Chapter 10

ELECTROMAGNETIC WAVES IN MATTER

10.1 Introduction

All materials contain ions and electrons. These charged particles experience Lorentz force when exposed to electromagnetic fields. Except in plasmas (ionized gases), currents are predominantly caused by electron motion because ions are practically immobile. In conductors, conduction electron response to externally applied electric field is particularly strong, and the current induced by electron motion tends to prevent the external field from penetrating into conductors. This phenomena is known as skin effect. In dielectrics, electrons are bound to molecules and are thus not free as in conductors. However, those bound electrons can still react to external fields, and induce microscopic current. In an oscillating field, the effective permittivity becomes frequency dependent, and electromagnetic waves in dielectrics are strongly dispersive in contrast to waves in free space. In this Chapter, we will study electromagnetic wave propagation in conductors, dielectrics and plasmas.

10.2 Skin Effects in Conductors

In conductors, electrons are free to move around, although they suffer rather frequent collisions with ions (lattice). The equation of motion for free (unbound) electrons in an electric field is

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - m\nu\mathbf{v} \tag{10.1}$$

where ν is the collision frequency and the magnetic force term $\mathbf{v} \times \mathbf{B}$ has been ignored because it is of second order. If the electric field is assumed to be

$$\mathbf{E} = \mathbf{E}_0 e^{j(\omega t - kz)} \tag{10.2}$$

the solution for the velocity is given by

$$\mathbf{v}(t) = -\frac{e}{m}\mathbf{E}_0\frac{1}{j\omega+\nu}e^{j(\omega t-kz)} \tag{10.3}$$

save the transient initial phase. (As we will see shortly, the wavenumber k is rather ill defined for conductors. It will turn out to be complex.) Multiplying by the electron charge density $-ne$ (n being the electron density), we find the current density,

$$\mathbf{J}_e = \frac{ne^2}{m}\frac{\mathbf{E}_0}{j\omega+\nu}e^{j(\omega t-kz)} \tag{10.4}$$

and the Maxwell's equation

$$\nabla\times\mathbf{H} = \mathbf{J}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{10.5}$$

becomes

$$\nabla\times\mathbf{H} = \left[\sigma(\omega)+j\omega\epsilon_0\right]\mathbf{E}_0 \tag{10.6}$$

where the ac conductivity $\sigma(\omega)$ is defined by

$$\sigma(\omega) = \frac{1}{j\omega+\nu}\frac{ne^2}{m} \tag{10.7}$$

Let us compare the conduction current conductivity σ and the displacement current conductivity $j\omega\epsilon_0$ in Eq. (10.6). They become comparable when

$$\frac{ne^2}{m}\frac{1}{|j\omega+\nu|} \simeq \omega\epsilon_0 \tag{10.8}$$

In copper, for example, $n \simeq 10^{29}/\mathrm{m}^3, \nu \simeq 10^{14}/\mathrm{sec}$. Therefore, the two terms become comparable when $\omega \simeq 10^{16}$ rad/sec. Below this frequency, which is in the ultraviolet regime, the displacement current is negligible compared with the conduction current. Furthermore, in practical problems, the frequency ω is much smaller than the collision frequency even in the microwave frequency range ($\omega \lesssim 10^{12}$/sec). Therefore, the simple dc conductivity

$$\sigma = \frac{ne^2}{m\nu} \tag{10.9}$$

can be used as long as ω is much less than 10^{14} rad/sec. The displacement current is ignorable in ordinary conductors at low frequencies.

The basic Maxwell's equations to describe electromagnetic fields in a conductor are therefore

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{10.10}$$

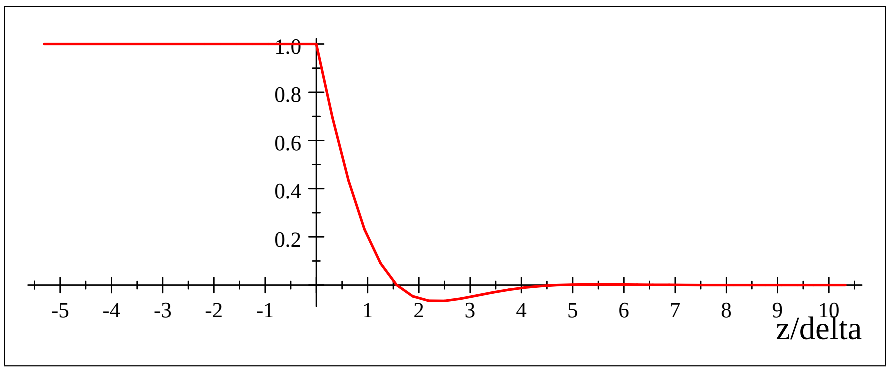
$$\nabla\times\mathbf{E} = \mu_0\sigma\mathbf{E} \tag{10.11}$$

Eliminating the magnetic field, and ignoring charge accumulation ($\boldsymbol{\nabla} \cdot \mathbf{E} = 0$), we obtain

$$\nabla^2 \mathbf{E} = \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla^2 \mathbf{H} = \mu_0 \sigma \frac{\partial \mathbf{H}}{\partial t} \tag{10.12}$$

These are diffusion equations, rather than wave equation. Electromagnetic fields do not propagate in conductors as freely as in vacuum or dielectrics. In fact, they encounter strong resistance when they try to penetrate into a conductor.

$\text{Heaviside}\,(-x) + \text{Heaviside}\,(x)\, e^{-x} \cos x$



Magnetic field $E_0 \exp(-z/\delta) \cos(z/\delta)$ penetration into a conductor. δ is the skin depth.

The diffusion equation can be analyzed in two extreme situations. The first case we consider is electromagnetic field penetration when an oscillating external field is applied. After a steady state is established, the electric field in a conductor is described by

$$\nabla^2 \mathbf{E} = j\omega\mu_0\sigma \mathbf{E} \tag{10.13}$$

Let us assume a one-dimensional problem shown in Fig. 10.1, wherein an external field $H_0 e^{j\omega t}$ is applied parallel to the conductor surface ($z = 0$). Eq. (10.13) in this case becomes

$$\frac{d^2}{dz^2} E(z) = j\omega\mu_0\sigma E(z) \tag{10.14}$$

which can be readily solved as

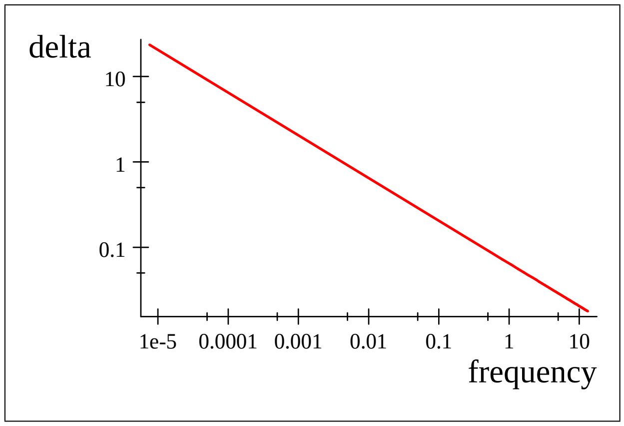
$$E(z) = E_0 \exp\left(-\frac{j+1}{\delta} z\right) \tag{10.15}$$

where

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \ \text{(m)} \tag{10.16}$$

is the skin depth. The skin depth is a measure of how deep the electric field, current and magnetic field can penetrate into a conductor. Observe that the skin depth is inversely proportional to $\sqrt{\omega\sigma}$, which indicates that field penetration is more difficult at higher frequency, and in better conductors. In copper, $\sigma \simeq 6 \times 10^7$ S/m. The skin depth in copper is shown in Fig. 10.2 as a function of the frequency. Even at $f = 60\,\text{Hz}$, the copper skin depth is only 8 mm.

$$\frac{0.064795}{\sqrt{x}}$$



Skin effects modify the resistance and internal inductance of a wire from those in dc fields. For dc fields, skin effects are obviously absent ($\delta \to \infty$), and the current flows uniformly across a wire cross section. For dc currents, the resistance and internal inductance per unit length of a wire having a radius a and conductivity σ are

$$\frac{R}{l} = \frac{1}{\sigma \pi a^2} \qquad (\Omega/\text{m}) \tag{10.17}$$

$$\frac{L}{l} = \frac{\mu_0}{8\pi} \qquad (\text{H/m}) \tag{10.18}$$

respectively. With skin effects, the resistance is expected to increase because the effective cross section area for current flow decreases. The internal inductance is expected to decrease because the magnetic energy stored in the wire becomes smaller.

To analyze field penetration into a wire, we employ the cylindrical coordinates (Fig. 10.3). The diffusion equation for the axial electric field $E_z(\rho)\mathrm{e}^{j\omega t}$ thus reduces to

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - j\omega\mu_0\sigma \right) E_z = 0 \tag{10.19}$$

provided the frequency ω is sufficiently low so that the variation along the wire itself is negligible ($\partial/\partial z = 0$). There is no angular (ϕ) dependence either because the field is symmetric. Eq. (10.19) is the Bessel's equation which we encountered in Chapter 3, except $k^2 = -j\omega\epsilon_0\sigma$ is now complex. The solution for $E_z(\rho)$ can still be formally written as

$$E_z(\rho) = E_0 \frac{J_0(k\rho)}{J_0(ka)} \tag{10.20}$$

where

$$k^2 = -j\omega\mu_0\sigma = -j\frac{2}{\delta^2} \tag{10.21}$$

and E_0 is the field at the surface $\rho = a$. The axial current density $J_z(\rho)$ is given by

$$J_z(\rho) = \sigma E_z(\rho) = \sigma E_0 \frac{J_0(k\rho)}{J_0(ka)} \tag{10.22}$$

Then, the total current through the wire cross section can be found by integrating the current density over the wire cross section,

$$\begin{aligned} I &= \int_0^a J_z(\rho) 2\pi\rho d\rho \\ &= \sigma E_0 \frac{2\pi}{J_0(ka)} \int_0^a \rho J_0(k\rho) d\rho \\ &= \sigma E_0 \frac{2\pi}{J_0(ka)} \frac{a}{k} J_1(ka) \end{aligned} \tag{10.23}$$

where use has been made of the following indefinite integral

$$\int x J_0(x) dx = x J_1(x).$$

Therefore, the internal impedance per unit length of the wire is found as

$$\frac{Z}{l} = \frac{E_0}{I} = \frac{k}{2\pi a\sigma} \frac{J_0(ka)}{J_1(ka)} \tag{10.24}$$

It should be noted that the impedance Z/l should be defined in terms of the Poynting flux at the surface,

$$2\pi a E_z(\rho = a) H_\phi(\rho = a) \equiv \frac{Z}{l} I^2 \quad \text{(W/m)} \tag{10.25}$$

Since $2\pi a H_\phi(\rho = a) = I$ from Ampere's law, we find $Z/l = E_z(\rho = a)/I$. In the dc limit $(\omega \to 0, k \to 0)$, $J_0(0) = 1$, $J_1(ka) \simeq ka/2$, and we recover the dc resistance. In the high frequency limit (or in the limit of strong skin effect), $|k|$ becomes large. Since k is complex, both $J_0(ka)$ and $J_1(ka)$ diverge when $|k|a \to \infty$, but their ratio approaches unity. Therefore, the resistance is of order

$$\frac{R}{l} \simeq \frac{1}{2\pi a\delta\sigma} \quad (a \gg \delta) \tag{10.26}$$

which is an expected result because the current penetrates at most the skin depth from the surface, and the area over which the current flows is about $2\pi a\delta$. In between the two extreme cases, we have to resort to numerical analysis to evaluate Eq. (10.25).

The imaginary part of the impedance (per unit length) should yield the internal inductance,

$$\frac{L_i}{l} = \frac{1}{\omega} \text{Im}\left(\frac{Z}{l}\right) \tag{10.27}$$

In the low frequency limit ($|k|a \ll 1$), the Bessel functions may be approximated by (see Chapter 3)

$$J_0(ka) \simeq 1 - \frac{1}{4}(ka)^2 + \frac{1}{64}(ka)^4 \tag{10.28}$$

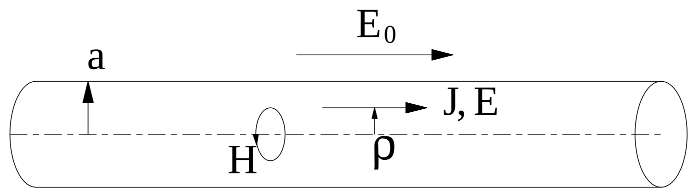


Figure 10-1: Skin effect in a conductor rod for ac field oscillating at ω.

$$J_1(ka) = -J_0'(ka) \simeq \frac{1}{2}ka - \frac{1}{16}(ka)^3 \tag{10.29}$$

Then, in Eq. (10.25),

$$\begin{aligned} \lim_{ka\to 0} ak\frac{J_0(ka)}{J_1(ka)} &= \frac{1-\frac{1}{4}(ka)^2}{\frac{1}{2}-\frac{1}{16}(ka)^2} \\ &\simeq 2\left[1-\frac{1}{4}(ka)^2\right]\left[1+\frac{1}{8}(ka)^2\right] \\ &\simeq 2\left(1-\frac{1}{8}(ka)^2\right) \\ &= 2\left[1+\frac{a^2}{8}j\omega\mu_0\sigma\right] \end{aligned} \tag{10.30}$$

and the impedance may be approximated by

$$\frac{Z}{l} \simeq \frac{1}{\pi a^2\sigma}\left(1+j\frac{1}{8}\omega\mu_0\sigma a^2\right) \tag{10.31}$$

The internal inductance per unit length is therefore

$$\frac{L_i}{l} = \frac{\mu_0}{8\pi} \quad \text{(low frequency limit)} \tag{10.32}$$

which is the familiar low frequency internal inductance.

In the high frequency (strong skin effect) limit, the inductance becomes

$$\frac{L_i}{l} \simeq \frac{2\delta}{a}\frac{\mu_0}{8\pi} \quad \text{(high frequency limit)} \tag{10.33}$$

(Derivation of this result is left for an exercise.)

$$\left|\frac{J_0\left((1-i)\,1.5x\right)}{J_0\left((1-i)\,1.5\right)}\right|$$

Figure 10.4 shows the radial distribution of the current density amplitude normalized by that at the conductor surface for different values of a/δ. The current density on the axis ($\rho = 0$) becomes

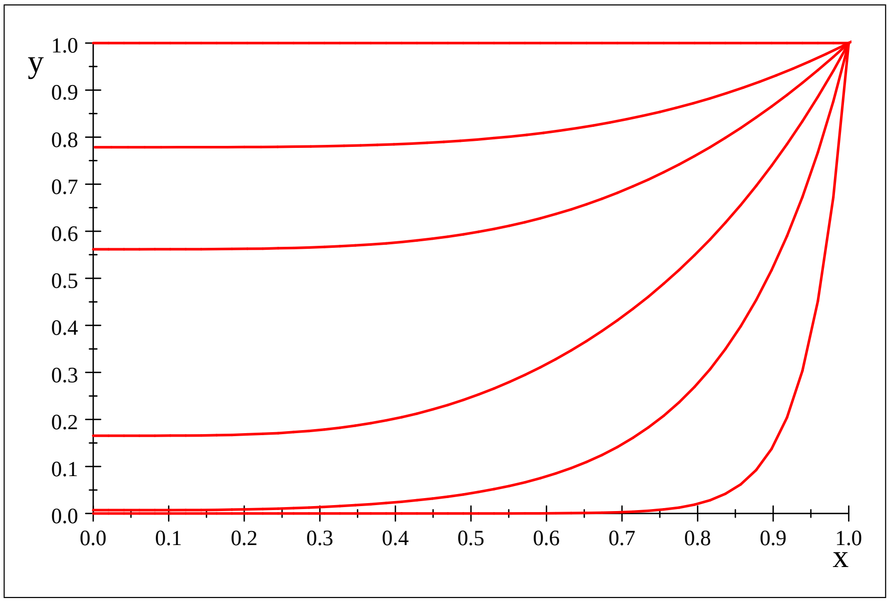


Figure 10-2: Radial profile of the magnitude of the current density for (from top) $a/\delta = 0$ (dc), 1.5, 2, 3.5, 7, and 20.

one half of the surface value when $a/\delta \simeq 2.5$. Figure 10.5 shows how the resistance and inductance per unit length vary with the parameter a/δ. Note that the inductive impedance itself increases with the frequency and is equal to the resistance

$$\frac{\omega L_i}{l} = \frac{2\omega\delta}{a}\frac{\mu_0}{8\pi} = \frac{1}{2\pi a\sigma\delta} = \frac{R}{l} \propto \sqrt{\omega} \tag{10.34}$$

$$\mathrm{Re}\left((1-i)\,x\frac{J_0((1-i)\,x)}{J_1((1-i)\,x)}\right)$$

10.3 Transient Field Penetration (Diffusion) into a Conductor

Even when a dc electric field is suddenly applied to a conductor, the field cannot instantaneously penetrate into the conductor, but gradually diffuses into the conductor being governed by the diffusion equation

$$\nabla^2\mathbf{E} = \mu_0\sigma\frac{\partial\mathbf{E}}{\partial t} \tag{10.35}$$

If the characteristic dimension of the conductor is a, an order of magnitude estimate for the field penetration time across the distance a is

$$\tau \simeq \mu_0\sigma a^2$$

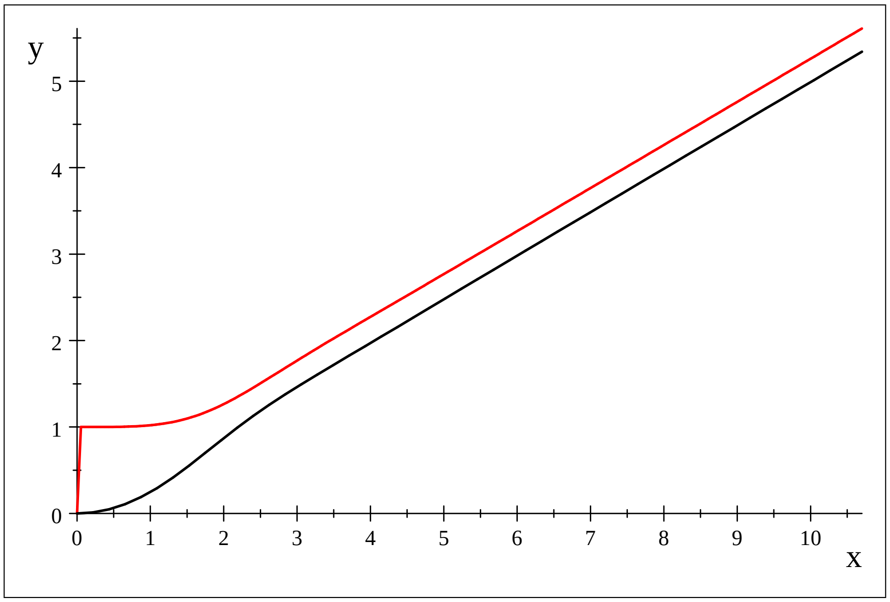


Figure 10-3: Re(Z/l) (red) and Im(Z/l) (black) of a conductor rod as functions of a/δ. The impedance is normalized by $R_{dc} = \pi a^2 \sigma$ (Ω).

where $|\nabla| \simeq 1/a$ for orders of magnitude estimate. For example, if a copper plate is 1 cm thick, the approximate penetration time is

$$\tau \simeq 4\pi \times 10^{-7} \times 6 \times 10^{7} \times (0.01)^2 \simeq 8 \text{ msec}$$

To describe transient field penetration rigorously, the diffusion equation must be solved. Let us consider a wide conductor slab having a thickness d. A dc external field is suddenly applied at $t = 0$ parallel to the slab, as shown in Fig. 10.6. Then, the diffusion equation becomes one dimensional,

$$\frac{\partial^2 E_z}{\partial x^2} = \mu_0 \sigma \frac{\partial E_z}{\partial t} \tag{10.36}$$

with the initial condition

$$E_z(x) = 0 \;, \quad t = 0 \tag{10.37}$$

and the boundary condition

$$E_z\left(x = \pm\frac{d}{2}\right) = E_0 \;, \quad \text{at any } t \; (> 0) \tag{10.38}$$

This can be solved by the method of separation of variables. If we assume

$$E_z(x,t) = X(x)\; T(t) \tag{10.39}$$

Equation (10.36) separates as

$$\frac{1}{X}\frac{d^2X}{dx^2} = \mu_0\sigma\frac{1}{T}\frac{dT}{dt} = -k^2 \text{ (const.)} \tag{10.40}$$

Then, applying the boundary conditions, Eq. (10.38), the solution for $E_z(x,t)$ may be assumed as

$$E_z(x,t) = \sum_n A_n \cos k_n x e^{-t/\tau_n} + E_0 \tag{10.41}$$

where

$$\tau_n = \mu_0\sigma/k_n^2\ , \quad k_n = \frac{n\pi}{d} \tag{10.42}$$

is the penetration time of n-th spatial harmonic mode. The coefficient A_n can be determined from the initial condition,

$$E_z(x, t=0) = 0 \tag{10.43}$$

Then

$$\sum_n A_n \cos\left(\frac{n\pi}{d}x\right) = -E_0 \tag{10.44}$$

Multiplying by $\cos\left(\frac{m\pi}{d}x\right)$ and integrating from $x = -d/2$ to $d/2$, we find

$$\begin{aligned} A_n &= \frac{2}{d}E_0\int_{-d/2}^{d/2}\cos\left(\frac{n\pi}{d}x\right)dx \\ &= -\frac{4}{n\pi}E_0\sin\left(\frac{n\pi}{2}\right) \end{aligned} \tag{10.45}$$

The final solution for $E_z(x,t)$ is

$$\begin{aligned} E_z(x,t) = E_0 - \frac{4E_0}{\pi}&\left[\cos\left(\frac{\pi}{d}x\right)e^{-t/\tau_1} - \frac{1}{3}\cos\left(\frac{3\pi}{d}x\right)e^{-t/\tau_3}\right] \\ &+\frac{1}{5}\cos\left(\frac{5\pi}{d}x\right)e^{-t/\tau_5} - \cdots \end{aligned} \tag{10.46}$$

with

$$\tau_n = \mu_0\sigma\left(\frac{d}{n\pi}\right)^2 \quad (n = 1,\ 3,\ 5,\ \ldots) \tag{10.47}$$

DC field penetration into a conductor rod of a radius a can be analyzed in a similar manner. This is left for an exercise. The result is

$$E_z(\rho,t) = E_0\left[1 - \sum_n A_n J_0\left(k_n\rho\right)e^{-t/\tau_n}\right] \tag{10.48}$$

where

$$A_n = \frac{2}{ak_n}\frac{1}{J_1(k_n a)} \tag{10.49}$$

with $k_n a$ the n-th root of $J_0(ka) = 0$, and

$$\tau_n = \mu_0 \sigma / k_n^2 \tag{10.50}$$

10.4 Electromagnetic Waves in a Plasma

A plasma is an ionized gas consisting of free ions and electrons. Charge neutrality is still maintained if the positive charge density of ions and negative charge density of electrons are equal. Waves in a plasma are quite diversified particularly when the plasma is in a magnetic field as those for magnetic fusion research and in the magnetosphere and ionosphere. Plasmas are in general far from thermal equilibrium in the sense that plasma production requires a significant amount of energy. In the case of magneto/ionospheric plasma, the energy source is the solar wind, while laboratory plasmas can only be maintained when sufficient energy is continuously fed. A hot, high density plasma tends to be electromagnetically unstable whereby it releases energy to approach thermal equilibrium with the environment. A positive aspect of plasma instabilities is that a plasma can be a source of high frequency, high power micro (or mm) waves. For example, with electrons having relativistic energy, generation of extremely high power ($>$ 1 GW) mm waves has been achieved (Gyrotrons). Free electron lasers, based on relativistic electron dynamics in a corragated magnetic field, are promising high power sub mm (even visible) source with a wide wavelength tunability. It is not our purpose here to review all kinds of waves in a plasma. (To do so, a few volumes of books will be required.) Rather, we will be interested how a plasma will modify the wave equation in vacuum. In particular, we wish to understand how the ionospheric plasma can reflect shortwave radio, but not higher frequency electromagnetic waves.

In Section 10.2, we have seen that the ac conductivity in a conducting material is given by

$$\sigma(\omega) = \frac{1}{j\omega + \nu}\frac{ne^2}{m} \tag{10.51}$$

In a gaseous plasma, the collision frequency ν is negligibly small because of a much lower particle density than in solids. Then,

$$\sigma(\omega) \simeq \frac{1}{j\omega}\frac{ne^2}{m} \tag{10.52}$$

and the Maxwell's equation, Eq. (10.6), becomes

$$\nabla \times \mathbf{H} = \left(1 - \frac{\omega_p^2}{\omega^2}\right) j\omega\epsilon_0 \mathbf{E} \tag{10.53}$$

where

$$\omega_p^2 = \frac{ne^2}{m\epsilon_0} \tag{10.54}$$

is the square of the plasma frequency. Eq. (10.53) implies that the effective permittivity of a plasma is given by

$$\epsilon(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\epsilon_0 \tag{10.55}$$

which is *smaller* than ϵ_0. The dispersion relation of electromagnetic waves in a plasma is therefore given by

$$\frac{\omega^2}{k^2} = \frac{1}{\epsilon(\omega)\mu_0} = \frac{c^2}{1 - \left(\frac{\omega_p}{\omega}\right)^2}$$

or

$$\omega^2 = \omega_p^2 + (ck)^2 \tag{10.56}$$

Note that for $\epsilon(\omega)$ to be positive, the frequency ω has to be larger than the plasma frequency, ω_p. If $\omega < \omega_p$, the phase constant (or the wavenumber) k becomes pure imaginary,

$$k = \pm j\frac{1}{c}\sqrt{\omega_p^2 - \omega^2} \tag{10.57}$$

If we recall the spatial dependence e^{-jkz}, we see that waves with $\omega < \omega_p$ is strongly damped, and cannot propagate in a plasma. The characteristic impedance of the plasma

$$Z = \sqrt{\frac{\mu_0}{\epsilon(\omega)}} = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{\sqrt{1 - (\omega_p/\omega)^2}} \tag{10.58}$$

also becomes pure imaginary (pure reactive) when $\omega < \omega_p$ and low frequency waves are completely reflected.

When the frequency ω is sufficiently small compared with the plasma frequency ω_p, ($\omega \ll \omega_p$), the field in a plasma becomes proportional to

$$\exp\left(-\frac{\omega_p}{c}z\right) \tag{10.59}$$

where $z = 0$ is chosen at the air (or vacuum)-plasma boundary. Spatial exponential damping length is therefore given by

$$\delta = \frac{c}{\omega_p} \tag{10.60}$$

This quantity is called the (collisionless) skin depth of a plasma. It is a measure of how deep a low frequency electromagnetic field can penetrate into a plasma. Although no net energy propagates into a plasma when $\omega < \omega_p$, the plasma reactively exchanges energy with the medium outside the plasma in the form of a reactive power. (In a superconductor, the current and magnetic field penetration depth is also given by the skin depth.)

The electron density of the ionospheric plasma is typically $10^{13}/\mathrm{m}^3$. Then, the plasma frequency is $f_p = \omega_p/2\pi \simeq 30$ MHz. Electromagnetic waves having a frequency less than about 30 MHz are thus reflected, while waves with a higher frequency can penetrate. Shortwave radio communication depends on reflection by the ionospheric plasma and the earth surface, which form a global waveguide for low frequency electromagnetic waves. The earth conductivity is typically $\sigma \simeq 10^{-2}$ S/m. Although this is much smaller than the conductivities of metals, the earth can be regarded as a good conductor as long as the impedance

$$Z = \sqrt{\frac{j\omega\mu_0}{\sigma}}$$

is much smaller than that of free space, 377 Ω, so that large reflection occurs. This has an important implication to an antenna erected from the ground as we will see in Chapter 11.

10.5 Phase and Group Velocities

Electromagnetic waves in free space are described by the wave equation

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{10.61}$$

Based on this equation, we have seen in Chapter 9 that TEM modes obey a simple dispersion relation,

$$\omega = ck \qquad \text{or} \qquad f\lambda = c \tag{10.62}$$

The frequency ω is linearly proportional to the wavenumber k over the entire frequency range. Waves obeying such a dispersion relation is called non- dispersive, in the sense that the waveform originally sent out does not deform as it propagates. The linear relation between ω and k ensures that the propagation speed is independent of the frequency, and low frequency and high frequency modes all propagate at the same speed c. Since any waveforms, no matter how complicated they may be, can be Fourier decomposed into many frequency components, we can see that a nondispersive dispersion relation guarantees that there is no deformation in the waveform after propagation over a long distance.

Electromagnetic waves in a plasma, in contrast, obey the dispersion relation

$$\omega^2 = \omega_p^2 + (ck)^2$$

Proportionality between ω and k is broken, as can be seen in Fig. 10.9(a).

The propagation velocity is given by

$$\frac{\omega}{k} = \frac{c}{\sqrt{1-(\omega_p/\omega)^2}} \tag{10.63}$$

which is evidently frequency dependent, lower frequency mode propagating faster than higher frequency mode.

The propagation velocity given in Eq. (10.63) is always larger than c. How do we interpret such a velocity? The relativity theory demands that nothing can travel faster than the speed of light.

For dispersive waves, it is necessary to distinguish two propagation velocities, one the phase velocity given by

$$v_{ph} = \frac{\omega}{k} \quad \text{(phase velocity)} \tag{10.64}$$

and the other the group velocity defined by

$$v_{gr} = \frac{d\omega}{dk} \quad \text{(group velocity)} \tag{10.65}$$

For nondispersive waves characterized by $\omega = ck$, both are identical. The group velocity of the wave described by Eq. (10.63) is

$$\frac{d\omega}{dk} = c\sqrt{1 - (\omega_p/\omega)^2} \tag{10.66}$$

which remains smaller than c. In fact, energy associated with waves is carried at the group velocity, not at the phase velocity.

In a nondispersive medium, the waveform remains unchanged as a wave propagates, as shown in Fig. 10.9(b) (nondispersive). In a dispersive medium, the waveform is bound to change. For example, a wave packet originally sent out becomes spread spatially or "dispersed" as shown in Fig. 10.9(b) (dispersive). The example shown has a phase velocity larger than the group velocity. The ripples propagate with the phase velocity, while the global envelope propagates with the group velocity. Since the wave amplitude is maximum at the envelope peak, we can qualitatively understand that wave energy is carried at the group velocity. As long as the group velocity remains smaller than c, no contradiction with the relativity theory occurs.

Let us convince ourselves of this statement by actually calculating the energy associated with the electromagnetic wave in a plasma. We assume a mode propagating along the z coordinate, with an electric field in x direction, and a magnetic field in y direction,

$$E_x(z,t) = E_0 \mathrm{e}^{j(\omega t - kz)} \tag{10.67}$$

$$H_y(z,t) = \frac{1}{Z} E_0 \mathrm{e}^{j(\omega t - kz)} \tag{10.68}$$

where Z is the impedance given in Eq. (10.58), and ω/k is the phase velocity given in Eq. (10.63). The energy densities associated with these fields are

$$\begin{aligned} &\frac{1}{2}\epsilon_0 \left|E_x\right|^2 + \frac{1}{2}\mu_0 \left|H_y\right|^2 \\ &= \left[1 + 1 - \left(\frac{\omega_p}{\omega}\right)^2\right] \frac{1}{2}\epsilon_0 E^2 \end{aligned} \tag{10.69}$$

However, in a plasma the kinetic energy of electrons must be incorporated, too. Electrons undergo oscillatory motion being accelerated by the electric field. The kinetic energy density associated with electron motion is

$$\begin{aligned} \frac{1}{2} n m v_e^2 &= \frac{1}{2} n m \left(\frac{eE_0}{m\omega}\right)^2 \\ &= \frac{1}{2}\left(\frac{\omega_p}{\omega}\right)^2 \epsilon_0 E_0^2 \end{aligned} \tag{10.70}$$

which cancels one term in Eq. (10.69). Therefore, the total energy density associated with the wave is

$$u = \epsilon_0 E_0^2 \quad (\mathrm{J/m^3}) \tag{10.71}$$

which is formally identical to that of waves in free space.

Now, the energy propagation velocity is by definition

$$\text{energy propagation velocity} = \frac{\text{Poynting flux}}{\text{Energy density}} \tag{10.72}$$

The Poynting flux in the present example is

$$\begin{aligned} S_z &= E_x H_y^* \\ &= \frac{1}{Z} E_0^2 = \sqrt{\frac{\epsilon_0}{\mu_0}} \sqrt{1 - (\omega_p/\omega)^2} E_0^2 \end{aligned} \tag{10.73}$$

Therefore, we find

$$\text{energy propagation velocity} = c\sqrt{1 - (\omega_p/\omega)^2} < c \tag{10.74}$$

which is identical to the group velocity given in Eq. (10.66).

It should be noted that the energy density found in Eq. (10.73) is quite inconsistent with what we (incorrectly) expect in analogy to the dc fields,

$$\begin{aligned} u &= \frac{1}{2}\mathbf{D} \cdot \mathbf{E}^* + \frac{1}{2}\mu_0 \mathbf{H} \cdot \mathbf{H}^* \\ &= \frac{1}{2}\epsilon(\omega) \left|E_x\right|^2 + \frac{1}{2}\mu_0 \left|H_y\right|^2 \\ &= \left(1 - \frac{\omega_p^2}{\omega^2}\right) \epsilon_0 E^2 \quad \text{(incorrect)} \end{aligned} \tag{10.75}$$

When the permittivity is frequency dependent, a correct form for the electric energy density is

$$u_e = \frac{1}{2} \frac{\partial}{\partial \omega} \left[\omega \epsilon(\omega)\right] \left|\mathbf{E}\right|^2 \tag{10.76}$$

which has been first formulated by Landau-Lifshits. Since

$$\frac{\partial}{\partial \omega} \left(\omega \epsilon(\omega)\right) = \epsilon_0 \left[1 + \left(\frac{\omega_p}{\omega}\right)^2\right] \tag{10.77}$$

the energy density predicted by Eq. (10.76) is

$$\frac{1}{2}\epsilon_0 \left[1 + \left(\frac{\omega_p}{\omega}\right)^2\right] E_0^2 \tag{10.78}$$

which consists of the electric field energy density and the electron kinetic energy density. Likewise, if the magnetic permeability is frequency dependent (which occurs in gyrotropic media), the magnetic energy density is to be evaluated from

$$\frac{1}{2}\frac{\partial}{\partial\omega}\left(\omega\mu(\omega)\right)|\mathbf{H}|^2 \tag{10.79}$$

In Chapter 12, we will encounter a dispersion relation similar (or identical) to Eq. (10.56) when analyzing electromagnetic waves confined in a waveguide. The same concept of phase and group velocities is applicable to guided waves. As will be shown, electromagnetic waves confined in a conductor tube cannot be pure TEM modes, and the deviation from the nondispersive dispersion relation, $\omega = kc$, is strictly due to the deviation from TEM modes even though the wave medium is still air in waveguides.

10.6 AC Permittivity of Dielectrics

The permittivity of dielectrics found in Chapter 4

$$\epsilon = \epsilon_0\left(1+\frac{\omega_p^2}{\omega_0^2}\right) \tag{10.80}$$

is for dc fields without time variation. In ac fields, this has to be modified as

$$\epsilon(\omega) = \epsilon_0\left(1+\frac{\omega_p^2}{\omega_0^2-\omega^2}\right) \tag{10.81}$$

In Eqs. (10.80) and (10.81), ω_p is the effective plasma frequency corresponding to the bound (rather than free as in plasmas) electron density, and ω_0 is the frequency of bound harmonic motion. (See Chapter 4.) Let us see how Eq. (10.81) emerges.

In the absence of external electromagnetic fields, an electron bound to an atom or molecules undergoes harmonic motion,

$$m\frac{d^2\mathbf{x}}{dt^2} = m\frac{d\mathbf{v}}{dt} = -\omega_0^2\mathbf{x} \tag{10.82}$$

When an oscillating electric field $\mathbf{E}\mathrm{e}^{j\omega t}$ is applied, the equation of motion is modified as

$$m\frac{d\mathbf{v}}{dt} = -\omega_0^2\mathbf{x} - e\mathrm{E}\mathrm{e}^{j\omega t} \tag{10.83}$$

Differentiating once, we obtain

$$m\frac{d^2\mathbf{v}}{dt^2} = -\omega_0^2\mathbf{v}-j\omega e\mathbf{E}e^{j\omega t} \tag{10.84}$$

The steady oscillatory solution is

$$\mathbf{v}(t) = -\frac{j\omega e\mathbf{E}}{m\left(\omega_0^2-\omega^2\right)}e^{j\omega t} \tag{10.85}$$

Consequent electron current density is therefore

$$\mathbf{J}_e(t) = \frac{ne^2/m\varepsilon_0}{\omega_0^2 - \omega^2}\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \tag{10.86}$$

and the Maxwell's equation becomes

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{J}_e + \varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \\ &= \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right)\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{10.87}$$

Thus, the effective ac permittivity of a dielectric is given by

$$\varepsilon(\omega) = \varepsilon_0\left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right) \tag{10.81}$$

Note that the permittivity of a plasma in Eq. (10.54) corresponds to the special case $\omega_0 = 0$ in Eq. (10.82). This is understandable because electrons in a plasma are free and they are not bound to ions or molecules. Therefore, in a plasma $\omega_0 = 0$. However, a plasma in an external magnetic field exhibits a permittivity similar to Eq. (10.81)

$$\varepsilon_\perp(\omega) = \varepsilon_0\left(1 + \frac{\omega_p^2}{\omega_c^2 - \omega^2}\right) \tag{10.88}$$

where $\omega_c = eB/m$ is the electron cyclotron frequency. The subscript $\perp$ indicates that the permittivity given in Eq. (10.88) pertains to wave electric fields perpendicular to the external magnetic field. The permittivity for electric field parallel to the magnetic field is still given by Eq. (10.55). In other words, the permittivity of a magnetized plasma is anisotropic and depends on the direction of the electric field with respect to the magnetic field. The appearance of the cyclotron frequency in the perpendicular permittivity is expected because for perpendicular electric field, electrons undergoing cyclotron motion are equivalent to those undergoing bound harmonic motion in dielectrics.

Some crystals exhibit anisotropic dielectric properties. This is because electron polarizability is not uniform (anisotropic), and polarizability along a certain crystal axis can be larger than in other directions. Double refraction phenomena, which has been briefly explained