CHAPTER 3. Elementary Fluid Dynamics

- Understanding the physics of fluid in motion
- Derivation of the Bernoulli equation from Newton’s second law

Basic Assumptions of fluid stream, unless a specific comment

1st assumption: Inviscid fluid (Zero viscosity = Zero shearing stress)

→ No force by wall of container and boundary

→ Applied force = Only Gravity + Pressure force

※ Newton’s Second Law of Motion of a Fluid Particle

\[ \sum F = (\text{Net pressure force}) + (\text{Gravity}) = m\ddot{a} = (\text{Fluid mass}) \times (\text{Acceleration}) \]

2nd assumption: Steady flow (?)

→ No Change of flowing feature with time at a given location

→ Every successive particle passing though the same point

: Same path (called streamline) &

Same velocity (tangential to the streamline)
Additional Basic Terms in Analysis of Fluid Motion

- **Streamline** (Path of a fluid particle)
  - Position of a particle
    \[ f(r_o, \vec{v}) \]
    where \( r_o \): Initial position,
    \( \vec{v} \): Velocity of particle
  - No streamlines intersecting each other

- Two Components in *Streamline Coordinates* (See the figure)
  1. **Tangential** coordinate: \( s = s(t) \)
     - **Moving distance** along streamline,
     - Related to Particle’s speed \( (v = ds/dt) \)
  2. **Normal** coordinate: \( n = n(t) \)
     - **Local radius of curvature** of streamline \( R = R(s) \)
     - Related to Shape of the streamline

- Two Accelerations of a fluid particle along \( s \) and \( n \) coordinates
  1. Streamwise acceleration \( (\vec{a}) \) Change of the speed
      \[ a_s = \frac{d^2v}{dt^2} = \frac{\partial \vec{v}}{\partial s} \frac{ds}{dt} = \frac{\partial \vec{v}}{\partial s} \]
      using the Chain rule
  2. Normal acceleration \( (\vec{a}) \) Change of the direction
     \[ a_n = \frac{\vec{v}^2}{R} \]
     (: Centrifugal acceleration)
Q. What generate these $a_s$ and $a_n$?  (Pressure force and Gravity)

Part 1. Newton’s second law along a streamline ($\hat{s}$ direction)

Consider a small fluid particle of size $\delta s \times \delta n \times \delta y$ as shown

Newton’s second law in $\hat{s}$ direction

$$\sum \delta F_s = \delta m a_s = \delta m v \frac{\partial v}{\partial s} = \rho \delta V \frac{\partial v}{\partial s} \text{ along } \hat{s} \text{ direction}$$

$$= \text{Gravity force} + \text{Net Pressure force}$$

where $\delta V$: Volume of a fluid particle $= \delta s \times \delta n \times \delta y$
(i) **Gravity force** along $\hat{s}$ direction

\[
\delta W_s = -\delta W \sin \theta = -(\gamma \delta V) \sin \theta
\]

(ii) **Pressure force** along $\hat{s}$ direction

Let $p$: Pressure at the center of $\delta V$

\[
p \pm \frac{\partial p}{\partial s} \frac{\delta s}{2}: \text{Average pressures at Left face (Decrease)}
\]

\[
p + \frac{\partial p}{\partial s} \frac{\delta s}{2}: \text{Average pressures at Right face (Increase)}
\]

Then, *Net pressure force* along $\hat{s}$ direction, $\delta F_{ps} = (\text{Pressure}) \times (\text{Area})$

\[
\delta F_{ps} = (p - \frac{\partial p}{\partial s} \frac{\delta s}{2}) \delta n \delta y - (p + \frac{\partial p}{\partial s} \frac{\delta s}{2}) \delta n \delta y = -\frac{\partial p}{\partial s} \delta s \delta n \delta y = -\frac{\partial p}{\partial s} \delta V
\]

: Depends not on $p$ itself, but on $\frac{\partial p}{\partial s}$ (*Rate of change in $p$*)

- Total force in $\hat{s}$ direction (Streamline)

\[
\sum \delta F_s = \delta W_s + \delta F_{ps}
\]

\[
\rho \delta V a_s = \rho \delta V \frac{\partial v}{\partial s} = (\gamma \sin \theta - \frac{\partial p}{\partial s}) \delta V
\]

Finally, Newton’s second law *along a streamline* ($\hat{s}$ direction)

\[
\therefore \rho a_s = \rho v \frac{\partial v}{\partial s} = -\gamma \sin \theta - \frac{\partial p}{\partial s}
\]

*Change of Particle’s speed*

\[
\therefore \text{Affected by Weight and Pressure Change}
\]
Making this equation more familiar

\[ \rho v \frac{\partial v}{\partial s} = -\gamma \sin \theta - \frac{\partial p}{\partial s} \]

\[ \frac{1}{2} \rho \frac{d(v^2)}{ds} = -\gamma \frac{dz}{ds} - \frac{dp}{ds} \]

because \( \sin \theta = \frac{dz}{ds} \) (See the figure above)

\[ \frac{\partial p}{\partial s} = \frac{dp}{ds} \]

using \( dp = \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial n} dh = \frac{\partial p}{\partial s} ds \) (why?)

\[ v \frac{\partial v}{\partial s} = v \frac{dv}{ds} = \frac{1}{2} \frac{dv^2}{ds} \]

or

\[ dp + \frac{1}{2} \rho d(v^2) + \gamma dz = 0 \]

(Divided by \( ds \))

or

\[ \int \frac{dp}{\rho} + \frac{1}{2} v^2 + gz = \text{constant} \]

(By integration)

By assuming a constant \( \rho \) (Incompressible fluid): 3\textsuperscript{rd} assumption

\[ : \quad p + \frac{1}{2} \rho v^2 + \gamma z = \text{Constant} \]

along streamline (\( \hat{s} \) direction)

: Bernoulli equation along a streamline

\( \cong \) Valid for (1) a steady flow of (2) incompressible fluid
(3) without shearing stress

c.f. If \( \rho \) is not constant (Compressible, e.g. Gases),

\[ \cong \rho = \rho(p) : \text{Must be known to integrate } \int \frac{dp}{\rho}. \]
What this *Bernoulli* Equation means? (Physical Interpretation)

For a *Steady* flow of *Inviscid* and *Incompressible* fluid,

\[ p + \frac{1}{2} \rho v^2 + \gamma z = \text{Constant along streamline} \quad (1) \]

: Mathematical statements of *Work-energy principle*

• Unit of Eq. (1): \([ \text{N/m}^2 ] = [\text{N} \cdot \text{m/m}^3 ] = [\text{Energy per unit volume}] \]

\[ p = \text{Works on unit fluid volume done by pressure} \]
\[ \gamma z = \text{Works on unit fluid volume done by weight} \]
\[ \frac{1}{2} \rho v^2 = \text{Kinetic energy per unit fluid volume} \]

※ Same *Bernoulli Equations in different units*

1. Eq (1) ÷ \( \gamma \) \( \Leftrightarrow \) \([\text{N} \cdot \text{m/m}^3] ÷ [\text{N} \cdot \text{m/m}^3] = [\text{m}] = [\text{Length unit}] \)

\[ \frac{p}{\gamma} + \frac{v^2}{2g} + z = \text{Constant} \quad \text{(Head unit)} \]

\[\frac{p}{\gamma}: \text{Depth of a fluid column produce } p \quad \text{(Pressure head)}\]
\[\frac{v^2}{2g}: \text{Height of a fluid particle to reach } v \text{ from rest by free falling} \quad \text{(Velocity head)}\]
\[z: \text{Height corresponding to Gravitational potential} \quad \text{(Elevation head)}\]
**Part 2.** Newton’s second law **normal to a streamline** ($\hat{n}$ direction)

Consider the same situation as Sec. 3.3 shown in figure

For a small fluid particle of size $\delta s \times \delta n \times \delta y$ as shown

Newton’s second law in $\hat{n}$ direction

\[
\sum \delta F_n = \delta m a_n = \delta m \frac{v^2}{R} = \rho \delta V \frac{v^2}{R} \quad \text{along } \hat{n} \text{ direction}
\]

\[= \text{Gravity force} + \text{Net Pressure force}\]
(i) *Gravity force* along \( \hat{n} \) direction

\[
\delta W_n = -\delta W \cos \theta = -(\gamma \delta V) \cos \theta
\]

(ii) *Pressure force* along \( \hat{n} \) direction

By the same manner in the previous case,

\[
\delta F_{pn} = (p - \frac{\partial p}{\partial n} \frac{\delta n}{2}) \delta s \delta y - (p + \frac{\partial p}{\partial n} \frac{\delta n}{2}) \delta s \delta y = -\frac{\partial p}{\partial n} \delta n \delta s \delta y = -\frac{\partial p}{\partial n} \delta V
\]

- **Total force in \( \hat{n} \) direction (Normal to Streamline)**

\[
\sum \delta F_n = \delta W_n + \delta F_{pn}
\]

\[
\therefore \rho \delta V a_n = \rho \delta V \frac{v^2}{R} = (-\gamma \cos \theta - \frac{\partial p}{\partial n}) \delta V
\]

\[\therefore \frac{v^2}{R} = -\gamma \cos \theta - \frac{\partial p}{\partial n} \text{ normal to streamline (} \hat{n} \text{ direction)}
\]

**Change of Particle’s direction of motion**

\[\therefore \text{Affected by Weight and Pressure Change along } \hat{n} \]

**Ex.** If a fluid flow: *Steep direction change* \((R \downarrow)\) or *fast flow* \((v \uparrow)\) or *heavy* \((\rho \uparrow)\) fluid

\[\therefore \text{Generate large force unbalance} \]

- **Special case: Standing close to a Tornado**

  i.e. Gas flow (Negligible \( \gamma \)) in horizontal motion \((\frac{dz}{dn} = 0)\)

\[-\gamma \frac{dz}{dn} - \frac{\partial p}{\partial n} = \rho \frac{v^2}{R} \quad \rightarrow \quad \frac{\partial p}{\partial n} = -\rho \frac{v^2}{R} < 0 \quad \text{(Attractive)}\]

  \[\therefore \text{Moving closer } (R \downarrow) \quad \therefore \text{More dangerous } (\frac{\partial p}{\partial n} \uparrow)\]
Making this equation more familiar

By the same manner as the previous case,

\[
\rho \frac{v^2}{R} = -\gamma \cos \theta - \frac{\partial p}{\partial n} \quad \Leftrightarrow \quad \rho \frac{v^2}{R} = -\gamma \frac{dz}{dn} - \frac{dp}{dn}
\]

because \( \cos \theta = \frac{dz}{dn} \) (See the figure)

\[
\frac{\partial p}{\partial n} = \frac{dp}{dn}, \text{ since } dp = \frac{\partial p}{\partial s} \, ds + \frac{\partial p}{\partial n} \, dn = \frac{\partial p}{\partial n} \, dn
\]

or \( \int \frac{dp}{\rho} + \int \frac{v^2}{R} \, dn + gz = \text{Constant} \) (normal to streamline)

By assuming a constant \( \rho \) (Incompressible fluid): \( 3^{\text{rd}} \) assumption

\[
\therefore \quad p + \rho \int \frac{v^2}{R} \, dn + \gamma z = \text{Constant} \quad \text{(normal to streamline)}
\]

\( \text{Bernoulli equation normal to streamline} \) \( (\hat{n} \text{ direction}) \)

\( \Leftrightarrow \) Valid for (1) a steady flow of (2) incompressible fluid

(3) without shearing stress