Purpose of Error Calculations

- The purpose of error calculations is to determine the uncertainty in a calculated quantity, based upon the estimated uncertainties in the measured data used to calculate the quantity.

Estimating Errors in Measurements

- The error in a measurement is at least as large as the smallest scale division of the measuring instrument.
- i.e. Don’t try to read between divisions.
- It is fine to assign an error of more than one division if you feel justified.
Calculus Method

- The Calculus method of Error Calculations uses differentials of mathematical expressions, which is similar, but not identical, to the derivatives of the same mathematical expressions.

Basic Rules of Derivatives and Differentials

- 1a) Derivative Exponent Rule
  \[ \frac{d}{dx}(cx^n) = ncx^{n-1} \]

- 1b) Differential Exponent Rule
  \[ d(cx^n) = ncx^{n-1} \, dx \]
Basic Rules of Derivatives and Differentials

• 2a) Derivative Addition/Subtraction Rule
\[ \frac{d}{dx} (u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx} \]

• 2b) Differential Addition/Subtraction Rule
\[ d(u + v - w) = du + dv - dw \]

Basic Rules of Derivatives and Differentials

• 3a) Derivative Product Rule
\[ \frac{d}{dx} (uvw) = v\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx} \]

• 3b) Differential Product Rule
\[ d(uvw) = vwdx + uwdx + uvdy \]
Basic Rules of Derivatives and Differentials

• 4a) Derivative Quotient Rule

\[
\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v(du/dx) - u(dv/dx)}{v^2}
\]

• 4b) Differential Quotient Rule

\[
d\left( \frac{u}{v} \right) = \frac{v(du) - u(dv)}{v^2}
\]

Tip – Avoid Quotient Rule

• For error calculation purposes, the quotient rule is clumsy.
• It is better to rewrite the original expression as a product and use the product and exponent rules:

\[
d\left( \frac{u}{v} \right) = d(uv^{-1}) = (du)v^{-1} + u(-1)v^{-2}dv
\]
Basic Rules of Derivatives and Differentials

• 5a) Derivative Chain Rule
\[
\frac{d}{dx} (\sin \theta) = (\cos \theta) \frac{d\theta}{dx}
\]

• 5b) Differential Chain Rule
\[d(\sin \theta) = (\cos \theta)d\theta\]

• 6a) Derivative Exponential Rule (base e)
\[
\frac{d}{dx} (e^x) = e^x
\]

• 6b) Differential Exponential Rule (base e)
\[d(e^x) = (e^x)dx\]
Basic Rules of Derivatives and Differentials

• 7a) Derivative Exponential Rule (base 10)

\[
\frac{d}{dx} \left( \frac{10^x}{10} \right) = \frac{1}{10} \frac{d}{dx} (e^{x \ln 10}) = \left( \frac{1}{10} \right) (\ln 10) (e^{x \ln 10})
\]

\[
\frac{d}{dx} \left( \frac{10^x}{10} \right) = \left( \frac{1}{10} \right) (\ln 10) (10^x) = (\ln 10) \left( \frac{10^x}{10} \right)
\]

• 7b) Differential Exponential Rule (base 10)

\[
d \left( \frac{10^x}{10} \right) = \frac{1}{10} d(e^{x \ln 10}) = \left( \frac{1}{10} \right) (\ln 10) (e^{x \ln 10}) dx
\]

\[
d \left( \frac{10^x}{10} \right) = \left( \frac{1}{10} \right) (\ln 10) (10^x) dx = (\ln 10) \left( \frac{10^x}{10} \right) dx
\]
Basic Rules of Derivatives and Differentials

- **8a) Derivative Logarithm Rule (base e)**
  \[
  \frac{d}{dx} (\ln x) = \frac{1}{x}
  \]

- **8b) Differential Logarithm Rule (base e)**
  \[
  d(\ln x) = \frac{dx}{x}
  \]

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Basic Rules of Derivatives and Differentials

- **9a) Derivative Logarithm Rule (base 10)**
  \[
  \frac{d}{dx} (10 \log x) = 10 \frac{d}{dx} (\log x) = 10 \frac{d}{dx} \left( \frac{\ln x}{\ln 10} \right)
  \]
  \[
  \frac{d}{dx} (10 \log x) = \frac{10}{\ln 10} \left( \frac{dx}{x} \right)
  \]
Basic Rules of Derivatives and Differentials

• 9a) Differential Logarithm Rule (base 10)

\[ d(10 \log x) = 10d(\log x) = 10d \left( \frac{\ln x}{\ln 10} \right) \]
\[ d(10 \log x) = \frac{10}{\ln 10} \left( \frac{dx}{x} \right) \]

Error Calculation Procedure

• Step 1a)
   Must ensure that original expression is in simplest form, with each variable appearing as few times as possible
• Simplify: \( Y = am^{2/m} \) to \( Y = am \)
• Simplify: \( Z = ax - bx \) to \( Z = (a - b)x \)
Error Calculation Procedure

• Step 1b)
  Ensure that each variable is independent of all other variables in the original expression
  • Independent variables are quantities that are measured separately
  • If \( y = ax \) and \( Z = yx \), then \( y \) and \( x \) are NOT independent
  • Solution: Substitute \( y = ax \) into \( Z \) equation, \( Z = (ax)x = ax^2 \)

Error Calculation Procedure

• Step 2)
  Using rules from calculus, calculate the differential of the original expression.
  (See previous calculus summary)
  E.g.:
  \[
  K = \frac{1}{2} mv^2
  \]
  \[
  dK = \frac{1}{2} (dm)v^2 + \frac{1}{2} m(2v)(dv)
  \]
  \[
  dK = \frac{1}{2} (dm)v^2 + mv(dv)
  \]
Error Calculation Procedure

• Step 3)
To avoid having to repeatedly insert both positive and negative error values into error equations, take the absolute value of each term and always ADD terms (even if calculus rules tell you to subtract terms)

\[ dK = \left| \frac{1}{2} (dm) v^2 \right| + \left| m v (dv) \right| \]

Error Calculation Procedure

• Step 4)
Replace each differential \((dx)\) with the corresponding absolute error \((\delta x)\)

\[ \delta K = \left| \frac{1}{2} (\delta m) v^2 \right| + \left| m v (\delta v) \right| \]

\[ \delta K = \left| \frac{1}{2} v^2 (\delta m) \right| + \left| m v (\delta v) \right| \]
Error Calculation Procedure - Quadrature

• Step 5)
  A better estimate of the uncertainty is obtained by adding the error equation terms “in quadrature”:

\[ \delta K = \left| \frac{1}{2} (\delta m) v^2 \right| + |mv(\delta v)| \text{ becomes} \]

\[ \delta K = \sqrt{\left( \frac{1}{2} v^2 (\delta m) \right)^2 + (mv(\delta v))^2} \]

Error Calculation Procedure - Quadrature

• Rather than calculating the maximum possible uncertainty, addition in quadrature accounts for the fact that it is unlikely that the uncertainties in measurements will always add in such a way as to give the worst possible result.

• The uncertainty calculated using quadrature is a better estimate of the likely uncertainty.
Error Calculation Procedure

• Step 5)
  When angles are used, express the absolute error in RADIANS
  • You should keep the angle in degrees.
  • Example: If $\theta = 30^\circ \pm 1^\circ$, and $Y = \sin \theta$,
    $\delta Y = |\cos (\theta) \delta \theta| = |\cos(30^\circ) (\pi / 180)|$

Error Formula Check

• Error formulae should ALWAYS have one and only one error factor ($\delta x, \delta m$, etc) in each term.
Sample Problem

• Suppose that the mass and speed of an object have been measured and you wish to calculate the momentum of the object, and the uncertainty in this momentum:
  • $m = 3.1 \text{ kg} \pm 0.1 \text{ kg}$,
  $v = 11.2 \text{ m/s} \pm 0.5 \text{ m/s}$
  • momentum, $p = mv$

Sample Problem Solution

• $m = 3.1 \text{ kg} \pm 0.1 \text{ kg}$,
  $v = 11.2 \text{ m/s} \pm 0.5 \text{ m/s}$
• Find uncertainty in $p = mv = 34.72 \text{ kg.m/s}$
• $dp = (dm)v + m (dv)$
• $\delta p = |(\delta m)v| + |m (\delta v)|$
• $\delta p = [(v(\delta m))^2 + (m (\delta v))^2]^{1/2}$ (quadrature)
• $\delta p = [((11.2 \text{ m/s})(.1 \text{ kg}))^2 + (3.1 \text{ kg}(.5 \text{ m/s}))^2]^{1/2}$
• $\delta p = 1.9123023 \text{ kg.m/s}$
• $p = 34.72 \pm 1.9123023 \text{ kg.m/s}$ (Reasonable?)
Rounding-off the Result

- Remember that the calculated value is based on measurements and that the uncertainties in the measurements are estimates.
- Must be careful to use a reasonable number of digits to express the result.
- A useful rule is to express the uncertainty in the result to two significant figures, and then round-off the value to the same number of decimal places.

Thus,

\[ p = 34.72 \pm 1.9123023 \text{ kg.m/s becomes:} \]

\[ p = 34.7 \pm 1.9 \text{ kg.m/s} \]

The uncertainty has been rounded to two digits, resulting in significance in the tenth place (but no further).

The value, 34.7, has then been rounded to the tenth place as well.
Check Units

- Always check that each term of an error equation has the same units.
- If not, your error formula is wrong.

Dummy Variable Technique for complicated formulae

- If you need to determine an error calculation for a complicated formula, then use dummy variables to represent part of the equation.
- Work out the differential for each part and then substitute everything back into the original expression to obtain the final error equation.
Example of Dummy Variable Technique

• 1) Find error equation for complex formula
  \[ Y = A(B - C) \] where \( A, B \) and \( C \) are
  variables. [This formula is complex
  because calculus has different rules for
  multiplication and subtraction.]

• 2) Define dummy variable \( D = B - C \)

• 3) Then: \( Y = AD \)

• 4) Find differential of \( Y \):
  \[ dY = (dA)D + A(dD) \]

• 5) Error Formula for \( Y \):
  \[ \delta Y = |(\delta A)D| + |A(\delta D)| \]
  continued next slide…

Example of Dummy Variable Technique

• 6) Find differential of \( D \):
  \[ dD = dB - dC \]

• 7) Error formula for \( D \):
  \[ \delta D = |\delta B| + |\delta C| \]

• 8) Substitute into \( \delta Y = |(\delta A)D| + |A(\delta D)| \):
  \[ \delta Y = |(\delta A)(B - C)| + |A(\delta B)| + |A(\delta C)| \]

• 9) Expand so that each term has only 1 error
  factor:
  \[ \delta Y = |(\delta A)(B - C)| + |A(\delta B)| + |A(\delta C)| \]