Today’s lecture

V. Magnetostatics (in vacuum)

- Gauge transformations
- Magnetostatic boundary conditions
- Examples
Last session

\[ \nabla \cdot \vec{B} = 0 \quad \oint_S \vec{B} \cdot d\vec{a} = 0 \]

magnetic fields of steady currents → **Ampère’s law**

\[ \nabla \times \vec{B} = \mu_0 \cdot \vec{J}(\vec{r}) \quad \oint_{\partial P} \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enclosed}} \]

Electrostatic: Magnetostatic:

Start with Coulomb started with Biot-Savart

⇒ Applied theorems and derived alternative form:

Gauss’s law Ampere’s law

• The electric field diverges from its sources the (positive) charges.

• The magnetic field is source free and curls around the electric current.
In magnetostatics the fact that \( \nabla \cdot \vec{B} = 0 \)

Plus \( \nabla \cdot (\nabla \times \vec{A}) = 0 \) invites the introduction of the **Magnetic Vector potential**:

\[
\vec{B} = \nabla \times \vec{A}
\]

Unfortunately a similar scalar potential would be incompatible with Ampère’s law:

\[
\nabla \times \vec{B} = \nabla \times \left( -\nabla \cdot \vec{A} \right) = 0 \neq \mu_0 \cdot \vec{J}
\]

The definition \( \vec{B} = \nabla \times \vec{A} \) specifies the curl of \( \vec{A} \), we still have the freedom to choose its divergence!

This leads to the magnetostatic equivalent of the Poisson equation:

\[
\nabla^2 \vec{A} = -\mu_0 \cdot \vec{J} \\
\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau'
\]

\[\vec{J}(\vec{r}') \to 0 \quad \text{for} \quad |\vec{r}'| \to \infty\]
Scalar potential $V$: we had the freedom to add any function with gradient zero (a constant) without altering $E = -\nabla V$

Magnetic vector potential $A$: We can add any vector field whose curl vanishes $B = \nabla \times A$ without affecting $B$

Due to $\nabla \times (\nabla \cdot \lambda) = 0$ this is equivalent with adding a gradient of a scalar: $A = A' - \nabla \lambda$

This transformation between the magnetic vector potentials $A$ and $A'$ is called **Gauge transformation**.

The magnetic field is invariant under such transformations ("gauge invariant").

Imposing the divergence of the magnetic field results in

$$
\nabla \cdot B = \nabla \cdot (\nabla \times A) = \nabla \cdot (\nabla \times (A' - \nabla \cdot \lambda)) = \nabla \cdot (\nabla \times A') - \nabla^2 \cdot \lambda = -\nabla^2 \cdot \lambda = 0
$$
Imposing Ampère’s law leads to
\[ \nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla \cdot (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \cdot \vec{J} \]

We still have the freedom to choose the divergence of \( \vec{A} \) in such a way that \( \nabla \cdot \vec{A} = 0 \).

This is called the **Coulomb Gauge**.

It is the suitable normalization for magnetostatics since it simplifies Ampère’s law to
\[ \nabla^2 \vec{A} = -\mu_0 \cdot \vec{J} \]

Analogue to Poisson’s equation in electrostatics, we can establish the solution for \( \vec{A} \):
\[ \nabla^2 V = -\frac{\rho}{\varepsilon_0} \quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau' \]

\[ \nabla^2 \vec{A} = -\mu_0 \cdot \vec{J} \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau' \]

This holds as long as
\[ \vec{J}(\vec{r}') \to 0 \quad \text{for} \quad |\vec{r}'| \to \infty \]
Boundary conditions for $\vec{B}$

Component perpendicular to the surface:

$$\oint_S \vec{B} \cdot d\vec{a} = 0 \quad B^\perp_{\text{above}} - B^\perp_{\text{below}} = 0$$

Component parallel to the surface

$$\oint_P \vec{B} \cdot d\vec{l} = \mu_0 \cdot I_{\text{enclosed}} = (B^\parallel_{\text{above}} - B^\parallel_{\text{below}}) \cdot \vec{l} = \mu_0 \cdot K \cdot \vec{l}$$

$$B^\parallel_{\text{above}} - B^\parallel_{\text{below}} = \mu_0 \cdot K$$

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0 \cdot (\vec{K} \times \vec{u}_{\text{normal}})$$

The vector Potential is continuous across any boundary!

$$\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow A^\perp_{\text{above}} - A^\perp_{\text{below}} = 0$$

$$\vec{\nabla} \times \vec{A} = \vec{B} \Rightarrow A^\parallel_{\text{above}} - A^\parallel_{\text{below}} = 0$$

$$\vec{A}_{\text{above}} - \vec{A}_{\text{below}} = 0$$

$$\frac{\delta \vec{A}_{\text{above}}}{\delta n} - \frac{\delta \vec{A}_{\text{below}}}{\delta n} = -\mu_0 \cdot \vec{K}$$
Three fundamental quantities of magnetostatics $\vec{J}(\vec{r})$, $\vec{B}(\vec{r})$ and $\vec{A}(\vec{r})$ are connected by the equations:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_\mathcal{V} \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_\mathcal{V} \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \times \vec{u}_d \, d\tau'$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \ ; \ \vec{\nabla} \cdot \vec{A} = 0$$

All equations are based on two experimental observations (the rest is math):

- The superposition principle and
- The law of Biot-Savart.
(Problem 5.21): Suppose there did exist magnetic monopoles. How would you modify Maxwell’s equations and the force Laws, to accommodate them?

Gauss
\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]
\[ \nabla \times \vec{E} = \alpha_0 \cdot \vec{J}_{mag} \]

Boundary condition \( \vec{E} \to 0 \) far away from all charges

Ampère
\[ \nabla \times \vec{B} = \mu_0 \cdot \vec{J}(\vec{r}) \]

Boundary condition \( \vec{B} \to 0 \) far away from all currents
(Problem 5.14): A thick slab extending from \( z = -a \) to \( z = +a \) carries a uniform volume current \( \vec{J} = J \vec{u}_x \). Find the magnetic field, as a function of \( z \), both inside and outside the slab.

(Problem 5.17): Show that the magnetic field of an infinite solenoid runs parallel to the axis, \textit{regardless of the cross-sectional shape of the coil}, as long as that shape is constant along the length of the solenoid.