Chapter 2

Electrostatics II
Boundary Value Problems

2.1 Introduction

In Chapter 1, we have seen that the static scalar potential $\Phi(\mathbf{r})$ satisfies the Poisson’s equation,

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$  \hspace{1cm} (2.1)

For a prescribed charge density distribution, $\rho(\mathbf{r})$, a formal solution for the potential has been,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

However, in many electrostatic problems, particularly those involving conductors, the charge density distribution is not known \textit{a priori}. Rather, the potential is prescribed on a given surface as a boundary value, and it is required to find the potential elsewhere off the boundary surface. Such formulation is called the boundary value problem for the scalar potential.

To become familiar with the principle behind the boundary value problem, let us consider an almost trivial case, a parallel plate capacitor configuration shown in Fig. ?? . If the plate size is much larger than the separation distance $d$, the edge effects can be ignored, and the potential inside the plates becomes one dimensional, that is, $\Phi$ becomes a function of $z$ only, $\Phi(\mathbf{r}) = \Phi(z)$. The boundary conditions are:

$$\Phi = 0 \quad \text{at} \quad x = 0$$ \hspace{1cm} (2.2)

$$\Phi = V \quad \text{at} \quad x = d$$ \hspace{1cm} (2.3)

provided the lower plate is grounded. Since inside the capacitor, there are no charges ($\rho = 0$), the Poisson’s equation reduces to the following Laplace equation,

$$\frac{d^2 \Phi}{dx^2} = 0,$$ \hspace{1cm} (2.4)
Figure 2-1: Potential $\Phi(x) = \frac{V}{d}x$ and electric field $E_x = -\frac{V}{d}$ in a parallel plate capacitor.

which is to be solved under the boundary conditions given above. A general solution is

$$\Phi(x) = ax + b$$  \hspace{1cm} (2.5)

where $a$ and $b$ are the integration constants. Applying the boundary conditions, we readily find

$$a = \frac{V}{d} \text{ and } b = 0$$

and the potential inside the capacitor is uniquely determined as

$$\Phi(x) = \frac{V}{d}x$$  \hspace{1cm} (2.6)

The electric field can be found from the gradient of the potential,

$$\mathbf{E} = -\nabla\Phi = -\frac{d\Phi(x)}{dz} = -\frac{V}{d}\mathbf{e}_x$$  \hspace{1cm} (2.7)

which is an expected result. Recalling the field due to charge sheets given in Eq. (2.23), we conclude that the surface charge density $\pm \sigma$ (C/m$^2$) residing on the plates are given by

$$\sigma = \epsilon_0 E$$  \hspace{1cm} (2.8)
and the total charge,
\[ q = \sigma S = \frac{\varepsilon_0 s}{d} V \]  \hspace{1cm} (2.9)

This defines the capacitance
\[ C = \frac{q}{V} = \frac{\varepsilon_0 s}{d} \] \hspace{1cm} (2.10)

which is the familiar formula.

Note that in this simple example, the potential is solved first, and then the charge is reduced from the electric field. The charge distribution is not known \textit{a priori}. The potential has been calculated wholly from the given boundary conditions.

A question arises as to the uniqueness of the potential. In other words, are boundary potentials sufficient and at the same time necessary to uniquely determine the potential? The answer is affirmative, as implied by the following theorem: If the potential on a closed surface is prescribed, then the potential in the space off the surface is uniquely determined. This formulation is known as the Dirichlet problem.

In this Chapter, some simple boundary value problems will be solved. In many practical applications, boundary potentials correspond to those on electrodes, as in the case of the one-dimensional capacitor. Only simple electrode shapes can be handled analytically. For complicated shapes, we have to resort to numerical analysis.

### 2.2 Elementary Solutions to Laplace Equation in Cartesian Coordinates (Two Dimensional)

The capacitor problem analyzed in the Introduction is a one-dimensional example for which the solution is almost trivial. As the number of dimensions increases, solutions become more complicated as we will see.

Let us first consider a typical two-dimensional problem, the potential inside a long rectangular tube. (2-2) We assume that the tube consists of four conductor walls which may be insulated from
each other. Here we assume that the bottom and two side walls are all grounded, but the top wall is maintained at a potential $V$. Inside the tube, there are no charges, $\rho = 0$. Therefore, the potential should satisfy the two-dimensional Laplace equation,

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi(x, y) = 0, \quad \left\{ \begin{array}{l} 0 < x < a \\ 0 < y < b \end{array} \right.$$  \hspace{1cm} (2.11)

Note that $z$ dependence is suppressed because of the assumption of sufficiently long cylinder in $z$ direction. Let us assume that the potential $\Phi(x, y)$ can be written as a product of two unknown functions, one depending on $x$ only, and the other on $y$ only,

$$\Phi(x, y) = X(x)Y(y) \hspace{1cm} (2.12)$$

(Whether such separation is possible or not depends on coordinates. However, in most coordinate systems familiar to us, such as cartesian, cylindrical and spherical, complete separation of variables is indeed always possible. In some odd (but still very useful) coordinates, separation is impossible, but we will not consider such cases.) Noting

$$\frac{\partial^2}{\partial x^2} \Phi(x, y) = Y(y) \frac{d^2 x}{dx^2} \hspace{1cm} (2.13)$$

$$\frac{\partial^2}{\partial y^2} \Phi(x, y) = X(x) \frac{d^2 Y}{dy^2} \hspace{1cm} (2.14)$$

and dividing by $X(x)Y(y)$, we rewrite Eq. (2.11) as

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0 \hspace{1cm} (2.15)$$

The first term is now a function of $x$ only, and the second term is a function of $y$ only. Therefore, each must be constant, and cancel each other,

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -k^2 \hspace{1cm} (2.16)$$

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = +k^2 \hspace{1cm} (2.17)$$

where $k$ is a constant having the dimensions of $1/\text{length}$. $k^2$ is called a separation constant, and can be either positive or negative. When $k^2 > 0$, the general solution to Eq. (2.16) is,

$$X(x) = A \sin kx + B \cos kx \quad \text{or} \quad Ae^{ikx} + Be^{-ikx} \hspace{1cm} (2.18)$$

while when $k^2 < 0$,

$$X(x) = A \sinh \kappa x + B \cosh \kappa x \quad \text{or} \quad Ae^{\kappa x} + Be^{-\kappa x} \hspace{1cm} (2.19)$$
where \( \kappa^2 = -k^2 \ (> 0) \). The sign for \( k^2 \) must be chosen by inspecting the boundary conditions. Since \( \Phi \) vanishes on the side walls, \( x = 0 \) and \( a \), the hyperbolic solutions in Eq. (2.17) must be discarded. Therefore, \( k^2 > 0 \) for this particular problem. Furthermore, the cosine solution in Eq. (2.18) must be discarded, since the cosine function does not vanish at \( x = 0 \). Then,

\[
X(x) = A \sin \left( \frac{n\pi}{a} x \right), \quad k = \frac{n\pi}{a}
\]  

(2.20)

is our desired solution for the \( x \) part with \( n \) an integer.

For the \( y \) part, \( Y(y) \) should now satisfy

\[
\frac{d^2 Y}{dy^2} - \left( \frac{n\pi}{a} \right)^2 Y = 0
\]

(2.21)

which has a general solution in the form,

\[
Y(y) = C \sinh \left( \frac{n\pi}{a} y \right) + D \cosh \left( \frac{n\pi}{a} y \right)
\]

(2.22)

However, the potential should vanish at \( y = 0 \) regardless of the value of \( x \). Therefore, we discard the \( \cosh \) solution, and write the potential \( \Phi(x, y) \) in the form of sum of spatial harmonics,

\[
\Phi(x, y) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi}{a} x \right) \sinh \left( \frac{n\pi}{a} y \right)
\]

(2.23)

where \( A_n \) is an yet undetermined coefficient for each harmonic. This can be achieved by using the last boundary condition,

\[
\Phi = V \quad \text{at} \quad y = b \quad (0 < x < a)
\]

(2.24)

or

\[
V = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi}{a} x \right) \sinh \left( \frac{n\pi}{a} b \right)
\]

(2.25)

As the reader might guess, this is nothing but Fourier series expansion of a square wave, and determining the coefficients \( A_n \) can be achieved by exploiting the orthogonality of the sinusoidal functions. Multiplying both sides by \( \sin \left( \frac{m\pi}{a} x \right) \) and integrating from \( x = 0 \) to \( a \), we obtain

\[
V \int_{0}^{a} \sin \left( \frac{m\pi}{a} x \right) dx = \sum_{n=1}^{\infty} A_n \sinh \left( \frac{n\pi}{a} b \right) \int_{0}^{a} \sin \left( \frac{n\pi}{a} x \right) \sin \left( \frac{m\pi}{a} x \right) dx
\]

(2.26)

However, the integral in the RHS is nonvanishing only when \( m = n \),

\[
\int_{0}^{a} \sin \left( \frac{n\pi}{a} x \right) \sin \left( \frac{m\pi}{a} x \right) dx = \frac{a}{2} \delta_{mn}
\]

(2.27)
where $\delta_{mn}$ is Kronecker’s delta,

$$
\delta_{mn} = \begin{cases} 
1 & m = n \\
0 & m \neq n
\end{cases}
$$

(2.28)

The integral in the LHS is elementary,

$$
\int_0^a \sin \left( \frac{n\pi}{a} x \right) dx = \left. -\frac{a}{n\pi} \cos \left( \frac{n\pi}{a} x \right) \right|_0^a = \frac{a}{n\pi} [1 - (-1)^n]
$$

(2.29)

Then,

$$
A_n = \frac{4V}{n\pi} \frac{1}{\sinh \left( \frac{n\pi}{a} b \right)} (n = 1, 3, 5, \ldots)
$$

(2.30)

and the final form for the potential becomes,

$$
\Phi(x, y) = \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n\sinh \left( \frac{n\pi}{a} b \right)} \sin \left( \frac{n\pi}{a} x \right) \sinh \left( \frac{n\pi}{a} y \right)
$$

(2.31)

This enables us to calculate the potential at an arbitrary point inside the cylinder. Figure 2-3 shows equipotential surfaces inside a rectangular cylinder when $a = b$, and the top plate is at 1 V.

$$
\frac{4}{\pi} \sum_{n=0}^{10} \frac{1}{2n + 1} \frac{1}{\sinh ((2n + 1)\pi)} \sin ((2n + 1)\pi x) \sinh ((2n + 1)\pi y) = 0.5
$$

Potential inside a square ($a = b$) cylinder. The top plate is at $V = 1$ volt. From top, $\Phi = 1, 0.5, 0.25, 0.1, 0.05$ V.

The method developed can be generalized for more complicated potential prescription on the surface. For example, if the top plate is at a potential $V_1$ and one of the side plates at $V_2$, the
principle of superposition can be applied, as shown in Fig. 2-3. Each case in RHS in the figure can be worked out separately, and the total potential is given by the sum of respective potentials.

2.3 Long Circular Cylinder

In the cylindrical coordinate \((\rho, \phi, z)\), Laplace’s equation becomes

\[
\left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] \Phi(\rho, \phi, z) = 0 \tag{2.32}
\]

If \(z\) dependence is ignorable \((\partial/\partial z = 0)\), such as in the case of sufficiently long cylinder, Eq. (2.32) reduces to a two-dimensional form,

\[
\left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right] \Phi(\rho, \phi) = 0 \tag{2.33}
\]

This has general solutions

\[
\Phi(\rho, \phi) = \left( A \rho^n + \frac{B}{\rho^n} \right) (C \cos n\phi + D \sin n\phi) \tag{2.34}
\]

where \(n\) is a constant. \(n\) may not be an integer for an incomplete cylinder, such as a wedge formed by two intersecting large conducting plates.

If \(\phi\) dependence is also ignorable, the potential becomes symmetric about the \(z\) axis, and thus one dimensional,

\[
\left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} \right) \Phi(\rho) = 0 \tag{2.35}
\]

Solutions for this equation are,

\[
\Phi(\rho) = A \ln \rho + B \tag{2.36}
\]

which we have encountered already in Chapter 1 (e.g., potentials of coaxial cables, parallel wire transmission lines).

**Potential due to a Boundary Potential on an Infinitely Long Cylinder**

When a cylinder is sufficiently long so that edge effects are ignorable, Laplace equation reduces
As explained earlier, the general solution to this equation is

\[ \Phi(\rho, \phi) = \left( A\rho^n + \frac{B}{\rho^n} \right) (C \cos n\phi + D \sin n\phi) \]  

(2.38)

as can be readily checked by direct substitution. The constant \( n \), which is a separation constant for Eq. (2.37), must be an integer \((n = 0, 1, 2, \ldots)\) if the cylinder is complete because the potential must be periodic with respect to the angle \( \phi \). (Otherwise, it would be unphysical.)

As a concrete example, we consider the case shown in Fig. 2-4 (a), in which the upper half of the cylinder is at a potential \(+V\), and the lower half at \(-V\). The boundary surface potential \( \Phi_s(\phi) \) is described by

\[ \Phi_s(\phi) = \begin{cases} 
+V & 0 < \phi < \pi \\
-V & -\pi < \phi < 0 
\end{cases} \]  

(2.39)

Since the function \( \Phi_s(\phi) \) is an odd function of \( \phi \), the cosine integral, Eq. (2.38), identically vanishes. The sine integral for \( D_n \) becomes

\[ D_n = \frac{2V}{\pi a^n} \int_0^\pi \sin n\phi d\phi \\
= \frac{2V}{n\pi a^n} \left[ 1 - (-1)^n \right] \quad (n = 1, 2, 3, \ldots) \\
= \frac{2V}{n\pi a^n} \quad (n = 1, 2, 3, \ldots) \]  

(2.40)

Therefore, the interior potential can be determined as

\[ \Phi(\rho, \phi) = \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \left( \frac{\rho}{a} \right)^n \sin n\phi = \frac{2V}{\pi} \tan^{-1} \left( \frac{2a\rho \sin \phi}{\rho^2 - a^2} \right) \]  

(2.41)

For the exterior \((\rho > a)\), the term \( \rho^n \) in Eq. (2.38) should be discarded. The remaining procedure is the same, and the exterior potential is given by,

\[ \Phi(\rho, \phi) = \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \left( \frac{\rho}{a} \right)^n \sin n\phi = \frac{2V}{\pi} \tan^{-1} \left( \frac{2a\rho \sin \phi}{\rho^2 - a^2} \right) \]  

(2.42)

### 2.4 Potential due to a charge off the origin

The potential due to a charge \( q \) at the origin is

\[ \Phi(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} \]
If the charge is moved to the position \( z = -a \), the potential at \((r, \theta)\) is

\[
\Phi(r, \theta) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{r^2 + a^2 + 2ar \cos \theta}}
\]  

(2.43)

since the distance between the charge and the observing point is

\[
|r - r'| = \sqrt{r^2 + a^2 + 2ar \cos \theta}
\]

In the region \( r > a \), the potential in Eq. (2.43) can be expanded as

\[
\Phi(r, \theta) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{r} - \frac{a \cos \theta}{r^2} + \frac{3}{2} \frac{\cos^2 \theta - 1}{r^3}a^2 + \cdots \right]
\]

This expansion consists of the monopole, dipole and quadrupole potentials. In general, in axially symmetric systems, the potential may be expanded in the form

\[
\Phi(r, \theta) = \sum_l \left( A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos \theta)
\]

where \( P_l(\cos \theta) \) is the Legendre polynomials,

\[
\begin{align*}
P_0 &= 1 \\
P_1 &= \cos \theta \\
P_2 &= \frac{1}{2} (3 \cos^2 \theta - 1)
\end{align*}
\]

**Potential due to a Uniformly Charged Insulating Disk**

Consider an insulating thin disk of radius \( a \) uniformly charged with a surface charge density \( \sigma \) \((C^2/m)\). (The problem of a charged conducting disk is an entirely different problem, and will be analyzed later.) We assume that the disk is placed on the \( x-y \) plane with its center at the origin.
The potential on the axis can be found easily,

\[
\Phi(z) = \frac{1}{4\pi\varepsilon_0} \int_0^a \frac{\sigma}{\sqrt{z^2 + \rho^2}} 2\pi \rho d\rho \\
= \frac{\sigma}{2\varepsilon_0} \left( \sqrt{z^2 + a^2} - |z| \right) 
\]  
(2.44)

In the region \(|z| > a\), the function \(\sqrt{z^2 + a^2}\) can be expanded as

\[
|z| \left[ 1 + \frac{1}{2} \left( \frac{a}{z} \right)^2 - \frac{1}{8} \left( \frac{a}{z} \right)^4 + \cdots \right] 
\]  
(2.45)

and the potential reduces to

\[
\Phi(|z| > a) = \frac{\sigma}{2\varepsilon_0} \left[ \frac{a^2}{|z|} - \frac{1}{2} \frac{a^4}{|z|^3} + \frac{1}{8} \frac{a^4}{|z|^5} + \cdots \right] 
\]  
(2.46)

For positive \(z\),

\[
\Phi(z > a) = \frac{\sigma \pi a^2}{4\pi\varepsilon_0} \left( \frac{1}{z} - \frac{1}{2} \frac{a^2}{z^3} + \frac{1}{8} \frac{a^4}{z^5} + \cdots \right) 
\]  
(2.47)

However, all Legendre functions \(P_l(\cos \theta)\) become unity at \(\theta = 0\), or \(\cos \theta = 1\) as we have seen earlier. Therefore, Eq. (2.47) can be regarded as the limiting case, \(\theta \to 0\), of the off axis potential

\[
\Phi_c(r, \theta) = \frac{q}{4\pi\varepsilon_0} \left( \frac{1}{r} - \frac{1}{2} \frac{a^2}{r^3} P_2(\cos \theta) + \frac{1}{8} \frac{a^4}{r^5} P_4(\cos \theta) - \cdots \right) 
\]  
(2.48)

which is indeed the solution for the potential in the region \(r > a\). Here, \(q = \pi a^2 \sigma\) is the total
charge, and the subscript $e$ indicates "exterior".

For the region $r < a$, the potential becomes (Exercise)

$$\Phi_e(r, \theta) = \frac{q}{4\pi \epsilon_0} \left( \frac{2r}{a^2} |P_1(\cos \theta)| + \frac{r^2}{a^2} P_2(\cos \theta) - \frac{1}{8} \frac{r^4}{a^4} P_4(\cos \theta) - \cdots \right)$$

(2.49)

In general, for axially symmetric potential problems, finding the axial potential is sufficient to reduce the off axis potential at an arbitrary point $(r, \theta)$. Remember that this technique works only for axially symmetric systems in which $\partial \Phi / \partial \phi = 0$.

On the surface of the sphere of radius $a$, the potential must be continuous. When $r = a$, Eq. 2.48 becomes

$$\Phi_e(a, \theta) = \frac{q}{4\pi \epsilon_0} \left[ 1 - \frac{1}{2} P_2(\cos \theta) + \frac{1}{8} P_4(\cos \theta) - \cdots \right]$$

(2.50)

and Eq. (2.49) becomes

$$\Phi_i(a, \theta) = \frac{q}{4\pi \epsilon_0} \left[ 2 - 2 |\cos \theta| + P_2(\cos \theta) + \frac{1}{4} P_4(\cos \theta) + \cdots \right]$$

(2.51)

where $P_1(\cos \theta) = \cos \theta$ is substituted. Equating these two, we find an identity

$$2 |\cos \theta| = 1 + \frac{3}{2} P_2(\cos \theta) - \frac{3}{8} P_4(\cos \theta) + \cdots$$

or

$$2 |\mu| = 1 + \frac{3}{2} P_2(\mu) - \frac{3}{8} P_4(\mu) + \cdots$$

which can be regarded as the expansion of the function $2 |\mu|$ in terms of the Legendre polynomials $P_l(\mu)(l \text{ even})$.

The appearance of the term $|\cos \theta|$ in the interior potential is due to the fact that the potential inside the sphere of radius $a$ must satisfy the Poisson’s equation, rather than the Laplace equation, because of the charge sheet extending up to $r = a$. (The exterior potential merely satisfies the Laplace equation.) The term $|\cos \theta|$ appears as a particular solution to the Poisson’s equation. Other solutions, $r^2 P_2(\cos \theta), r^4 P_4(\cos \theta)$, etc., are general solutions to the Poisson’s equation, that is, solutions to the Laplace equation. The charge density of the ideally thin charge sheet can be written as

$$\rho(r) = \sigma \delta(z), \quad r < a$$

where $\delta(z)$ is the delta function having the dimensions of $1/\text{argument}$. Therefore, in the Poisson’s equation written in the cartesian coordinates,

$$\nabla^2 \Phi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi = -\frac{\rho}{\epsilon_0}$$

the term $\frac{\partial^2 \Phi}{\partial z^2}$ must yield a delta function, $\delta(z)$, for the both sides of the Poisson’s equation to be
compatible. Indeed, the second order derivative of the function $|z|$ does yield a delta function,

$$\frac{d^2}{dz^2}|z| = 2\delta(z)$$

and the term $|z|$ in Eq. (2.44), and the corresponding $|\cos \theta|$ in Eq. (2.50) are particular solutions to the singular Poisson's equation.

**Potential due to Oppositely Charged Hemispheres**

As an example of boundary value problem in the spherical coordinates, we consider a spherical shell of radius $a$ consisting of two halves, upper semisphere at a potential $+V$, and the lower semisphere at $-V$, as shown in Fig. 2-6. The system is symmetric about the axis. Also, the potential on the spherical surface at $r = a$ is constant ($+V$ or $-V$) and continuous. Therefore, the potential may be assumed in the form

$$\Phi(r, \theta) = \begin{cases} 
\sum_l A_l \left( \frac{r}{a} \right)^l P_l(\cos \theta) & \text{for } r < a \\
\sum_l A_l \left( \frac{a}{r} \right)^{l+1} P_l(\cos \theta) & \text{for } r \geq a
\end{cases}$$

(2.52)

Note that the coefficients $A_l$ for both interior ($r < a$) and exterior ($r > a$) solutions are common, in contrast to the case of the preceding example, which is not really a boundary value problem. Here, the potential $\Phi(r, \theta)$ satisfies the Laplace equation throughout the space. The boundary condition $r = a$ is

$$\Phi_a = \begin{cases} 
+V & 0 < \theta < \frac{\pi}{2} \quad (0 < \mu < 1) \\
-\Phi & \frac{\pi}{2} < \theta < \pi \quad (-1 < \mu < 0)
\end{cases}$$

(2.53)
Therefore, on the sphere surface \((r = a)\),
\[
\Phi_s(\theta) = \sum_l A_n P_l(\cos \theta) \tag{2.54}
\]

Multiplying by \(\sin \theta P_l(\cos \theta)\), integrating over \(\theta\) from 0 to \(\pi\), and recalling the orthogonality of the Legendre polynomials
\[
\int_{-1}^{1} P_l(\mu) P_l(\mu) d\mu = \frac{2}{2l + 1} \delta_{ll'} \tag{2.55}
\]
we find,
\[
A_l = \frac{2l + 1}{2} \int_{-1}^{1} \Phi_s(\mu) P_l(\mu) d\mu \tag{2.56}
\]

Since the boundary function \(\Phi(\mu)\) is an odd function of \(\mu = \cos \theta\), only the odd order coefficients \((l = 1, 3, 5, \ldots)\) are nonvanishing. Then,
\[
A_l = \frac{2l + 1}{2} \times 2V \int_{0}^{1} P_l(\mu) d\mu = (2l + 1)V \int_{0}^{1} P_l(\mu) d\mu \quad (l = 1, 3, 5, \ldots) \tag{2.57}
\]

For \(l = 1\), \(P_1(\mu) = \mu\). Then,
\[
\int_{0}^{1} \mu d\mu = \frac{1}{2}.
\]

For \(l = 3\), \(P_3(\mu) = (5\mu^3 - 3\mu)/2\), and
\[
\int_{0}^{1} \frac{5\mu^3 - 3\mu}{2} d\mu = -\frac{1}{8}.
\]

In general, for odd \(l\),
\[
\int_{0}^{1} P_l(\mu) d\mu = (-1)^{\frac{l+1}{2}} \frac{(l - 2)!!}{(l + 1)!!} \tag{2.58}
\]
(For even \(l\), except \(l = 0\), the integral vanishes.) Then,
\[
A_l = (-1)^{\frac{l+1}{2}} \frac{(l - 2)!!}{(l + 1)!!} (2l + 1)V,
\]
and the first few terms of the potential are
\[
\Phi_i(r, \theta) = V \left[ \frac{3}{2a} r \cos \theta - \frac{7}{8} \left( \frac{r}{a} \right)^3 P_3(\cos \theta) + \cdots \right] \quad r < a \tag{2.59}
\]
\[
\Phi_e(r, \theta) = V \left[ \frac{3}{2} \left( \frac{a}{r} \right)^2 \cos \theta - \frac{7}{8} \left( \frac{a}{r} \right)^4 P_3(\cos \theta) + \cdots \right] \quad r > a \tag{2.60}
\]
where the subscripts \(i\) and \(e\) are for the interior and exterior potentials.

Outside the sphere \((r > a)\), the leading (lowest order) term is of dipole form. This is an expected result because the two semispheres are oppositely charged. The total (net) charge is zero. Hence,
the monopole potential is absent.

The potential at the midplane, \( \theta = \pi/(2\mu = 0) \), vanishes, since

\[ P_l(0) = 0 \quad (l \text{ odd}) \]  

(2.61)

This is also expected because of the up-down antisymmetry.

The electric field can be calculated in the usual way,

\[
E = -\nabla\Phi(r, \theta) = -\left( e_r \frac{\partial}{\partial r} + e_\phi \frac{1}{r} \frac{\partial}{\partial \theta} \right) \Phi(r, \theta)
\]

The surface charge distribution on the sphere surfaces can be found from the normal (radial in this case) component of the electric field. On the outer surface, the surface charge is given by

\[
\sigma_e(\theta) = -\epsilon_0 \frac{\partial}{\partial r} \Phi(r, \theta) \bigg|_{r=a} = \frac{\epsilon_0 V}{a} \left[ 3P_1(\cos \theta) - \frac{7}{2} P_3(\cos \theta) + \cdots \right] 
\]

(2.62)

and on the inner surface,

\[
\sigma_i(\theta) = +\epsilon_0 \frac{\partial}{\partial r} \Phi(r, \theta) \bigg|_{r=a} = \frac{\epsilon_0 V}{a} \left[ 3P_1(\cos \theta) - \frac{21}{8} P_3(\cos \theta) + \cdots \right] 
\]

(2.63)

The total charge residing on the upper semisphere can therefore be calculated from

\[ q = \int_0^{\pi/2} (\sigma_e + \sigma_i) 2\pi a^2 \sin \theta \, d\theta \]  

(2.64)

However, the sum of the series after integration simply diverges. This situation is similar to the divergence we encountered in the problem of a cylinder in Sec. 3, and due to the assumption of an infinitesimally small gap at \( \theta = \pi/2 \) separating the hemispheres. If a finite gap width, \( \delta(\ll 1\text{rad}) \), is assumed, the divergence is removed, and the capacitance becomes well defined. This problem is left for an exercise.