. (5.38), yields

$$\int \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \oint \left(\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi\right) \cdot d\mathbf{S} \tag{5.40}$$

This is known as the Green's theorem. Its usefulness in potential problems can be appreciated as follows.

In Eq. (5.40), we identify ϕ as the desired scalar potential $\Phi(\mathbf{r})$, and ψ as a solution to the following singular Poisson's equation

$$\nabla^2 G = -\delta\left(\mathbf{r} - \mathbf{r}'\right) \tag{5.41}$$

Since the potential $\Phi(\mathbf{r})$ satisfies Poisson's equation,

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0}\rho \tag{5.42}$$

and

$$\int \Phi(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')dV' = \Phi(\mathbf{r}) \tag{5.43}$$

Eq. (5.40) yields,

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0}\int G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')dV' - \oint_s (\Phi_s \boldsymbol{\nabla}' G - G\boldsymbol{\nabla}'\Phi_s)\cdot d\mathbf{S}' \tag{5.44}$$

where $\Phi_s(\mathbf{r}')$ is the surface potential prescribed on S, and $\boldsymbol{\nabla}'$ indicates derivative with respect to the surface coordinates $\mathbf{r}'$. When there are no surfaces, and the potential is entirely due to a prescribed charge density distribution ρ, the Green's function is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi}\frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{5.45}$$

and the potential is given by the familiar form,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}dV' \tag{5.46}$$

In the absence of charges ($\rho = 0$), the potential is determined by the surface potential Φ_s and its derivative, $\boldsymbol{\nabla}\Phi_s$, which is of course related to the electric field on the surface.

The Green's function in Eq. (5.45) is essentially the potential due to a charge located at $\mathbf{r}'$. However, in addition to this particular solution, any general solutions satisfying

$$\nabla^2 G_g = 0 \tag{5.47}$$

can be added to the Green's function so that it vanishes on a given surface. In this case ($G = 0$ on S), the potential due to a prescribed surface potential distribution becomes,

$$\Phi(\mathbf{r}) = -\oint \Phi_s(\mathbf{r}')\frac{\partial G}{\partial n'}dS' \tag{5.48}$$

where $\mathbf{n}'$ is the normal vector on the surface, which is directed away from the region in which we wish to evaluate the potential. (Fig. 5.6) This is known as the Dirichlet's formulation for potential boundary value problems.

Solving a boundary value potential problem is now reduced to finding a suitable Green's function for a given surface shape, on which $G = 0$. This is where the method of image will be very useful.

Green's Function for a Sphere

Consider a spherical surface having radius a. We place a charge q at $\mathbf{r}'$. An image charge $-q' = -aq/r'$ at

$$\mathbf{r}'' = \frac{a^2}{r'^2}\mathbf{r}' \tag{5.49}$$

and the original charge q at $\mathbf{r}'$ make the surface potential vanish. Therefore, the Green's function for a sphere is immediately written as

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a/r'}{|\mathbf{r}-\mathbf{r}''|}\right) \tag{5.50}$$

Noting,

$$\left|\mathbf{r}-\mathbf{r}'\right| = \left(r^2 + r'^2 - 2rr'\cos\gamma\right)^{1/2} \tag{5.51}$$

$$\left|\mathbf{r}-\mathbf{r}''\right| = \left[r^2 + \left(\frac{a^2}{r'}\right)^2 - 2\frac{a^2}{r'}r\cos\gamma\right]^{1/2} \tag{5.52}$$

Fig. 5.6 Geometry for the Dirichlet boundary value problems,

Fig. 5.7 Greeen's function for a spherical boundary is essentially equivalent to the potential due to a charge near a grounded conducting sphere of the same radius.

we find

$$\begin{aligned}
G(\mathbf{r}, \mathbf{r}') &= \frac{1}{4\pi}\left(\frac{1}{(r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}} - \frac{\frac{a}{r'}}{\left[r^2 + \left(\frac{a^2}{r'}\right)^2 - 2\frac{a^2}{r'}r\cos\gamma\right]^{1/2}}\right) \\
&= \frac{1}{4\pi}\left(\frac{1}{(r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{r^2r'^2}{a^2} + a^2 - 2rr'\cos\gamma\right)^{1/2}}\right)
\end{aligned}$$

When we wish to evaluate the potential outside the sphere, the normal derivative is given by

$$\begin{aligned}
\frac{\partial G}{\partial n'} &= -\left.\frac{\partial G}{\partial r'}\right|_{r'=a} \\
&= \frac{1}{4\pi}\left(\frac{a - r\cos\gamma}{(r^2 + a^2 - 2ra\cos\gamma)^{3/2}} - \frac{\frac{r^2}{a} - r\cos\gamma}{(r^2 + a^2 - 2ra\cos\gamma)^{3/2}}\right) \\
&= \frac{1}{4\pi}\left(a - \frac{r^2}{a}\right)\frac{1}{(r^2 + a^2 - 2ra\cos\gamma)^{3/2}} \qquad (r > a)
\end{aligned} \tag{5.53}$$

For interior problems, the negative of the above equation is to be employed. Recall that $\cos\gamma$ in

Eq. (5.53) is given by

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \tag{5.54}$$

in terms of the angular locations of the vectors $\mathbf{r} = (r, \theta, \phi)$ and $\mathbf{r}' = (r', \theta', \phi')$. Substituting Eq. (5.53) into Eq. (5.48), we finally obtain the desired exterior potential,

$$\Phi(\mathbf{r}) = \frac{a^2}{4\pi}\left(\frac{r^2}{a} - a\right)\int_0^{\pi}\sin\theta' d\theta' \int_0^{2\pi} d\phi' \frac{\Phi_s(\theta', \phi')}{(r^2 + a^2 - 2ra\cos\gamma)^{3/2}} \tag{5.55}$$

Let us apply this formula to the problem of oppositely charged hemisphere which we solved in Chapter 3. The surface potential $\Phi_s(\theta)$ is described by,

$$\Phi_s(\theta) = \begin{cases} +V & 0 < \theta < \dfrac{\pi}{2} \\ \\ -V & \dfrac{\pi}{2} < \theta < \pi \end{cases} \tag{5.56}$$

and is independent of the azimuthal angle ϕ. On the axis, $\theta = 0$, and the potential becomes

$$\begin{aligned} \Phi(z) &= \frac{a^2}{4\pi}\left(\frac{z^2}{a} - a\right)2\pi\left[\int_0^{\pi/2}\frac{\sin\theta' d\theta'}{(z^2 + a^2 - 2az\cos\theta')^{3/2}}\right. \\ &\qquad \left. - \int_{\pi/2}^{\pi}\frac{\sin\theta' d\theta'}{(z^2 + a^2 - 2az\cos\theta')^{3/2}}\right] \\ &= V\left(\frac{a^2 - z^2}{z\sqrt{z^2 + a^2}} + \text{sign } z\right) \end{aligned} \tag{5.57}$$

where

$$\text{sign } z = \begin{cases} +1 & \text{for} \quad z > 0 \\ -1 & \text{for} \quad z < 0 \end{cases} \tag{5.58}$$

Note that the potential $\Phi(z)$ must be an odd function of z. When $z > a$, Eq. (5.57) may be expanded as

$$\Phi(z) = V\left[\frac{3}{2}\left(\frac{a}{z}\right)^2 - \frac{7}{8}\left(\frac{a}{z}\right)^4 + \cdots\right] \tag{5.59}$$

Therefore, at an arbitrary location $(r > a, \theta)$, the potential is given by,

$$\Phi(r, \theta) = V\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \cdots\right] \tag{5.60}$$

which agrees with the previous result in Eq. (3.175). The interior potential $(r < a)$ can immediately be written down by inspection as

$$\Phi(r, \theta) = V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \cdots\right] \tag{5.61}$$

Green's Function for a Long Cylinder (Two Dimensional)

The image line charge and corresponding potential due to a long line charge placed parallel to a grounded conducting cylindrical surface has been worked out in the preceding section. The potential is given by

$$\Phi\left(\mathbf{r},\mathbf{r}'\right)=\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{a\left|\mathbf{r}-\mathbf{r}''\right|}{\rho'\left|\mathbf{r}-\mathbf{r}'\right|}\right) \tag{5.62}$$

where

$$\begin{array}{ll} \mathbf{r}=(\rho,\phi) & \text{(observing point)} \\ \mathbf{r}'=(\rho',\phi') & \text{(location of the line charge)} \\ \mathbf{r}''=\left(\dfrac{a^2}{\rho},\phi'\right) & \text{(location of the image)} \end{array}$$

Then, the Green's function for a cylinder can be identified as

$$\begin{aligned} G(\mathbf{r},\mathbf{r}') &= \frac{1}{2\pi}\ln\left(\frac{a\left|\mathbf{r}-\mathbf{r}''\right|}{\rho'\left|\mathbf{r}-\mathbf{r}'\right|}\right) \\ &= -\frac{1}{2\pi}\ln\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right]^{1/2} \\ &\quad +\frac{1}{2\pi}\ln\left(\frac{\rho^2\rho'^2}{a^2}+a^2-2\rho\rho'\cos(\phi-\phi')\right)^{1/2} \end{aligned} \tag{5.63}$$

For exterior ($\rho>a$), the relevant normal derivative is

$$\begin{aligned} \frac{\partial G}{\partial n} &= -\left.\frac{\partial G}{\partial \rho'}\right|_{\rho'=a} \\ &= \frac{1}{2\pi}\frac{a-\dfrac{\rho^2}{a}}{\rho^2+a^2-2a\rho\cos(\phi-\phi')} \qquad (\rho>a) \end{aligned} \tag{5.64}$$

For interior, the negative of Eq. (5.64) is to be used.

It is left for an exercise to recover the potential due to a cylinder, its upper half being at $+V$ and lower half at $-V$, worked out in Chapter 3.